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TRIANGLES INSCRIBED IN SMOOTH CLOSED ARCS

by

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Concerning a more general problem stated by P. TURÁN (see in [2]), E. G. STRAUS has proved the following theorem (unpublished):

Given any continuum Σ (= compact connected set) in the Euclidean plane $\mathbf{R}^2$, such that for some triangle ABC in the plane no three points A', B', C' in Σ form a triangle similar to ABC, then Σ is a simple arc.

From this theorem in particular follows that if Γ is a closed arc, then there exists a triangle $A'B'C'$ similar to ABC, which is inscribed in Γ in the sense that $A', B', C' \in \Gamma$.

We ask now about the existence of a triangle $A'B'C'$ inscribed in the closed arc Γ, which has parallel sides with the sides of the triangle ABC and the same orientation. This special case appears in some problems in the geometry of convex sets (see our notes [1] and [2]). The answer in general may be negative. In our note we solve this problem in the special case when Γ is a closed arc of class C^1 and has additional conditions about its tangent lines. From our theorem we derive a characterization of the strictly convex closed arcs of the class C^1.

Lemma 1. *Let be Ω the strip formed by the points (x, y) of the Euclidean plane $\mathbf{R}^2$ with $a \leq x \leq b$ $(a < b)$. Denote by Γ' and Γ'' two simple arcs of the class C^1 in Ω, with disjoint interiors, having the parametric representations given by*

$$\Gamma' \begin{cases} x = \varphi(t'), \\ y = \psi(t'), \end{cases} t' \in [0, 1], \text{ and } \Gamma'' \begin{cases} x = \xi(t''), \\ y = \eta(t''), \end{cases} t'' \in [0, 1],$$

and suppose that $\varphi(0) = \xi(0) = a$ and $\varphi(1) = \xi(1) = b$. Suppose that Γ' and Γ'' have a finite number of points in which the tangent is parallel to Oy

and in the neighbourhood of which the arc is on the same side of the tangent line. Then:

(i) *There exists a set of segments $T'(t)T''(t)$ with the endpoints $T'(t) \in \Gamma'$ and $T''(t) \in \Gamma''$, depending continuously on the parameter $t \in [0, 1]$, such that for any t the segment is parallel to Oy and $T'(0) = (\varphi(0), \psi(0))$, $T'(1) = (\varphi(1), \psi(1))$ and $T''(0) = (\xi(0), \eta(0))$, $T''(1) = (\xi(1), \eta(1))$.*

(ii) *If $T' \in \Gamma'$ is arbitrary, then there exists a point $T'' \in \Gamma''$ such that $T' = T'(t)$ and $T'' = T''(t)$ for some t in $[0, 1]$.*

(iii) *There exist neighbourhoods U' and U'' of $T'(0)$ and $T''(0)$, such that if $T' \in \Gamma' \cap U'$ and $T'' \in \Gamma'' \cap U''$ and $T'T''$ is parallel to Oy, then there exists a t such that $T' = T'(t)$ and $T'' = T''(t)$.*

Proof. We proceed by induction on the number m of points of Γ' and Γ'' (different from their endpoints), in which the tangent is parallel to Oy and in the neighbourhood of which the arcs are on the same side of the tangent lines. (The number of these points is obviously even.)

Suppose that $m = 0$. In this case the arcs Γ' and Γ'' may be represented in the form

$$\Gamma' : y = f(x) \text{ and } \Gamma'' : y = g(x).$$

Then the set of segments $T'(t)T''(t)$ where $t = \frac{x-a}{b-a}$ and

$$T'(t) = (f(t(b-a)+a), t(b-a)+a), \quad T''(t) = (g(t(b-a)+a), t(b-a)+a),$$

satisfies all the conditions of the lemma.

Suppose that the lemma holds for $m \leq 2n$ and prove it for $m = 2(n+1)$.

Let be $T'_k \in \Gamma'$, the point in which the tangent is parallel to Oy and has the minimal abscissa relative to all the points with this property in the interior of Γ' and Γ''. Suppose that $T'_k = (\varphi(t'_k), \psi(t'_k))$ and let be $T'_i = (\varphi(t'_i), \psi(t'_i))$, $i = 1, \ldots, k-1$ all the points with $t'_i < t'_k$ and with the property that the tangents to Γ' are parallel to Oy. Suppose that $T'_l = (\varphi(t'_l), \psi(t'_l))$, $1 \leq l \leq k-1$ is a point with the maximal abscissa relative to the set of points T'_i, $i = 1, \ldots, k-1$, (see Fig. 1). For $i \leq k$ we shall denote by t''_i the minimal value of t'' with the property that $\xi(t''_i) = \varphi(t'_i)$. Put $T''_i = (\xi(t''_i), \eta(t''_i))$. Then the segments $T'_i T''_i$, $i = 1, \ldots, k$ are parallel to Oy.

Consider now the strip $a \leq x \leq \varphi(t'_l)$, and the part Γ'_1 of Γ' for $0 \leq t' \leq t'_l$ and the part Γ''_1 of Γ'' for $0 \leq t'' \leq t''_l$. The number of points on Γ'_1 and Γ''_1 in which the tangents are parallel to Oy is $\leq 2n$, because two such points, the point T'_k and T'_l, are eliminated (T'_k is eliminated because $t'_k > t'_l$ and then $T'_k \notin \Gamma'_1$ and T'_l is eliminated because it is an endpoint of Γ'_1.) By the induction hypothesis there exists then the set of segments $T'(t)T''(t)$ with the propeties (i), (ii) and (iii) with respect to the arcs Γ'_1 and Γ''_1.

By a similar way may be seen the existence of a set of segments $T'(t)T''(t)$ with the properties (i), (ii) and (iii) for the parts Γ'_2 of Γ' for

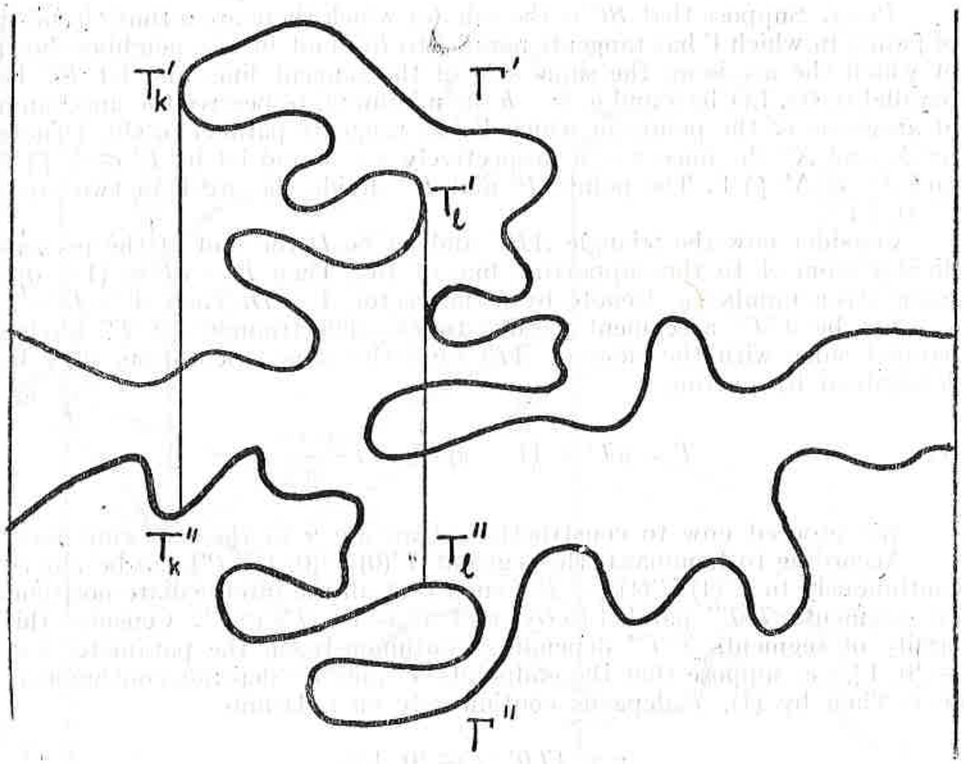

Fig. 1.

$t'_l \leq t' \leq t'_k$ and Γ''_2 of Γ'' for $t''_k \leq t'' \leq t''_l$. (Here we have $t''_k < t''_l$, from the definition of the points T''_k and T''_l, because $\xi(t''_k) < \xi(t''_l)$.)

We may consider now the part Γ'_3 of Γ' and Γ''_3 of Γ'' for $t'_k \leq t' \leq 1$ and $t''_k \leq t'' \leq 1$ respectively, and apply the induction hypothesis for Γ'_3 and Γ''_3.

As a final step, by a simple joining of the obtained families of segments $T'(t)T''(t)$ we obtain a family of segments which completes the proof of the assertion for $m = 2(n+1)$. This completes the proof of the lemma.

THEOREM 1. *Let be ABC a triangle in the Euclidean plane $\mathbf{R}^2$. Suppose that Γ is a simple closed arc of class C^1 in $\mathbf{R}^2$, which has the property that for at least one of the sides of ABC there is a finite number of points of Γ, in which the tangents are parallel to the respective side and in the neighbourhood of which the arc is on the same side of the tangent line. Then there exists a triangle $A'B'C'$ with sides parallel to sides of ABC and of the same orientation as ABC, which is inscribed in Γ, in the sense that A', B', $C' \in \Gamma$.*

Proof. Suppose that BC is the side for which there are a finite number of points in which Γ has tangents parallel to BC and in the neighbourhood of which the arc is on the same side of the tangent line, and let BC be parallel to Oy. Let be a and b, $a < b$ the minimum, respective the maximum of abscissas of the points in which Γ has tangents parallel to Oy. Denote by Δ' and Δ'' the lines $x = a$, respectively $x = b$ and let be $P' \in \Delta' \cap \Gamma$ and $P'' \in \Delta'' \cap \Gamma$. The points P' and P'' divide the arc Γ in two arcs: Γ' and Γ''.

Consider now the triangle ABC and let be D the foot of the perpendicular from A to the supporting line of BC. Then $D = qB + (1 - q)C$ for a given number q. Denote by r the vector $A - D$. Then $A = D + r$.

Let be $T'T''$ a segment parallel to BC. The triangle $TT'T''$ having parallel sides with the sides of ABC and the same orientation, may be determined by putting

$$T = qT' + (1 - q)T'' + r\frac{|T'T''|}{|BC|}. \tag{1}$$

We proceed now to construction of an arc γ in the following way:

According to Lemma 1, the segment $T'(0)T''(0)$ $(= P')$ can be moved continuously to $T'(1)T''(1)$ $(= P'')$ such that all the intermediate positions are segments $T'T''$ parallel to Oy, and $T' \in \Gamma'$, $T'' \in \Gamma''$. Consider this family of segments $T'T''$ depending continuously on the parameter $t \in [0, 1]$, i.e. suppose that the endpoints T' and T'' depends continuously on t. Then by (1), T depends continuously on t. Denote

$$\gamma = \{T(t) : t \in [0, 1]\}.$$

We have obviously $\gamma(0) = T(0) = P'$ and $\gamma(1) = T(1) = P''$.

According to the theorem of Jordan, Γ determines two domains of $\mathbf{R}^2$, which we will call the „inside'' and the „outside'' of Γ. In what follows we shall prove that there exists a value t_0 in (0,1) such that $T(t_0)$ is inside and a value t_1 in (0,1) such that $T(t_1)$ is outside Γ.

We observe that from the condition of the theorem it follows that P' is an isolated point with the property that the tangent to Γ is parallel to Oy. Let be T' a point on Γ' and let be T'' the point on Γ'' with minimal value of the parameter t'' such that $T'T''$ is parallel to Oy. Let be $T''R'$ and $T'R''$ the segments parallel to CA and BA respectively. Letting $T' \to P'$, the segment $T'T''$ tends to P'. From the property that Γ is of class C^1, it follows that the parallel line in P' to BA will intersect Γ in a point. Denote by Q'' the nearest point to P' with this property. Then the open segment $P'Q''$ will be inside Γ. Similarly, let $P'Q'$ be the segment with the same property, parallel to CA. Let be T_1' a point on Γ' and denote by T_1'' the point on Γ'' with minimal value of the parameter t'', such that $T_1'T_1''$ is parallel te Oy. Suppose that the abscissa of $T_1' \leq$ minimum of the abscissas of Q' and Q''. Then $T_1'T_1''$ will intersect $P'Q'$ in R' and $P'Q''$ in R''. Because Γ is a simple closed arc, for T_1' sufficiently near

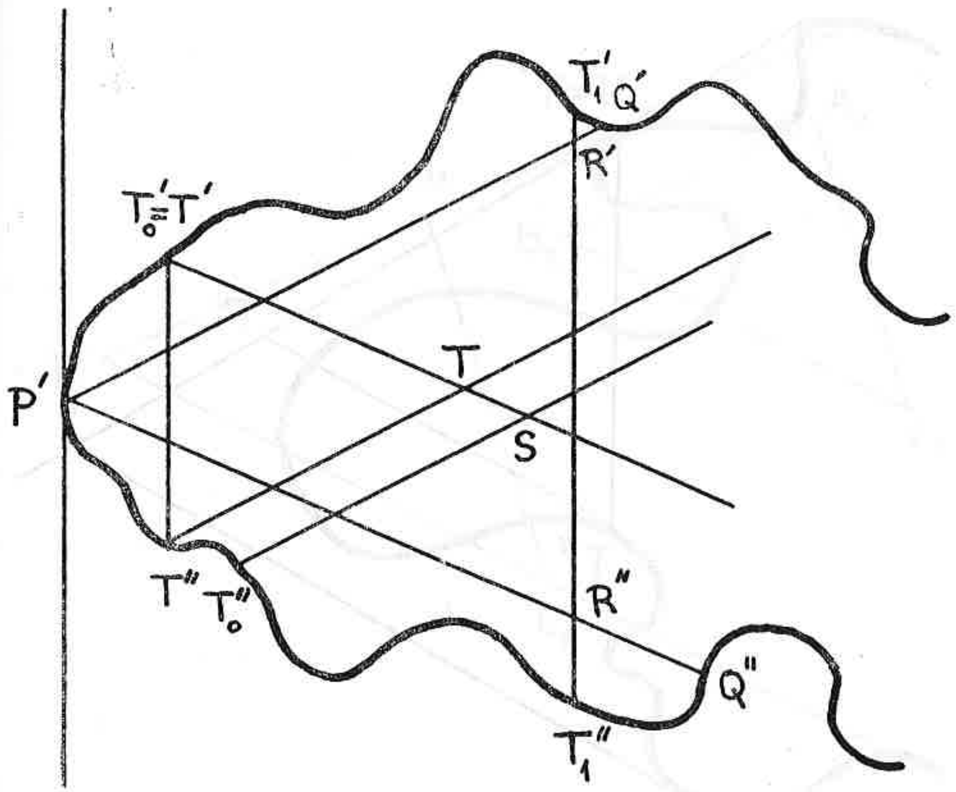

Fig. 2.

to P' the triangle $R'P'R''$ will be inside Γ. Consider a point S inside this triangle and let T_0' be the intersection point of Γ' and the parallel line in S to BA having the minimal value of the parameter t', and respectively, let T_0'' be the intersection of Γ'' and the parallel line in S to CA, with the minimum value of the parameter t''. Suppose that the abscissa of $T_0' \leq$ than the abscissa of T_0''. Put $T' = T_0'$ and denote by $T'T''$ the segment parallel to Oy with $T'' \in \Gamma''$ being the point with the minimal value of the parameter t'' with this property (see Fig. 2). T'' will be obviously „above'' the line $T_0''S$, and therefore the point T defined by (1) will be inside the triangle $R'P'R''$ and therefore T is also inside Γ. According (iii) in Lemma 1 it follows that for T' and T'' sufficiently near to P', there exists a value t_0 of the parameter t such that $T' = T'(t_0)$, $T'' = T''(t_0)$ and therefore $T = T(t_0)$.

Denote by Δ' and Δ'' the supporting lines of the convex hull Co (Γ) of Γ, parallel with BA and CA respectively, such that their intersection

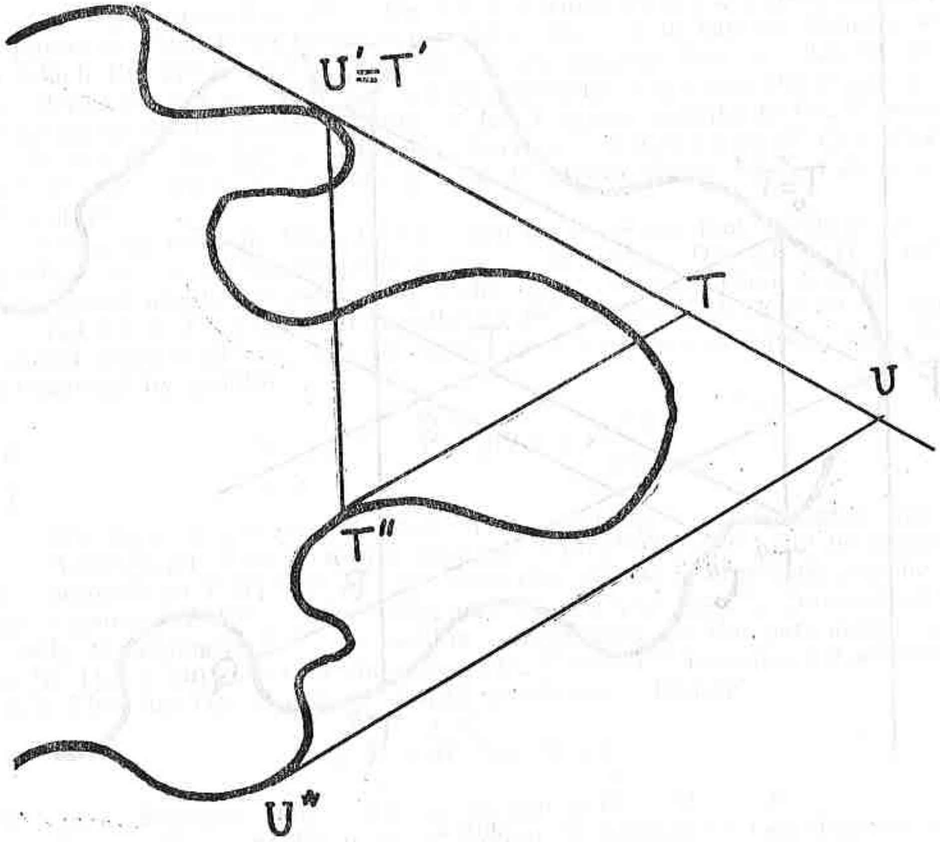

Fig. 3.

point U has the abscissa $> b$ (see Fig. 3). Denote by U' the point of Γ' on Δ' nearest to U and by U'' the point of Γ'' on Δ'' nearest to U. Suppose that the abscissa of $U' \geqq$ than the abscissa of U''. Put $U' = T'$ and let be T'' a point of Γ'' such that $T'T''$ is parallel to Oy and such that $T'T''$ occurs in the construction of γ (see (ii) in Lemma 1), i.e., there exists a parameter t_1 such that $T' = T'(t_1)$ and $T'' = T''(t_1)$. The point $T = T(t_1)$ defined by (1) will be on $T'U$, and because the segment $T'U$ is outside Γ (excepting T'), $T(t_1)$ will be outside Γ.

It follows that the open arc γ will intersect Γ, according the theorem of Jordan. Denote by T an intersection point. Then from the definition of γ it will follow that there exist the points $T' \in \Gamma'$ and $T'' \in \Gamma''$ such that the triangle $TT'T''$ has the required property. This completes the proof of the theorem.

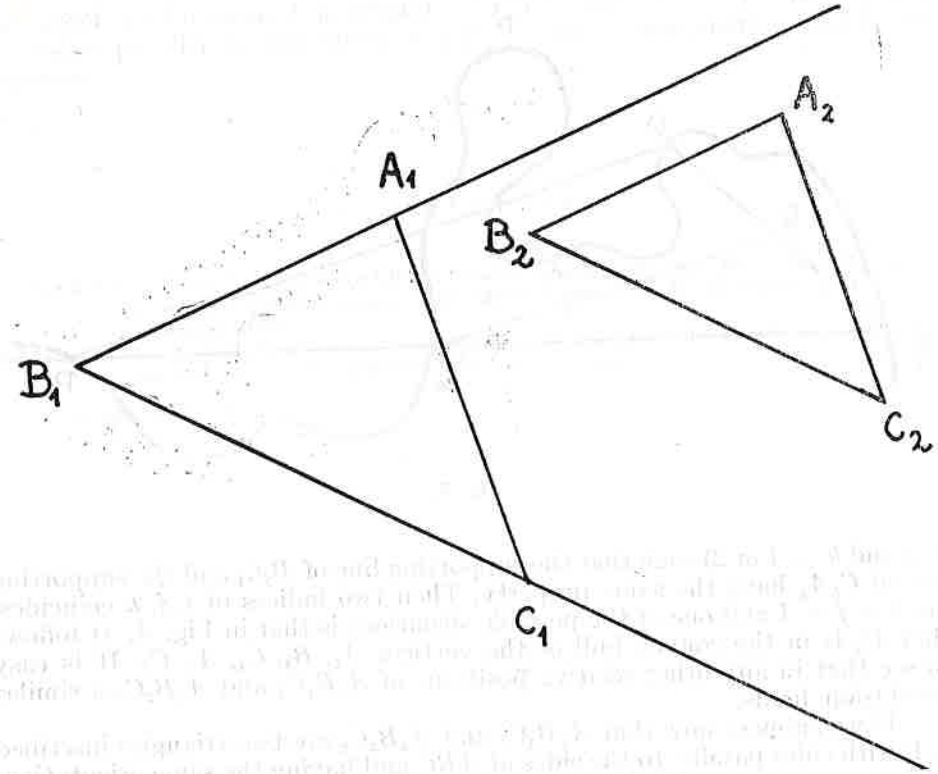

fig. 4.

R e m a r k. A special interest presents for us the case when Γ in Theorem 1 is a convex curve. Then an other proof may be given, which makes use of Brouwer's fixed point theorem. In this case our theorem follows from the more general Theorem 1 in the paper [1] (or [2]).

THEOREM 2. *Let be ABC a triangle in the Euclidean plane $\mathbf{R}^2$. Suppose that Γ is a strictly convex, closed arc of class C^1. Then there exists a single triangle $A_1B_1C_1$ with sides parallel to sides of ABC and of the same orientation as ABC, and which is inscribed in Γ, in the sense that $A_1, B_1, C_1 \in \Gamma$.*

Proof. The existence of a triangle $A_1B_1C_1$ with the required properties follows from the Theorem 1. It remains to show the unicity.

Suppose that $A_1B_1C_1$ and $A_2B_2C_2$ are two triangles with parallel sides and with the same orientation. Then one of their vertices is contained in the convex hull of the other five. Consider the supporting lines of A_1B_1 and A_2B_2. Because $A_1B_1C_1$ and $A_2B_2C_2$ have the same orientation, one of these lines, say A_iB_i ($i = 1$ or 2) determines a closed halfplane which contains both triangles. By the same reasoning there is a j and a k ($j = 1$

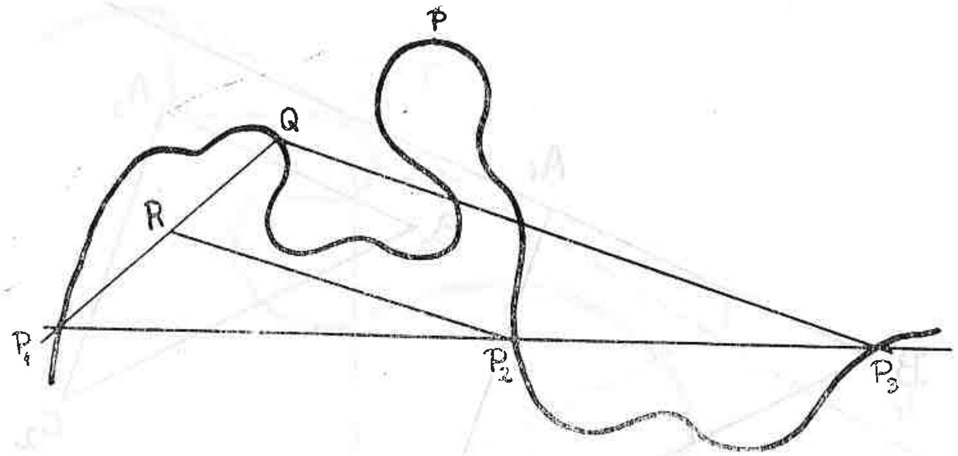

fig. 5.

or 2 and $k = 1$ or 2) such that the supporting line of B_jC_j and the supporting line of C_kA_k have the same property. Then two indices of i, j, k coincides, say $i = j = 1$ and one of the possible situations is that in Fig. 4. It follows that B_2 is in the convex hull of the vertices A_1, B_1, C_1, A_2, C_2. It is easy to see that in any other relative positions of $A_1B_1C_1$ and $A_2B_2C_2$ a similar conclusion holds.

If we suppose now that $A_1B_1C_1$ and $A_2B_2C_2$ are two triangles inscribed in Γ with sides parallel to the sides of ABC and having the same orientation, then we get a contradiction with the hypothesis that Γ is strictly convex.

THEOREM 3. *If Γ is a simple closed arc of class C^1 with the property that for any direction it has a finite number of tangents parallel to this direction, and for any triangle ABC there exists a single triangle $A_1B_1C_1$ inscribed in Γ having his sides parallel to the sides of ABC and the same orientation as ABC, then Γ is a strictly convex arc.*

Proof. Suppose that Γ is not strictly convex. Then there exists a segment which intersects Γ in the points P_1, P_2 and P_3. Without loss of generality we may suppose that we have the situation in Fig. 5. Because P_1, P_2, P_3 are isolated intersection points of Γ and P_1P_3 (this follows from the hypothesis that Γ has a finite number of points in which the tangent to Γ is parallel to P_1P_3), we may suppose that P_1P_3 is not tangent to Γ in the point P_1. Then there exists a segment P_1Q which is inside the closed arc Ω formed by the segment P_1P_2 and the part Γ_0 of Γ from P_1 to P_2 (see Fig. 5). Let be P_2R parallel to P_3Q and R in the segment P_1Q. Then R is inside the closed arc Ω. Denote now by P a farthest point from P_1P_3 on Γ_0. Then the tangent line in P is parallel to P_1P_3. Moving now the segment $T'T''$ continuously and parallel to P_1P_3, from P_1P_2 to P (see Lemma 1), we may construct with the same procedure as in

the proof of Theorem 1 a triangle $TT'T''$ with T, T', T'' on Γ_0 with the sides parallel to the sides of QP_1P_3. This contradiction proves the theorem.

REFERENCES

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