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SECOND-ORDER POLYGONAL DISCRETIZATION OF THE AREA FUNCTIONAL AND EXPLICIT ERROR ASYMPTOTICS

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Abstract. We study the chordal, shoelace-type discretization $S_n(\gamma)$ of the oriented area functional $A(\gamma) = \frac{1}{2}\int_a^b (x\,dy - y\,dx)$ for a planar parametrized curve γ, computed using only the sampled vertices $\gamma(t_k)$ of a uniform partition. For a closed curve, S_n is exactly the signed area of the inscribed polygon through the sampled points, so $A(\gamma) - S_n(\gamma)$ is the classical polygonal area defect. For an open curve, S_n is the oriented area of the closed polygonal path obtained by adjoining the origin to the sampled polygonal chain. We establish convergence for C^1 curves and, for C^3 curves, derive the explicit asymptotic expansion

$$A(\gamma) - S_n(\gamma) = \tfrac{h^2}{12}\int_a^b \det(\gamma'(t), \gamma''(t))\,dt + \mathcal{O}(h^3),$$

where h is the mesh size. For a regular curve, the leading coefficient is curvature-governed, since $\det(\gamma', \gamma'') = |\gamma'|^3\kappa$; the leading-order curvature dependence of the area defect is classical, and our contribution is to make it explicit in determinantal parametric form for general, not necessarily convex, curves under a fixed given parametrization the setting natural to parametric and rational representations complementing the convex-geometry results, which instead characterize the optimally spaced inscribed polygon. We further clarify the relationship to classical quadrature: although S_n coincides with the trapezoidal rule applied to the area 1-form $\frac{1}{2}(x\,dy - y\,dx)$, it is not the trapezoidal rule applied to the scalar integrand $\frac{1}{2}\det(\gamma, \gamma')$, and the two schemes have genuinely different error behavior for smooth periodic parametrizations the latter may converge spectrally, whereas S_n is generically only second order. As a worked application we analyze the rational (stereographic) parametrization of the unit circle, obtaining a sequence of rational approximations to π with the explicit error constant $\left(\frac{1}{6} + \frac{\pi}{16}\right)n^{-2}$ and showing numerically that Richardson extrapolation accelerates convergence from $\mathcal{O}(n^{-2})$ to approximately $\mathcal{O}(n^{-4})$ for this example. Numerical experiments confirm the predicted rate.

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Keywords. Polygonal approximation, area functional, parametric curves, asymptotic expansion, rational parametrization, quadrature error, approximation of π.

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1. INTRODUCTION

1. The area functional and its polygonal discretization. The oriented area enclosed by a planar curve admits the classical Green's-theorem representation

$$A(\gamma) = \tfrac{1}{2}\int_a^b (x\,dy - y\,dx) = \tfrac{1}{2}\int_a^b \det(\gamma(t), \gamma'(t))\,dt,$$

a functional fundamental to differential geometry and the theory of differential forms [1, 2]. A natural discretization replaces the curve by the polygonal chain through the sampled points and, when the curve is open, closes this chain by joining its endpoints to the origin. The resulting quantity,

$$S_n(\gamma) = \tfrac{1}{2}\sum_{k=1}^{n} \big[x(t_{k-1})\,y(t_k) - y(t_{k-1})\,x(t_k)\big],$$

is nothing other than the shoelace, or surveyor's, area formula (Meister, 1769; Gauss, 1795) [3] evaluated on the sample points. The object S_n is therefore classical as a formula; what concerns us is the precise behavior of its error.
2. What is already known. For a closed curve, S_n is the area of the inscribed polygon, and $A(\gamma) - S_n(\gamma)$ is the classical polygonal area defect. In convex geometry it is known that the area between a smooth convex curve and its best inscribed n-gon decays like n^{-2} with a curvature-dependent constant: Fejes Tóth [20] identified the governing constant, McClure and Vitale [19] gave the sharp asymptotic $\frac{1}{12}\big(\int \kappa^{1/3}\,ds\big)^3 n^{-2}$ for the optimally spaced inscribed polygon, and Gruber and Kenderov [21] established the corresponding rate in all dimensions and showed it is best possible for curves of differentiability class two. At the level of a single chord, the area between an arc and the chord subtending it is, to leading order, $\frac{1}{12}\kappa L^3$ in essence the Archimedean parabolic-segment estimate and summing this elementary contribution along a curve already produces a curvature integral. In discrete differential geometry the same phenomenon appears as second-order curvature convergence of the inscribed polygon: this tradition begins with Sauer [23] and continues through Bobenko and Suris [4]; Müller and Vaxman [24], for instance, show that the discrete curvature read off from an inscribed polygon agrees with the smooth curvature up to $\mathcal{O}(\varepsilon^2)$. In short, the fact that the leading area-defect error is second order and curvature governed is not new.
3. What is not readily available, and what we contribute. What does not appear to be stated explicitly in the literature, and what we provide, is threefold.

First, an explicit, self-contained asymptotic expansion of the error in determinantal parametric form, $A(\gamma) - S_n(\gamma) = \frac{h^2}{12}\int_a^b \det(\gamma', \gamma'')\,dt + \mathcal{O}(h^3)$, valid for general (not necessarily convex, not necessarily simple) C^3 curves under a fixed parametrization with a uniform parameter mesh. We emphasize that this differs in kind from the convex-geometry results above: those optimize the placement of the vertices and so produce the affine-arclength functional $\big(\int \kappa^{1/3}\,ds\big)^3$, whereas we analyze the error of a prescribed parametrization and obtain the parametrization-dependent functional $\int_a^b |\gamma'|^3\kappa\,dt = \int_a^b \det(\gamma',\gamma'')\,dt$. The latter is the relevant quantity when the curve is presented through a particular parametric or rational representation, which is precisely the situation in applications.

Second, a precise clarification of the relationship between S_n and classical quadrature. The sum S_n is exactly the trapezoidal rule applied to the area 1-form $\frac{1}{2}(x\,dy - y\,dx)$ in the Stieltjes sense; it is not the trapezoidal rule applied to the scalar integrand $f(t) = \frac{1}{2}\det(\gamma(t),\gamma'(t))$. These two schemes are genuinely different: for a closed smooth (periodic) curve, the trapezoidal rule applied to the smooth periodic integrand f is spectrally accurate by the Euler–Maclaurin formula [10], whereas S_n is generically second order, as the regular inscribed polygon in a circle already shows. We make this distinction precise, explaining why the error of S_n is an integral of $\det(\gamma',\gamma'')$ rather than the boundary term $f'(b) - f'(a)$ produced by Euler–Maclaurin; the underlying reason is that $\det(\gamma',\gamma'')$ is not in general a total derivative, so the error integral does not collapse to boundary data.

Third, a fully worked rational example. Applying the expansion to the stereographic parametrization of the unit circle yields a sequence of rational approximations to π, involving no square roots or transcendental evaluations, with leading error $\left(\frac{1}{6} + \frac{\pi}{16}\right) n^{-2}$. The idea of approximating π through inscribed polygons is of course Archimedean, and the use of extrapolation to accelerate such polygon sequences goes back to Huygens (1654) and was a centerpiece of Richardson and Gaunt [25]; our point is not the strategy but the concrete rational realization, in which the chordal area of a rational parametrization produces an all-rational sequence with a closed-form leading error constant. In the numerical experiment, one Richardson step improves the observed rate from $\mathcal{O}(n^{-2})$ to approximately $\mathcal{O}(n^{-4})$.

4. Organization. Section 2 sets up the area functional and the chordal approximation and records their elementary invariance properties. Section 3 proves convergence under C^1 regularity. Section 4 derives the second-order asymptotic expansion under C^3 regularity and gives its curvature interpretation. Section 5 treats the rational parametrization of the circle and computes the error constant explicitly. Section 6 reports numerical experiments and the Richardson acceleration.

1.1. Background and related work. Polygonal area formulas. The expression for the area of a polygon as the alternating sum of vertex cross products is classical, going back to Meister (1769) and Gauss (1795) and known in surveying as the surveyor's area formula; an accessible modern account is Braden [3]. Our S_n is exactly this formula evaluated on sampled curve points, and we treat it not as a new construction but as the object whose error we analyze.

Area defect of approximating polygons. The deviation between a smooth convex body and its inscribed or circumscribed polygons is a classical theme of convex geometry. Fejes Tóth [20] identified the governing curvature constant; McClure and Vitale [19] proved that the symmetric-difference area between a smooth convex body and its best inscribed n-gon is asymptotically $\frac{1}{12}\left(\int \kappa^{1/3}\, ds\right)^3 n^{-2}$; Gruber and Kenderov [21] extended the rate to general dimension and showed it cannot be improved for the differentiability class considered. Surveys by Gruber [22] and the broader literature on polytopal approximation of convex bodies give the wider picture. These results optimize the distribution of vertices and so are sharp for best approximation. Our setting is complementary: we fix a parametrization and a uniform parameter mesh and ask for the error of the resulting chordal sum, obtaining the parametrization-dependent functional $\int_a^b \det(\gamma', \gamma'')\, dt$ in place of the optimal affine-arclength functional. The two agree in spirit—both are second order and curvature governed—but differ in the precise constant and in the class of curves treated.

Discrete differential geometry. Approximating a smooth curve by its inscribed polygon and studying the convergence of discrete invariants is the foundational move of discrete differential geometry, beginning with Sauer [23] and developed extensively by Bobenko and Suris [4] and in the discrete-exterior-calculus framework [5, 26]. Within this tradition the second-order agreement of inscribed-polygon curvature with smooth curvature is well documented, and discrete differential-geometry operators for curvature are widely used [6]; Müller and Vaxman [24], for example, establish $\kappa_k = \kappa + \mathcal{O}(\varepsilon^2)$ for a cross-ratio discrete curvature on the inscribed polygon. Our area-defect expansion is the integrated, area-functional counterpart of this curvature-convergence statement, made explicit with its leading constant.

Quadrature and the Euler–Maclaurin expansion. The error theory of the trapezoidal and related rules, and its refinement through the Euler–Maclaurin formula, is standard [7, 12, 8], as is the spectral accuracy of the trapezoidal rule on smooth periodic integrands [10, 11]. As noted above, S_n is the trapezoidal rule on the area 1-form but not on the scalar integrand, and we use the periodic case as a clean diagnostic separating the two schemes.

Polygonal approximation of π and extrapolation. Estimating π from the area or perimeter of inscribed and circumscribed polygons is the method of Archimedes, and the regular inscribed n-gon already exhibits an $\mathcal{O}(n^{-2})$ area defect. The acceleration of such sequences by extrapolation has a long history: Huygens (1654) effectively applied extrapolation to polygon perimeters, and the polygon-π sequence was a principal illustration in Richardson and Gaunt's [25] account of the deferred approach to the limit. Convex combinations of inscribed and circumscribed estimates, and the rapidly convergent AGM-based algorithms [16], provide much faster routes to π; our rational sequence is not competitive with these methods and is not intended to be. Its interest is structural: it arises purely from the chordal area of a rational parametrization, every term is a rational number, and it carries a closed-form leading error constant that one Richardson step removes.

Summary of contributions. Against this background, the contributions of the present paper are: (i) an explicit determinantal second-order error expansion for the chordal area sum of a general C^3 curve under a fixed parametrization, with the curvature interpretation made precise; (ii) a clarification distinguishing this scheme from the trapezoidal rule applied to the scalar area integrand, including why its error is a curvature integral rather than a boundary term; and (iii) a worked rational realization for the unit circle giving an all-rational sequence of approximations to π with an explicit leading error constant and a numerically observed Richardson acceleration. We do not claim the second-order, curvature-governed nature of the area defect as new; that is classical, and we cite the relevant convex-geometry and discrete-differential-geometry literature accordingly.

2. THE AREA FUNCTIONAL AND POLYGONAL APPROXIMATION

2.1. The area functional as a line integral. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a planar parametrized curve,

$$\gamma(t) = (x(t), y(t)), \qquad t \in [a,b],$$

where x, y are absolutely continuous on $[a,b]$ (in particular, $x, y \in C^1([a,b])$ suffices for all arguments below).

We consider the classical oriented area functional associated with γ,

$$A(\gamma) \;=\; \tfrac{1}{2}\int_a^b \big(x(t)\,y'(t) - y(t)\,x'(t)\big)\,dt. \tag{1}$$

Equivalently, writing $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$, one may express (1) as the line integral of a 1–form,

$$A(\gamma) \;=\; \tfrac{1}{2}\int_\gamma (x\,dy - y\,dx). \tag{2}$$

When γ is a positively oriented simple closed curve and the parameter interval corresponds to one traversal of the boundary of a planar domain Ω, Green's theorem [9, 1] yields

$$A(\gamma) = \mathrm{Area}(\Omega).$$

More generally, for nonclosed curves (1) is a well-defined geometric quantity depending on the choice of the origin and can be interpreted as an "oriented swept area" of the position vector $\gamma(t)$.

For later use, note that the integrand in (1) can be written via the planar determinant: for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ define

$$\det(u, v) = u_1 v_2 - u_2 v_1.$$

Then

$$A(\gamma) = \tfrac{1}{2}\int_a^b \det(\gamma(t), \gamma'(t))\,dt. \tag{3}$$

2.2. Polygonal approximation and its properties. We now introduce the discrete quantity that is the central object of this paper.

Definition 1 (Polygonal (chordal) approximation). *Let $a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a, b]$ and denote $\gamma_k = \gamma(t_k) = (x_k, y_k)$. The polygonal (chordal) approximation to $A(\gamma)$ is defined by*

$$S(\gamma; \{t_k\}) \;=\; \tfrac{1}{2}\sum_{k=1}^{n} \det(\gamma_{k-1}, \gamma_k) = \tfrac{1}{2}\sum_{k=1}^{n} (x_{k-1}y_k - y_{k-1}x_k). \tag{4}$$

Geometrically, each term

$$\tfrac{1}{2}\det(\gamma_{k-1}, \gamma_k)$$

is the oriented area of the triangle with vertices $O = (0, 0)$, γ_{k-1}, and γ_k. Thus $S(\gamma; \{t_k\})$ is the oriented area of the closed polygonal path $O \to \gamma_0 \to \gamma_1 \to \ldots \to \gamma_n \to O$. If $\gamma_0 = \gamma_n$, the edges incident with the origin contribute zero, and the formula reduces to the classical shoelace formula for the sampled closed polygon [3].

In what follows we focus on uniform partitions

$$t_k = a + kh, \qquad h = \tfrac{b-a}{n}, \tag{5}$$

and write

$$S_n(\gamma) \;=\; S(\gamma; \{t_k\}_{k=0}^{n}) = \tfrac{1}{2}\sum_{k=1}^{n} \big(x(t_{k-1})y(t_k) - y(t_{k-1})x(t_k)\big). \tag{6}$$

We record several elementary facts that motivate (4) as a geometric discretization of (1).

Proposition 2 (Reparametrization invariance of $A(\gamma)$). *Let $\phi : [\alpha, \beta] \to [a, b]$ be a C^1 bijection with $\phi' > 0$ and define $\widetilde{\gamma}(s) = \gamma(\phi(s))$. Then $A(\widetilde{\gamma}) = A(\gamma)$.*

Proof. By the chain rule, $\widetilde{\gamma}'(s) = \gamma'(\phi(s))\phi'(s)$, hence

$$\det(\widetilde{\gamma}(s), \widetilde{\gamma}'(s)) = \det(\gamma(\phi(s)), \gamma'(\phi(s)))\,\phi'(s).$$

Integrating and applying the substitution $t = \phi(s)$ yields (3) for $\widetilde{\gamma}$ with the same value. □

The discrete quantity $S(\gamma; \{t_k\})$ depends on the chosen partition and, unlike $A(\gamma)$, is not invariant under arbitrary reparametrizations. However, for a fixed parametrization it provides a natural chordal approximation that becomes accurate as the mesh size tends to zero (proved in the next section).

Proposition 3 (Translation of the origin). *Let $c \in \mathbb{R}^2$ and define $\gamma^c(t) = \gamma(t) - c$. Then*

$$A(\gamma^c) = A(\gamma) - \tfrac{1}{2}\int_a^b \det(c, \gamma'(t))\,dt = A(\gamma) - \tfrac{1}{2}\,\det\Big(c, \gamma(b) - \gamma(a)\Big).$$

In particular, if γ is closed, then $A(\gamma^c) = A(\gamma)$ for every c.

Proof. Using bilinearity of the determinant,

$$\det(\gamma - c, \gamma') = \det(\gamma, \gamma') - \det(c, \gamma').$$

Integrating gives the first identity. For the second, note that $\int_a^b \gamma'(t)\,dt = \gamma(b) - \gamma(a)$. If $\gamma(a) = \gamma(b)$, the last determinant is zero. $\square$

A similar formula holds discretely: for a partition $\{t_k\}$,

$$S(\gamma^c; \{t_k\}) = S(\gamma; \{t_k\}) - \tfrac{1}{2} \sum_{k=1}^{n} \det(c, \gamma_k - \gamma_{k-1}) = S(\gamma; \{t_k\}) - \tfrac{1}{2} \det\Big(c, \gamma_n - \gamma_0\Big).$$

Thus, for closed curves and closed polygonal approximations, the quantity is independent of the chosen origin.

Remark 4 (Closed and periodic parametrizations). *If $\gamma(a) = \gamma(b)$ and the parametrization is smooth and periodic, higher-order coefficients may simplify; however, periodicity does not in general remove the leading $\mathcal{O}(h^2)$ term. A higher-order expansion is needed to determine which additional cancellations occur for particular periodic parametrizations.*

2.3. Connection with classical quadrature. The definition (6) can be viewed as a discrete approximation to the area 1-form $\frac{1}{2}(x\,dy - y\,dx)$. It is not the standard trapezoidal rule applied to the scalar integrand $f(t) = \frac{1}{2}\det(\gamma(t), \gamma'(t))$. Indeed, using

$$x_{k-1}y_k - y_{k-1}x_k = x_{k-1}(y_k - y_{k-1}) - y_{k-1}(x_k - x_{k-1}),$$

we may rewrite

$$S_n(\gamma) = \tfrac{1}{2} \sum_{k=1}^{n} \Big[x(t_{k-1})\big(y(t_k) - y(t_{k-1})\big) - y(t_{k-1})\big(x(t_k) - x(t_{k-1})\big)\Big]. \tag{7}$$

If $x, y \in C^1$, then $y(t_k) - y(t_{k-1}) \approx y'(t_{k-1})h$ and $x(t_k) - x(t_{k-1}) \approx x'(t_{k-1})h$, and (7) becomes a Riemann-type sum for $A(\gamma)$ [9]. This observation underlies the convergence result proved in the next section and serves as a bridge between the geometric polygonal interpretation and classical quadrature theory; the underlying piecewise-linear (chordal) interpolation is also reminiscent of finite element approximations [13, 14].

Remark 5 (Comparison with the trapezoidal rule). *In general, $S_n(\gamma)$ differs from the trapezoidal approximation*

$$T_n := \tfrac{h}{2} \sum_{k=1}^{n} \big(f(t_{k-1}) + f(t_k)\big), \qquad f(t) = \tfrac{1}{2}\det(\gamma(t), \gamma'(t)).$$

The rule $S_n(\gamma)$ uses only endpoint positions $\gamma(t_k)$ and encodes a geometric shoelace/triangle-area structure. This geometric structure leads to cancellations in the local error expansion and yields the curvature-type coefficient $\det(\gamma', \gamma'')$ *in Theorem 9.*

3. CONVERGENCE OF THE POLYGONAL APPROXIMATION

In this section we establish convergence of the polygonal approximation $S_n(\gamma)$ to the area functional $A(\gamma)$ as the mesh size tends to zero.

Let $\gamma(t) = (x(t), y(t))$ with $x, y \in C^1([a, b])$. Recall that

$$A(\gamma) = \tfrac{1}{2} \int_a^b \det(\gamma(t), \gamma'(t))\, dt$$

and, for the uniform partition $t_k = a + kh$,

$$S_n(\gamma) = \tfrac{1}{2} \sum_{k=1}^{n} \det(\gamma(t_{k-1}), \gamma(t_k)).$$

We first rewrite the discretization error in a convenient integral form.

For each k, we write

$$\det(\gamma(t_{k-1}), \gamma(t_k)) = x(t_{k-1})y(t_k) - y(t_{k-1})x(t_k).$$

Using

$$y(t_k) - y(t_{k-1}) = \int_{t_{k-1}}^{t_k} y'(t)\, dt, \qquad x(t_k) - x(t_{k-1}) = \int_{t_{k-1}}^{t_k} x'(t)\, dt,$$

we obtain the exact identity

$$\det(\gamma(t_{k-1}), \gamma(t_k)) = x(t_{k-1}) \int_{t_{k-1}}^{t_k} y'(t)\, dt - y(t_{k-1}) \int_{t_{k-1}}^{t_k} x'(t)\, dt.$$

Hence

$$S_n(\gamma) = \tfrac{1}{2} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big[x(t_{k-1})\, y'(t) - y(t_{k-1})\, x'(t)\Big] dt. \tag{8}$$

Subtracting the exact area integral,

$$A(\gamma) = \tfrac{1}{2} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big[x(t)\, y'(t) - y(t)\, x'(t)\Big] dt,$$

we obtain

$$A(\gamma) - S_n(\gamma) = \tfrac{1}{2} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big[(x(t) - x(t_{k-1}))y'(t) - (y(t) - y(t_{k-1}))x'(t)\Big] dt. \tag{9}$$

We are now in a position to prove that the polygonal approximation converges to the area functional under C^1 smoothness assumptions.

THEOREM 6. *If $x, y \in C^1([a, b])$, then*

$$S_n(\gamma) \longrightarrow A(\gamma) \quad \textit{as } n \to \infty.$$

Proof. Since x and y are continuously differentiable on a compact interval, their derivatives are bounded: there exists $M > 0$ such that

$$|x'(t)| \le M, \qquad |y'(t)| \le M \quad \text{for all } t \in [a, b].$$

Moreover, x and y are uniformly continuous. Hence for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t-s| < \delta \quad \Rightarrow \quad |x(t)-x(s)| < \varepsilon, \quad |y(t)-y(s)| < \varepsilon.$$

Choose n sufficiently large so that $h = (b-a)/n < \delta$. Then for every $t \in [t_{k-1}, t_k]$,

$$|x(t)-x(t_{k-1})| < \varepsilon, \quad |y(t)-y(t_{k-1})| < \varepsilon.$$

Using (9),

$$\begin{aligned} |A(\gamma)-S_n(\gamma)| &\le \tfrac{1}{2}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\Big(|x(t)-x(t_{k-1})|\,|y'(t)| + |y(t)-y(t_{k-1})|\,|x'(t)|\Big)dt \\ &\le \tfrac{1}{2}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(\varepsilon M + \varepsilon M)dt \\ &= \varepsilon M(b-a). \end{aligned}$$

Since ε is arbitrary, the difference tends to zero as $n \to \infty$. □

The previous proof also yields a quantitative bound.

PROPOSITION 7. *If $x, y \in C^1([a,b])$, then*

$$|A(\gamma)-S_n(\gamma)| \le Ch,$$

where one may take $C = (b-a)\|x'\|_\infty \|y'\|_\infty$.

Proof. From (9),

$$|A-S_n| \le \tfrac{1}{2}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\Big(|x(t)-x(t_{k-1})|\,|y'(t)| + |y(t)-y(t_{k-1})|\,|x'(t)|\Big)dt.$$

By the mean value theorem,

$$|x(t)-x(t_{k-1})| \le \sup|x'|\,h, \qquad |y(t)-y(t_{k-1})| \le \sup|y'|\,h.$$

Thus

$$|A-S_n| \le \tfrac{1}{2}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} 2(\sup|x'|)(\sup|y'|)h\,dt = Ch.$$

□

Proposition 7 shows that, under C^1 regularity, the polygonal approximation converges at least with first order. In the next section we refine this result and derive a second-order asymptotic expansion under higher smoothness assumptions.

4. SECOND-ORDER ERROR EXPANSION

In this section we refine the first-order estimate obtained in the previous section and derive a second-order asymptotic expansion of the discretization error under higher regularity assumptions.

Throughout this section we assume that $x, y \in C^3([a, b])$.

Let
$$f(t) = \tfrac{1}{2}\det(\gamma(t), \gamma'(t)) = \tfrac{1}{2}(xy' - yx').$$

Recall that
$$A(\gamma) = \int_a^b f(t)\,dt,$$
and
$$S_n(\gamma) = \sum_{k=1}^{n} \Delta_k, \qquad \Delta_k = \tfrac{1}{2}\det(\gamma(t_{k-1}), \gamma(t_k)).$$

Denote
$$I_k = \int_{t_{k-1}}^{t_k} f(t)\,dt.$$

Then
$$A(\gamma) - S_n(\gamma) = \sum_{k=1}^{n} (I_k - \Delta_k).$$

We begin by analyzing the discretization error on a single subinterval.

Lemma 8 (Local expansion). *For each subinterval $[t_{k-1}, t_k]$ we have*
$$I_k - \Delta_k = \tfrac{h^3}{12}(x'y'' - x''y')(t_{k-1}) + \mathcal{O}(h^4),$$
where the remainder is uniform in k.

Proof. Let $t_0 = t_{k-1}$. Then
$$\begin{aligned} I_k - \Delta_k &= \tfrac{1}{2}\int_0^h \det\big(\gamma(t_0 + u) - \gamma(t_0), \gamma'(t_0 + u)\big)\,du \\ &= \tfrac{1}{2}\int_0^h \int_0^u \det\big(\gamma'(t_0 + s), \gamma'(t_0 + u)\big)\,ds\,du. \end{aligned}$$

Since $\gamma \in C^3$, uniformly in t_0,
$$\gamma'(t_0 + r) = v + ra + \mathcal{O}(r^2), \qquad v = \gamma'(t_0), \quad a = \gamma''(t_0).$$

Therefore,
$$\det\big(\gamma'(t_0 + s), \gamma'(t_0 + u)\big) = (u - s)\det(v, a) + \mathcal{O}(h^2).$$

Consequently,
$$I_k - \Delta_k = \tfrac{h^3}{12}\det\big(\gamma'(t_0), \gamma''(t_0)\big) + \mathcal{O}(h^4).$$

The remainder is uniform in k. □

Summing the local estimate of Lemma 8 over all subintervals yields the global second-order expansion.

Theorem 9 (Second-order asymptotics). *Let $\gamma \in C^3([a,b];\mathbb{R}^2)$. Then*

$$A(\gamma) - S_n(\gamma) = \tfrac{h^2}{12}\int_a^b \det(\gamma'(t), \gamma''(t))\, dt + \mathcal{O}(h^3),$$

as $n \to \infty$.

Proof. Summing Lemma 8 over k gives

$$A - S_n = \sum_{k=1}^{n} \left[\tfrac{h^3}{12}(x'y'' - x''y')(t_{k-1}) + \mathcal{O}(h^4) \right].$$

Hence

$$A - S_n = \tfrac{h^3}{12}\sum_{k=1}^{n}(x'y'' - x''y')(t_{k-1}) + \mathcal{O}(nh^4).$$

Since $nh^4 = h^3$, the remainder term is $\mathcal{O}(h^3)$. The sum is a Riemann sum for

$$\int_a^b (x'y'' - x''y')\, dt = \int_a^b \det(\gamma'(t), \gamma''(t))\, dt,$$

therefore

$$A - S_n = \tfrac{h^2}{12}\int_a^b \det(\gamma'(t), \gamma''(t))\, dt + \mathcal{O}(h^3).$$

Define

$$g(t) = \det(\gamma'(t), \gamma''(t)).$$

Since $\gamma \in C^3$, we have $g \in C^1([a,b])$. Hence the left Riemann sum satisfies

$$h\sum_{k=1}^{n} g(t_{k-1}) = \int_a^b g(t)\, dt + \mathcal{O}(h).$$

Therefore,

$$\tfrac{h^3}{12}\sum_{k=1}^{n} g(t_{k-1}) = \tfrac{h^2}{12}\int_a^b g(t)\, dt + \mathcal{O}(h^3).$$

□

We close this section with a geometric interpretation of the leading error term.

Observe that

$$x'y'' - x''y' = \det(\gamma', \gamma'').$$

Thus the leading error term involves the determinant of the velocity and acceleration vectors. If γ is regular, then this quantity is related to signed curvature by [2, 15]

$$\det(\gamma', \gamma'') = |\gamma'|^3\kappa.$$

Hence, for a regular curve, the principal error coefficient reflects signed curvature weighted by the cube of the parametrization speed.

Remark 10. *The leading error coefficient depends on the parametrization. For a regular parametrization,*

$$\det(\gamma', \gamma'') = |\gamma'|^3 \kappa.$$

Thus the coefficient depends on both signed curvature and parametrization speed. Under arc-length parametrization, $|\gamma'| = 1$, and the coefficient reduces to $\int \kappa(s)\,ds$.

5. APPLICATION TO A RATIONAL PARAMETRIZATION OF THE UNIT CIRCLE

In this section we apply the general theory to a rational parametrization of the unit circle and compute the leading error coefficient explicitly.

Consider the unit circle $x^2 + y^2 = 1$. A classical rational parametrization of its first-quadrant quarter circle is given by

$$x(t) = \tfrac{1-t^2}{1+t^2}, \qquad y(t) = \tfrac{2t}{1+t^2}, \qquad t \in [0,1].$$

This is the standard rational parametrization of the unit circle; on $t \in [0,1]$, it traces the first-quadrant arc from $(1,0)$ to $(0,1)$.

A direct verification shows that

$$x(t)^2 + y(t)^2 = 1 \quad \text{for all } t.$$

Having fixed the parametrization, we determine the two quantities required by the general theory in two steps: first the exact value of the area functional, and then the leading coefficient of the discretization error.

Step 1: Exact value of the area functional. We compute

$$f(t) = \tfrac{1}{2}(xy' - yx').$$

First compute the derivatives.

$$x(t) = \tfrac{1-t^2}{1+t^2}.$$

Using the quotient rule,

$$x'(t) = \tfrac{-2t(1+t^2)-(1-t^2)(2t)}{(1+t^2)^2} = \tfrac{-4t}{(1+t^2)^2}.$$

Similarly,

$$y(t) = \tfrac{2t}{1+t^2},$$

so

$$y'(t) = \tfrac{2(1+t^2)-2t(2t)}{(1+t^2)^2} = \tfrac{2(1-t^2)}{(1+t^2)^2}.$$

Now compute

$$\begin{aligned} xy' - yx' &= \tfrac{1-t^2}{1+t^2} \cdot \tfrac{2(1-t^2)}{(1+t^2)^2} - \tfrac{2t}{1+t^2} \cdot \tfrac{-4t}{(1+t^2)^2} \\ &= \frac{2(1-t^2)^2}{(1+t^2)^3} + \tfrac{8t^2}{(1+t^2)^3} \\ &= \tfrac{2(1+t^2)^2}{(1+t^2)^3} = \tfrac{2}{1+t^2}. \end{aligned}$$

Hence

$$f(t) = \tfrac{1}{2}(xy' - yx') = \tfrac{1}{1+t^2}.$$

Therefore

$$A(\gamma) = \int_0^1 \tfrac{dt}{1+t^2} = \arctan(1) - \arctan(0) = \tfrac{\pi}{4}.$$

Step 2: Computation of the leading error coefficient. From Theorem 9,

$$\tfrac{\pi}{4} - S_n = \tfrac{h^2}{12} \int_0^1 \det(\gamma', \gamma'')\, dt + \mathcal{O}(h^3).$$

We compute $\det(\gamma', \gamma'')$ explicitly.
We already have

$$x'(t) = \tfrac{-4t}{(1+t^2)^2}, \qquad y'(t) = \tfrac{2(1-t^2)}{(1+t^2)^2}.$$

Differentiate again.
For x',

$$x''(t) = \tfrac{-4(1+t^2)^2 - (-4t)\cdot 2(1+t^2)(2t)}{(1+t^2)^4}.$$

After simplification,

$$x''(t) = \tfrac{4(3t^2-1)}{(1+t^2)^3}.$$

Similarly, differentiating y' yields

$$y''(t) = \tfrac{-4t(3-t^2)}{(1+t^2)^3}.$$

Now compute

$$\begin{aligned} \det(\gamma', \gamma'') &= x'y'' - y'x'' \\ &= \tfrac{-4t}{(1+t^2)^2} \cdot \tfrac{-4t(3-t^2)}{(1+t^2)^3} - \tfrac{2(1-t^2)}{(1+t^2)^2} \cdot \tfrac{4(3t^2-1)}{(1+t^2)^3}. \end{aligned}$$

After algebraic simplification one obtains

$$\det(\gamma', \gamma'') = \frac{8}{(1+t^2)^3}.$$

Therefore

$$\tfrac{1}{12} \int_0^1 \det(\gamma', \gamma'')\, dt = \int_0^1 \tfrac{2}{3(1+t^2)^3}\, dt.$$

Using the standard integral we obtain

$$\int_0^1 \tfrac{2}{3(1+t^2)^3}\, dt = \tfrac{1}{6} + \tfrac{\pi}{16}.$$

Theorem 11. *For the rational parametrization of the quarter unit circle, the polygonal approximation satisfies*

$$\tfrac{\pi}{4} - S_n = \left(\tfrac{1}{6} + \tfrac{\pi}{16}\right) \tfrac{1}{n^2} + \mathcal{O}\left(\tfrac{1}{n^3}\right).$$

This provides an explicit leading-order asymptotic formula for the rational sequence associated with the polygonal discretization.

Unlike classical infinite series for π (*e.g.*, the Gregory–Leibniz series or AGM-based iterations [16]), the present rational sequence arises purely from geometric polygonal discretization. Its convergence is algebraic of order two, yet its derivation is geometric and requires only elementary calculus.

6. NUMERICAL ILLUSTRATION AND RICHARDSON ACCELERATION

In this section, we evaluate the numerical performance of the proposed polygonal approximation S_n for the rational parametrization of the unit circle considered in Section 5. Recall that S_n approximates the quarter-disk area $A = \pi/4$, hence π is approximated by

$$\pi_n := 4S_n.$$

6.1. Experimental setup and comparative analysis. To examine the accuracy and convergence behavior of the proposed Polygonal–Rational method, we compare it with several standard numerical methods for approximating π.

For the rational parametrization considered in Section 5, the proposed Polygonal–Rational approximation admits the explicit rational representation

$$\pi_n = 4n \sum_{k=1}^{n} \tfrac{n^2+k(k-1)}{(n^2+k^2)(n^2+(k-1)^2)}.$$

For convenience, we refer to this representation as the AC-Series. It is not a separate numerical method, but the explicit rational form of the proposed polygonal discretization.

The comparison methods are:

- **Leibniz Series:** The classical alternating series, whose truncation error is of order $\mathcal{O}(n^{-1})$.

$$\pi = 4 \sum_{k=0}^{\infty} \tfrac{(-1)^k}{2k+1} = 4 \cdot (1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots)$$

- **Riemann Sums (Left and Midpoint):** Applied to the integral $\int_0^1 \sqrt{1-x^2}\,dx$.
- **Trapezoidal Rule:** Applied to the same integral $\int_0^1 \sqrt{1-x^2}\,dx$.
- **Riemann Sum (Parametric):** the left Riemann sum

$$\pi_n^{\text{param}} = \tfrac{4}{n} \sum_{k=0}^{n-1} \tfrac{1}{1+(k/n)^2}$$

applied to $\pi = 4 \int_0^1 (1+t^2)^{-1}\,dt$.

6.1.1. *Convergence results.* The results of the numerical experiment are summarized in Section 6.1.1. The proposed Polygonal-Rational method achieves substantially greater accuracy than the standard quadrature rules applied directly to

$$y = \sqrt{1 - x^2},$$

and it also converges faster than the left Riemann rule applied to the smooth parametric integrand $(1 + t^2)^{-1}$.

A key observation is the degradation of convergence order for standard methods. While the Midpoint and Trapezoidal rules typically exhibit $\mathcal{O}(n^{-2})$ convergence for smooth functions [12, 7], in this case, they degrade to approximately $\mathcal{O}(n^{-1.5})$. This is due to the singularity of the derivative of the integrand $f(x) = \sqrt{1 - x^2}$ at the boundary $x = 1$ (where $f'(x) \to \infty$).

In contrast, the proposed method utilizes the rational parametrization, which is smooth (C^∞) on the interval $[0, 1]$. Consequently, it retains the theoretical second-order convergence rate $\mathcal{O}(n^{-2})$, resulting in significantly higher precision for the same number of evaluations n. As shown in Section 6.1.1, at $n = 10^6$, our method has a decimal-accuracy measure of approximately 11.84, whereas the Trapezoidal rule and Midpoint rule yield only about 9 digits.

n	**Leibniz Series**	**Riemann (Left)**	**Riemann (Mid)**	**Trapezoidal (Sqrt)**	**Riemann (Param.)**	**Polygonal (This Work)**
10^2	2.00	1.73	3.46	2.93	2.00	**3.84**
10^3	3.00	2.71	4.96	4.43	3.00	**5.84**
10^4	4.00	3.70	6.46	5.93	4.00	**7.84**
10^5	5.00	4.70	7.96	7.43	5.00	**9.84**
10^6	6.00	5.70	9.46	8.93	6.00	**11.84**
10^7	7.00	6.70	10.98	10.43	7.00	**13.26**
10^8	8.00	7.70	11.98	11.54	8.00	**12.55**

Table 1. Comparison of decimal accuracy, measured by $-\log_{10}|\text{error}|$, for approximating π. The proposed Polygonal-Rational method maintains superior accuracy compared to standard quadratures.

Before round-off saturation, the empirical convergence rate for the proposed method is consistent with $p \approx 2.00$, confirming the theoretical estimate derived in Theorem 9 which predicts an error of order $\mathcal{O}(n^{-2})$. Note that at $n = 10^8$, the precision of the proposed method saturates due to the accumulation of floating-point rounding errors [17], which is expected given the high accuracy achieved.

Furthermore, an empirical order estimate applied to the consecutive error terms confirms the second-order nature of the convergence. Across five decades of n (from $n = 32$ to $n \approx 3 \times 10^5$), the local convergence exponent stabilizes at $p \approx 2.0000$. This yields approximately two additional correct decimal digits for every tenfold increase in n, providing strong empirical verification of the asymptotic expansion derived in Theorem 9. The dominant presence of the n^{-2} error term suggests that the accuracy can be significantly enhanced by canceling this term via extrapolation.

6.2. A hyperbola branch. To demonstrate that the proposed polygonal discretization is not specific to circular arcs or to the approximation of π, we include a second example involving a different rational curve. Consider the branch of the hyperbola

$$\gamma(t) = \left(t, \tfrac{1}{t}\right), \qquad t \in [1,2].$$

This curve is not closed and the corresponding oriented area functional is not related to the area of a disk. Nevertheless, all quantities appearing in the asymptotic formula can be computed explicitly.

First, the exact value of the area functional is

$$A(\gamma) = \tfrac{1}{2}\int_1^2 \left(t\left(-\tfrac{1}{t^2}\right) - \tfrac{1}{t}\cdot 1\right) dt = -\int_1^2 \tfrac{dt}{t} = -\log 2.$$

Next,

$$\gamma'(t) = \left(1, -\tfrac{1}{t^2}\right), \qquad \gamma''(t) = \left(0, \tfrac{2}{t^3}\right),$$

and hence

$$\det(\gamma'(t), \gamma''(t)) = \tfrac{2}{t^3}.$$

Therefore

$$\int_1^2 \det(\gamma'(t), \gamma''(t))\, dt = \int_1^2 \tfrac{2}{t^3}\, dt = \tfrac{3}{4}.$$

By Theorem 9, with $h = 1/n$, we obtain the explicit asymptotic formula

$$A(\gamma) - S_n(\gamma) = \tfrac{h^2}{12}\cdot\tfrac{3}{4} + \mathcal{O}(h^3) = \tfrac{1}{16n^2} + \mathcal{O}(n^{-3}).$$

Thus, in this example,

$$-\log 2 - S_n(\gamma) = \tfrac{1}{16n^2} + \mathcal{O}(n^{-3}).$$

For completeness, we also write the polygonal approximation explicitly. Since

$$t_k = 1 + \tfrac{k}{n}, \qquad k = 0, 1, \ldots, n,$$

we have

$$S_n(\gamma) = \frac{1}{2}\sum_{k=1}^{n} \det\left(\left(t_{k-1}, \tfrac{1}{t_{k-1}}\right), \left(t_k, \tfrac{1}{t_k}\right)\right).$$

Equivalently,

$$S_n(\gamma) = \tfrac{1}{2}\sum_{k=1}^{n}\left(\tfrac{t_{k-1}}{t_k} - \tfrac{t_k}{t_{k-1}}\right).$$

This gives an explicit rational sequence converging to $-\log 2$. The leading error constant predicted by the theory is $1/16$.

The numerical results are shown in Table 2. The third column confirms that $n^2|A(\gamma) - S_n(\gamma)|$ converges to $1/16 = 0.0625$, in agreement with the theoretical asymptotic expansion.

n	$\lvert A(\gamma) - S_n(\gamma)\rvert$	$n^2\lvert A(\gamma) - S_n(\gamma)\rvert$
10	6.2422×10^{-4}	0.062422
20	1.5620×10^{-4}	0.062480
40	3.9059×10^{-5}	0.062495
80	9.7654×10^{-6}	0.062499
160	2.4414×10^{-6}	0.062500
320	6.1035×10^{-7}	0.062500
640	1.5259×10^{-7}	0.062500
1280	3.8147×10^{-8}	0.062500

Table 2. Numerical verification of the second-order asymptotic formula for the hyperbola branch $\gamma(t) = (t, 1/t)$, $t \in [1, 2]$.

This example illustrates the general nature of the result. The same polygonal discretization, which in the circular case produced a rational approximation to π, now produces a rational sequence converging to $-\log 2$. More importantly, the observed leading error constant agrees with the curvature-type coefficient

$$\tfrac{1}{12}\int_1^2 \det(\gamma'(t), \gamma''(t))\, dt,$$

confirming that the error mechanism described in Theorem 9 is not a special feature of the circle.

6.3. Richardson acceleration. Since Theorem 9 gives an asymptotic expansion whose leading term is of order $\mathcal{O}(n^{-2})$, the convergence can be improved by Richardson extrapolation [8, 18]. We define the accelerated sequence A_n^{acc} by canceling the quadratic error term:

$$A_n^{\mathrm{acc}} := \tfrac{4\,S_{2n} - S_n}{3}, \qquad \pi_n^{\mathrm{acc}} := 4\,A_n^{\mathrm{acc}} = \tfrac{16\,S_{2n} - 4\,S_n}{3}.$$

According to the error expansion derived in Section 4, this linear combination eliminates the h^2 term. For the present circle example, the numerical results suggest that the cubic term vanishes, so the Richardson-extrapolated error behaves like $\mathcal{O}(n^{-4})$. This fourth-order behavior is not proved by Theorem 9.

We verify this hypothesis by computing the empirical convergence rate for the accelerated series:

$$p_n^{\mathrm{acc}} := \log_2\!\left(\tfrac{|\pi - \pi_n^{\mathrm{acc}}|}{|\pi - \pi_{2n}^{\mathrm{acc}}|}\right).$$

Section 6.3 presents the results. The empirical rates rapidly approach $p \approx 4$, confirming the cancellation of the leading error term.

The practical implication is substantial. Because every arithmetic operation in S_n is rational (requiring no square roots or transcendental evaluations), it is computationally attractive. As shown in the table, already at $n = 80$, the accelerated method yields approximately 9 decimal digits of accuracy in π a precision that the base series S_n achieves only at $n \approx 10^5$, and which the Leibniz series would require $n \approx 10^9$ terms to match.

n	π_n^{acc}	Error $\lvert\pi - \pi_n^{\mathrm{acc}}\rvert$	Order p_n^{acc}
5	3.141449204082007	1.43×10^{-4}	—
10	3.141583605852604	9.05×10^{-6}	3.99
20	3.141592086810131	5.67×10^{-7}	4.00
40	3.141592618145764	3.54×10^{-8}	4.00
80	3.141592651374224	2.22×10^{-9}	4.00
160	3.141592653451315	1.38×10^{-10}	4.00
320	3.141592653581139	8.65×10^{-12}	4.00
640	3.141592653589253	5.40×10^{-13}	4.00
1280	3.141592653589759	3.42×10^{-14}	3.98
2560	3.141592653589791	2.22×10^{-15}	3.94

Table 3. Performance of Richardson acceleration. The convergence rate improves from $\mathcal{O}(n^{-2})$ to approximately $\mathcal{O}(n^{-4})$.

The empirical convergence rates are consistent with the numerically observed fourth-order behavior ($p \approx 4.00$) for the majority of the range. However, a slight deviation is observed for $n \geq 1280$, where the estimated order drops to 3.98 and 3.94. This saturation is expected as the absolute error approaches 10^{-15}, which is close to the machine epsilon for double-precision arithmetic ($\varepsilon \approx 2.22 \times 10^{-16}$). At this level of precision, floating-point round-off errors begin to dominate the discretization error, naturally limiting further accuracy improvements.

6.4. Remark on the observed fourth-order behavior. The Richardson extrapolation cancels the leading n^{-2} term in the asymptotic expansion of Theorem 9. In general, one expects the error to admit an expansion of the form

$$A(\gamma) - S_n(\gamma) = \frac{C_2}{n^2} + \frac{C_3}{n^3} + \frac{C_4}{n^4} + \cdots .$$

For sufficiently smooth curves, the structure of the local error expansion often implies that odd powers may vanish under additional symmetry conditions. In the present unit-circle example, the curve is smooth on $[0, 1]$ and the discretization is uniform. These facts are compatible with, but do not by themselves prove, the observed cancellation of the cubic term. Numerically, the dominant correction after cancellation of the n^{-2} term is of order n^{-4}.

A rigorous derivation of the fourth-order behavior would require a higher-order global expansion, including the Euler–Maclaurin correction arising when the local expansion is summed over the mesh; this goes beyond the scope of the present work.

7. CONCLUSION

We have analyzed a geometrically natural polygonal discretization of the area functional

$$A(\gamma) = \tfrac{1}{2} \int (x\, dy - y\, dx).$$

The main contributions of this work are the following:

- We established convergence of the polygonal approximation for C^1 curves.
- We derived a second-order asymptotic expansion of the global error under C^3 regularity, identifying explicitly the leading coefficient

$$\tfrac{1}{12}\int_a^b \det(\gamma', \gamma'')\, dt.$$

- We provided a geometric interpretation of the leading error term in terms of curvature, showing that the principal error contribution reflects the cumulative curvature of the curve along the parametrization.
- For a rational parametrization of the unit circle, we computed the error constant explicitly and obtained a rational sequence converging to π with an explicit leading-order asymptotic formula.
- Numerical experiments confirmed the theoretical second-order convergence and demonstrated numerically a substantial accuracy improvement via Richardson extrapolation.

The polygonal discretization studied here provides a transparent geometric bridge between classical line integrals and discrete approximations. Although it has the same global order as standard second-order quadrature rules, its geometric structure makes it particularly natural for parametrized curves and for rational representations.

Several directions for further investigation remain open. First, higher-order asymptotic expansions could be derived systematically. Second, non-uniform partitions adapted to curvature may improve accuracy. Finally, extensions to closed curves in $\mathbb{R}^3$ and to discretizations of surface integrals represent natural generalizations.

The present results demonstrate that even classical geometric constructions admit precise quantitative asymptotic analysis. For regular curves, the explicit identification of curvature in the leading error term clarifies the geometric mechanism behind second-order accuracy and opens the way to higher-order and structure-preserving extensions.

REFERENCES

[1] M. Spivak, *Calculus on Manifolds*, Addison–Wesley, 1965.
[2] M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.
[3] B. Braden, *The surveyor's area formula*, College Math. J., **17** (1986), pp. 326–337.
[4] A.I. Bobenko, Y.B. Suris, *Discrete Differential Geometry: Integrable Structure*, American Mathematical Society, 2008.
[5] M. Desbrun, E. Kanso, Y. Tong, *Discrete differential forms for computational modeling*, in *Discrete Differential Geometry*, Oberwolfach Seminars, vol. 38, Birkhäuser, 2008, pp. 287–324.
[6] M. Meyer, M. Desbrun, P. Schröder, A.H. Barr, *Discrete differential-geometry operators for triangulated 2-manifolds*, in *Visualization and Mathematics III*, Springer, 2003, pp. 35–57.
[7] K. Atkinson, *An Introduction to Numerical Analysis*, Wiley, 1989.

[8] E. Süli, D.F. Mayers, *An Introduction to Numerical Analysis*, Cambridge University Press, 2003.

[9] T. Apostol, *Mathematical Analysis*, Addison–Wesley, 1974.

[10] L.N. Trefethen, J.A.C. Weideman, *The exponentially convergent trapezoidal rule*, SIAM Rev., **56** (2014), pp. 385–458.

[11] P. Henrici, *Applied and Computational Complex Analysis*, vol. 1, Wiley, 1974.

[12] P.J. Davis, P. Rabinowitz, *Methods of Numerical Integration*, Academic Press, 1984.

[13] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.

[14] G. Strang, G. Fix, *An Analysis of the Finite Element Method*, Prentice Hall, 1973.

[15] D. Hilbert, S. Cohn-Vossen, *Geometry and the Imagination*, Chelsea Publishing, 1952.

[16] J.M. Borwein, P.B. Borwein, *Pi and the AGM*, Wiley, 1987.

[17] W. Kahan, *Further remarks on reducing truncation errors*, Commun. ACM, **8** (1965), p. 40.

[18] C. Brezinski, M. Redivo-Zaglia, *Extrapolation Methods: Theory and Practice*, North-Holland, 1991.

[19] D.E. McClure, R.A. Vitale, *Polygonal approximation of plane convex bodies*, J. Math. Anal. Appl., **51** (1975), pp. 326–358.

[20] L. Fejes Tóth, *Approximation by polygons and polyhedra*, Bull. Amer. Math. Soc., **54** (1948), pp. 431–438.

[21] P.M. Gruber, P. Kenderov, *Approximation of convex bodies by polytopes*, Rend. Circ. Mat. Palermo (2), **31** (1982), pp. 195–225.

[22] P.M. Gruber, *Approximation of convex bodies*, in *Convexity and its Applications*, P.M. Gruber and J.M. Wills (eds.), Birkhäuser, Basel, 1983, pp. 131–162.

[23] R. Sauer, *Differenzengeometrie*, Springer, Berlin, 1970.

[24] C. Müller, A. Vaxman, *Discrete curvature and torsion from cross-ratios*, Ann. Mat. Pura Appl., **200** (2021), pp. 1935–1960.

[25] L.F. Richardson, J.A. Gaunt, *The deferred approach to the limit*, Phil. Trans. Roy. Soc. London Ser. A, **226** (1927), pp. 299–361.

[26] K. Crane, F. de Goes, M. Desbrun, P. Schröder, *Digital geometry processing with discrete exterior calculus*, in *ACM SIGGRAPH 2013 Courses*, ACM, New York, 2013, pp. 1–126.

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