REVUE D'ANALYSE NUMÉRIQUE ET DE LA THÉORIE DE L'APPROXIMATION, Tome 1, 1972, pp. 37—43

ON RESTRICTIVE UNIFORM APPROXIMATION

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1. Introduction.

In [14] an unified treatment of the theory of approximation by functions having restricted ranges was developed by G. D. TAYLOR. A generalization of this problem and other results were obtained in [5], [3].

Several authors have dealt with the problems of monotone approximation: O. SHISHA [13], J. A. ROULIER [12], G. G. LORENTZ and K. L. ZELLER [6], [7], G. G. LORENTZ [8], R. A. LORENTZ [9].

The paper [10] presents the problem of positive approximation, a combination of the theory of monotone approximation and the theory of approximation with positive convex constraints of J. R. RICE [11].

The aim of this paper is twofold: to indicate a new generalization of the problem of uniform approximation by functions having restricted ranges and to give a result in a more general problem of positive approximation.

2. Basic definitions and notations

Let T be a compact Hausdorff space and let $C(T)$ denote the Banach space of all real-valued continuous functions defined on T with norm:

$$||f|| = \max \{|f(t)|,\ t \in T\}, \qquad f \in C(T).$$

Let $H = [\varphi_1, \ldots, \varphi_n]$ be an n-dimensional Haar subspace of $C(T)$, where the functions $\varphi_1, \ldots, \varphi_n$ form a basis for H.

Recall that an n-dimensional subspace H of $C(T)$ is called a Haar subspace if every function in $H \sim \{0\}$ has at most $n-1$ zeros in T.

We shall fix away a couple (u, v) of extended real-valued functions:

$$u : T \to \overline{R} \text{ and } v : T \to \overline{R}$$

subject to the following restrictions:

(i) $\{t \in T : u(t) = -\infty\} = \emptyset$ but

$\{t \in T : u(t) = +\infty\} = T_{+\infty}$ may be nonempty,

(ii) $\{t \in T : v(t) = +\infty\} = \emptyset$ but

$\{t \in T : v(t) = -\infty\} = T_{-\infty}$ may be nonempty

(iii) $T_{+\infty}$ and $T_{-\infty}$ are open subsets of T,

(iv) u is continuous on $T \sim T_{+\infty}$ and v is continuous on $T \sim T_{-\infty}$,

(v) $\{t \in T : u(t) > v(t)\} = T$.

Now let ω be a linear operator:

$$\omega : C(T) \to C(T)$$

and if $f \in C(T)$ we denote $f_\omega = \omega(f)$.

Then define the set of approximating functions:

$$\overline{H} = \{p \in H : v(t) \leq p_\omega(t) \leq u(t),\ t \in T\}.$$

Definition 1. *A function $p^\circ \in \overline{H}$ is said to be a best restrictive approximation to $f \in C(T)$ provided:*

$$\|f - p^\circ\| = \inf \ \{\|f - p\|,\ p \in \overline{H}\}.$$

The following terminology will be used in the statement of the results on best restrictive approximation.

Definition 2. *The operator ω is said to be monoton if it satisfies the condition:*

(1) *If* $f(t) \leq g(t),\ t \in T$ *then* $f_\omega(t) \leq g_\omega(t),\ t \in T$.

Now fix $f \in C(T)$. Let $p^\circ \in \overline{H}$ and define the following critical point sets:

$$\gamma_0^+ = \{t \in T : f(t) - p^\circ(t) = \|f - p^\circ\|\}$$

$$\gamma_0^- = \{t \in T : f(t) - p^\circ(t) = -\|f - p^\circ\|\}$$

$$\gamma_0^u = \{t \in T : p^0_\omega(t) = u(t)\}$$

$$\gamma_0 = \{t \in T : p^0_\omega(t) = v(t)\}$$

$$\gamma_0 = \gamma_0^+ \cup \gamma_0^- ; \quad \gamma_0^\tau = \gamma_0^u \cup \gamma_0^v, \ \Gamma_0^+ = \gamma_0^+ \cup \gamma_0^v,$$

$$\Gamma_0^- = \gamma_0^- \cup \gamma_0^u ; \quad \Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$$

3. Results on best restrictive approximation

In this section we shall ascertain that there is a complete analogy between our problem and the problem of approximating a continuous function by functions having restricted ranges. By this we mean that for each result which is valid for the Taylor problem, there is a corresponding result valid for the new more general problem. In particular all our results reduce to well known ones in the case when the operator ω is the identity map [14].

THEOREM 1. *If $\overline{H}$ is nonempty for the couple (u, v) then there exists at least one element of best restrictive approximation $p^\circ \in \overline{H}$ to $f \in C(T)$.*

It follows immediately from the facts that $\overline{H}$ is a closed subset of a finit dimensional subspace of $C(T)$ and:

$$\inf \{\|f - p\|,\ p \in \overline{H}\} = \inf \ \{\|f - p\|,\ p \in \overline{H} \text{ and } \|p\| \leq 2\|f\|\}.$$

THEOREM 2. *If $f \in C(T)$ then the set of the elements of best restrictive approximation in $\overline{H}$ to f is a convex set.*

It is easy to prove:

Lemma 1. *If $\Gamma_0^+ \cap \Gamma_0^- \neq \emptyset$ and ω is a monotone operator then $p^\circ \in \overline{H}$ is a best restrictive approximation to $f \in C(T)$.*

For the remainder of this section we assume that the function $f \in C(T)$ is taken such that $\Gamma_0^+ \cap \Gamma_0^- = \emptyset$ for all $p^\circ \in \overline{H}$.

Lemma 2. *If for a function $p^\circ \in \overline{H}$ there exists a function $p \in H$ satisfying*

(2) $p(t) > 0$ *for all* $t \in \gamma_0^+$

(3) $p(t) < 0$ *for all* $t \in \gamma_0^-$

(4) $p_\omega(t) > 0$ *for all* $t \in \gamma_0^v$

(5) $p_\omega(t) < 0$ *for all* $t \in \gamma_0^u$

then p° *is not an element of best restrictive approximation to* $f \in C(T)$.

The proof of this statement is immediate [5].

A first characterization theorem is a corolar of Theorem 8 from [1]:

THEOREM 3. *The element* $p^\circ \in \overline{H}$ *is a best restrictive approximation to* $f \in C(T)$ *if and only if for any element* $p \in \overline{H}$ *there exists a point* $t' \in \gamma_0^+$ *such that:*

(6) $$p(t') \leq p^\circ(t')$$

or a point $t'' \in \gamma_0^-$ *such that:*

(7) $$p(t'') \geq p^\circ(t'')$$

Remark. If ω is a monotone operator the theorem 3 holds by replacing of γ_0^+ by Γ_0^+ and respectively γ_0^- by Γ_0^-.

Once again referring to the techniques of [5] we immediately obtain the following:

THEOREM 4. *Suppose that:*

(8) ω *is monotone operator*

(9) *There exists at least one* $p \in H$ *such that:*

$$v(t) < p_\omega(t) < u(t), \ t \in T.$$

The element $p^\circ \in \overline{H}$ *is a best restrictive approximation to* $f \in C(T)$ *if and only if there exist* k *points* ($k \leq n+1$) $t_1, \ldots, t_k$ *in* Γ_0 *(at least one in* γ_0*) and a functional of the form:*

$$\Phi(p) = \sum_{t_i \in \gamma_0} \alpha_i p(t_i) + \sum_{t_i \in \gamma_0^*} \alpha_i p_\omega(t_i), \qquad p \in C(T),$$

such that:

(10) $$\Phi(p) = 0 \ \textit{if} \ p \in H$$

(11) $$\alpha_i > 0 \ \textit{if} \ t_i \in \Gamma_0^+ \ \textit{and} \ \alpha_i < 0 \ \textit{if} \ t_i \in \Gamma_0^-$$

If T is a compact subset of $[a, b]$ containing at least $n+1$ points we find an analogous result with Theorem 3.2 from [14].

THEOREM 5. *Let* $f \in C(T)$, $p^\circ \in \overline{H}$ *and suppose that* ω *is a monotone operator. Then the following statements are equivalent:*

(12) p° *is a best restrictive approximation to* f.

(13) *The origin of Eucliden n-space belongs to the convex hull of*

$$\{\xi(t)\hat{t}, \ t \in \Gamma_0\}, \textit{where } \xi(t) = +1 \textit{ if } t \in \Gamma_0^+ \ \ \xi(t) = -1 \textit{ if } t \in \Gamma_0^- \textit{ and}$$
$$\hat{t} = (\varphi_1(t), \ldots \varphi_n(t)).$$

(14) *There exists* $n+1$ *distinct points* $t_1 < \ldots < t_{n+1}$ *in* Γ_0 *satisfying*

$$\xi(t_i) = (-1)^{i+1} \xi(t_1), \ i = 2, \ldots, n+1.$$

There may be get easy a uniqueness theorem and a strong unicity theorem [2]:

THEOREM 6. *If* $p^\circ \in \overline{H}$ *is a best restrictive approximation to* $f \in C(T)$ *then* p° *is unique.*

THEOREM 7. *Let* $p^\circ \in \overline{H}$ *be the best restrictive approximation to* $f \in C(T)$. *Then there exists a constant* $\delta > 0$ *such that for any* $p \in \overline{H}$:

$$\|f - p\| \geq \|f - p^\circ\| + \delta\|p^\circ - p\|.$$

Remarks 1°. An exemple of problem of restrictive uniform approximation is treated in [4].

2° The problem of restrictive uniform approximation may be stated more general using the idea of [3].

4. Statement of the probleme of partitional positive approximation

Let T, $\Omega_1, \Omega_2, \ldots, \Omega_m$ be closed subsets not necessarily disjoint, of a compact Haussorff space S and let $C(X)$ denote again the linear space of all real-valued continuous functions defined on a compact Hausdorff space X.

Let $\omega_1, \omega_2, \ldots, \omega_m$ be linear not necessarily bounded operators:

$$\omega_i : C(\Omega_i) \to C(\Omega_i), \ i = 1, \ldots, m$$

and let H be a finit dimensional subspace of $C(S)$.

Denote H_{Ω_i} the set of restrictions of all functions $p \in H$ to the subset $\Omega_i (i = 1, \ldots, m)$ and

$$\overline{H}_{\Omega_i} = \{p \in H_{\Omega_i} : p_{\omega_i}(s) \geq 0, \ s \in \Omega_i\}.$$

Then define the set of approximating functions:

$$\overline{H} = \bigcap_{i=1}^{m} \overline{H}_{\Omega_i}.$$

Let

$$||f|| = \max \{|f(s)|, \quad s \in T\}, \text{ for } f \in C(S).$$

D e f i n i t i o n 3. *The function $p^\circ \in \overline{H}$ is said to be an element of best partitional positive approximation to $f \in C(S)$ if:*

$$||f - p^\circ|| = \inf \{||f - p||, \ p \in \overline{H}\}.$$

R e m a r k. If $T = \Omega_1 = \ldots = \Omega_m = S$ we obtain the problem of positive approximation studied in [10].

Compactness and convexity arguments show that: if $H \neq \emptyset$, for each $f \in C(S)$ there exists at least one best partitional positive approximation.

Next we define for $p^\circ \in \overline{H}$ and $f \in C(S)$ the following critical point sets:

$$\Gamma_0 = \{s \in T : |f(s) - p^\circ(s)| = ||f - p^\circ||\}.$$

$$\gamma_0^i = \{s \in \Omega_i : p^0_{\omega_i}(s) = 0\}, \ i = 1, \ldots, m,$$

and we say that p° is a constraint interior element of $\overline{H}$ if and only if γ_0^i is nonempty for some i.

The following characterization theorem, of the Kolmogoroff type, may be proved only if we assume that $\overline{H}$ has a constraint interior point.

THEOREM 8. *Let $f \in C(S)$ and $p^\circ \in \overline{H}$ with $||f - p^\circ|| \neq 0$. Then $p^\circ \in \overline{H}$ is an element of best partitional positive approximation to f if and only if there is no element $p \in \overline{H}$ for which:*

(15) $$\max \{(f(s) - p^\circ(s))p(s), \ s \in \Gamma_0\} < 0$$

and

(16) $$p_{\omega_i}(s) < 0, \quad s \in \gamma_0^i, \ i = 1, \ldots, m$$

We omit the proof which is similar to the proof of the Theorem 3.1 [10].

The uniqueness of the element of best partitional positive approximation is an open question.

R E F E R E N C E S

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Received 16. IX. 1971