REVUE D'ANALYSE NUMÉRIQUE ET DE LA THÉORIE DE L'APPROXIMATION, Tome 1, 1972, pp. 73—108

GENERALIZED JUXTAPOLYNOMIALS WITH SOME APPLICATIONS

by

I. MARUŞCIAC

(Cluj)

§ 1. Introduction

Let K be a compact pointset in the complex plane and $F, f_1, f_2, \ldots, f_n$ — given complex functions defined and continuous on K. We denote by $\mathcal{P}(f)$ the set of generalized polynomials with respect to the system $f = (f_k)_1^n$:

$$p(z) = p(f\,;\,z) = a_1 f_1(z) + \ldots + a_n f_n(z). \tag{1.1}$$

Sometimes we shall omit the word ,,generalized'' and so instead of saying ,,generalized polynomial'' we shall simply say ,,polynomial''.

Definition 1.1. *A polynomial $\pi \in \mathcal{P}(f)$ is called juxtapolynomial to F on K if there is no polynomial $p \in \mathcal{P}(f)$, $p \neq \pi$ such that*

$$F(z) - \pi(f\,;\,z) = 0,\ z \in K \Rightarrow p(f\,;\,z) = \pi(f\,;\,z), \tag{1.2}$$

$$F(z) - \pi(f\,;\,z) \neq 0,\ z \in K \Rightarrow |F(z) - p(f\,;\,z)| < |F(z) - \pi(f\,;\,z)|. \tag{1.3}$$

The set of generalized juxtapolynomials to F on K will be denoted by $\mathcal{J}(K\,;F\,;f)$.

When $f_k(z) = z^{n-k}$, then we obtain the set $\mathcal{J}_n(K, F)$ of algebraic juxtapolynomials of degree $n-1$ to the function F on K, defined by MOTZKIN, T. S. and WALSH, J. L. [14].

The juxtapolynomials to a continuous function F are a special case of the generalized infrapolynomials defined by the author [10].

If $F(z) = z^n$, and $f_k = z^{n-k}$, $k = 1, 2, \ldots, n$, then

$$\mathcal{I}_n(K) = \{z^n - \pi(z) | \pi \in \mathcal{J}_n(K; z^n)\}$$

is the set of infrapolynomials of degree n on K, defined in 1922 by FEJÉR, L. [1].

Remark 1. If A, $B \subset K$ are two compact sets and $A \subset B$, then from $p \in \mathcal{J}(A; F; f)$ it follows $p \in \mathcal{J}(B; F; f)$, because if there is no polynomial of $\mathcal{P}(f)$ satisfying (1.2)—(1.3) on A, then obviously there is no such a polynomial satisfying the same conditions on B.

The purpose of this paper is to point out some general properties of this generalized juxtapolynomials and also to give certain applications in approximation theory.

Among the best known juxtapolynomials to a function F are those which approximate F in a given norm. We are going to present some examples.

1° If $\pi \in \mathcal{J}(K; F; f)$, and

$$W(z) = F(z) - \pi(f; z) \neq 0, \quad z \in K,$$

then π is a polynom of best approximation to F on K with respect to the norm

$$||u|| = \max \frac{1}{|W(z)|} |u(z)|.$$

2° The polinomial $\pi \in \mathcal{P}(f)$ of best approximation to F on K in the uniform norm, i.e.

$$\max_{z \in K} |F(z) - \pi(f; z)| = \inf_{p \in \mathcal{P}(f)} \max_{z \in K} |F(z) - p(f; z)|$$

is a juxtapolynomial to F on K, too.

Indeed, if we suppose that there exists a polynomial $p \in \mathcal{P}(f)$, $p \neq \pi$, satisfying the conditions (1.2)—(1.3), then we have

$$\max_{z \in K} |F(z) - p(f; z)| < \max_{z \in K} |F(z) - \pi(f; z)|,$$

i.e. a contradiction.

Similarly we can verify that:

3° If $Z_m = \{z_j\}_1^m$ and $\pi \in \mathcal{P}(f)$ minimizes the weighted mean

$$[\Sigma \mu_j |F(z_j) - p(f; z_j)|^r]^{1/r}, r \geq 1, \quad p \in \mathcal{P}(f),$$

where

$$\mu_i \geq 0, \ \Sigma \mu_i = 1, \text{ then } \pi \in \mathcal{J}(Z_m; F; f)$$

4° If K is a rectifiable arc, and $\pi \in \mathcal{P}(f)$ is the best approximation polynomial to F in norm $L_p(K)$, then $\pi \in \mathcal{J}(K; F; f)$.

5° A more interesting example is the following. Let $K = [0, 1]$ and F, f_k be real valued continuous functions on $[0, 1]$. Moreover, if $f = (f_k)_1^n$ is an orthonormal system on $[0, 1]$, then $t \in \mathcal{P}(f)$ being Fourier's polynomial to F on $[0, 1]$ of the order n, i.e.

$$t(f; x) = a_1 f_1(x) + \ldots + a_n f_n(x),$$

where

$$a_k = \int_0^1 F(x) f_k(x) dx, \qquad k = 1, 2, \ldots, n,$$

it follows $t \in \mathcal{J}([0, 1]; F; f)$.

Indeed, if for $\forall x \in [0, 1]$, $F(x) = t(f; x)$, then obviously t is the unique juxtapolynomial to F on $[0, 1]$. If $F(x) \not\equiv f(f; x)$, $x \in [0, 1]$, then there exists a point $x_0 \in [0, 1]$ on which $F(x_0) - t(f; x_0) \neq 0$.

Now, let us suppose on the contrary that $t \notin \mathcal{I}([0, 1], F; f)$. Then there exists a polynomial

$$q(f, x) = b_1 f_1(x) + \ldots + b_n f_n(x), \quad (b_k = b_k' + i b_k'')$$

satisfying (1.2)—(1.3). But from (1.3) we obtain

$$|F(x_0) - q(f; x_0)| < |F(x_0) - t(f; x_0)|$$

and by continuity of F, f_k, $k = 1, 2, \ldots, n$, we have

$$\int_0^1 [F(x) - \sum_{k=1}^n b_k' f_k(x)]^2 dx \leq \int_0^1 |F(x) - \sum_{k=1}^n b_k f_k(x)|^2 dx <$$

$$< \int_0^1 [F(x) - \sum_{k=1}^n a_k f_k(x)]^2 \, dx,$$

which contradicts the well known property of the Fourier's coefficient a_k of the function F. Hence $t \in \mathcal{J}([0, 1]; F, f)$.

§ 2. Basic properties

We are going to give some general properties of the juxtapolynomials to a given continuous function on K, which do not coincide with the function on any point of K.

Thus, let $p \in \mathcal{J}(K; F; f)$ be a fixed juxtapolynomial to F on K, such that

$$F(z) - p(f; z) \neq 0, \quad z \in K.$$

We consider the continuous function

$$Z(z) = \frac{q(z)}{F(z) - p(f, z)}, \quad q \in \mathcal{P}(f),$$

which maps the compact set K onto a compact set K_q of the plane (Z). Let $\mathcal{K}_q$ be the convex hull of K_q and

$$\mathcal{K} = \bigcap_{q \in \mathcal{P}(f)} \mathcal{K}_q.$$

THEOREM 2.1. *A polynomial $p \in \mathcal{P}(f)$, $p(f; z) \neq F(z)$, $z \in K$, is a juxtapolynomial to F, i.e. $p \in \mathcal{J}(K; F; f)$ if and only if $\mathcal{K}$ contains the origin O_z of the plane (Z).*

Proof. We shall prove the equivalent statement: there exists a polynomial $q \in \mathcal{P}(f)$, $q \neq p$, satisfying (1.2)–(1.3), if and only if there exists $r \in \mathcal{P}(f)$ such that $\mathcal{K}_r \in O_Z$.

Indeed, if such a polynomial $q \in \mathcal{P}(f)$ exists, i.e. the conditions (1.2)–(1.3) are satisfied, then for $r = q - p$, and $Z(z) = \dfrac{r(f; z)}{F(z) - p(f; z)}$, we have

$$|Z - 1| = \left| \frac{q(f; z) - p(f; z)}{F(z) - p(f; z)} - 1 \right| = \frac{|F(z) - q(f; z)|}{|F(z) - p(f; z)|} < 1.$$

Therefore $K_r \subset \Gamma = \{Z : |Z - 1| < 1\}$ and hence $\mathcal{K}_r \subset \Gamma$. So it follows that $O_z \notin \mathcal{K}_r$.

Conversely, if for some $r \in \mathcal{P}(f)$, $\mathcal{K}_r$ does not contain O_z, then there exists a line L trough O_Z which does not intersect $\mathcal{K}_r$. Thus there exists a complex number $a \neq 0$, such that

$$|Z - a| \leq |a|, \quad \forall z \in K.$$

This inequality implies that $Z = \dfrac{r(f; z)}{a[F(z) - p(f; z)]}$ lies in the disk $\Gamma = \{Z : |Z - 1| \leq 1\}$, and hence that

$$q(f; z) = p(f; z) + \frac{1}{a} r(f; z)$$

verifies de conditions (1.2)–(1.3), because we have

$$\left| \frac{F(z) - q(f; z)}{F(z) - p(f; z)} \right| = \left| 1 - \frac{r(f; z)}{a[F(z) - p(f; z)]} \right| = |1 - Z| < 1.$$

The Theorem 2.1 has the following counterpart in Euclidian space of $2n$ – dimensions.

Lemma 2.1. *Let K be a compact set and $p \in \mathcal{P}(f)$, $p(f; z) \neq F(z)$, $z \in K$. Let E be the corresponding $2n$ – dimensional set whose points are expressed in the n complex valued coordinates $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$, where*

$$\zeta_j = \frac{f_j(z)}{F(z) - p(f; z)}, \quad z \in K.$$

Then $p \in \mathcal{J}(K; F; f)$ if and only if the origin of $\mathbf{R}^{2n}$ lies in the convex hull $\mathcal{E}$ of E.

Proof. Using the notation of Theorem 2.1, we may write

$$(2.1) \qquad Z = \frac{r(f; z)}{F(z) - p(f; z)} = \frac{c_1 f_1(z) + \ldots + c_n f_n(z)}{F(z) - p(f; z)} = c_1\zeta_1 + \ldots + c_n\zeta_n.$$

Now, by Theorem 2.1 it follows that $p \in \mathcal{J}(K; F; f)$ if and only if $O_Z \in \mathcal{K}$. Thus from (2.1) it follows that $p \in \mathcal{J}(K; F; f)$ if and only if the origin o $\mathbf{R}^{2n}$ is contained in the convex hull $\mathcal{E}$ of E.

THEOREM 2.2. *A polynomial $p \in \mathcal{P}(f)$, $p(f; z) \neq F(z)$, $z \in K$ is a juxtapolynomial to F on K, i.e. $p \in \mathcal{J}(K; F; f)$, if and only if there exist an integer m with $1 \leq m \leq 2n + 1$, a set of positive constants δ_j and a set of m points $Z_m = \{z_j\}_1^m \subset K$, such that*

$$(2.2) \qquad \sum_{j=1}^{m} \delta_j \frac{f_k(z_j)}{F(z_j) - p(f; z_j)} = 0, \quad \forall k \in \{1, 2, \ldots, m\}.$$

If $p \in \mathcal{J}(K; F; f)$ then $p \in \mathcal{J}(Z_m; F; f)$.

Proof. By Lemma 2.1, $p \in \mathcal{P}(f)$, $p(f; z) \neq F(z)$, $z \in K$ is a juxtapolynomial to F on K if and only if the origin O of $\mathbf{R}^{2n}$ is contained in $\mathcal{E}$. Hence the origin O is the centroid of m points $\zeta_j \in E$, corresponding to m points $z_j \in K$, with $m \leq 2n + 1$. That is we may find positive constants δ_i such that

$$\sum_{j=1}^{m} \delta_j \frac{f_k(z_j)}{F(z_j) - p(f; z_j)} = 0, \quad \forall k \in \{1, 2, \ldots, n\}.$$

i.e. (2.2) is satisfied.

The last part of Theorem 2.2 follows from the fact that $O_z \in \mathcal{M} = \bigcap_{q \in \mathcal{P}(f)} \mathcal{M}_q$, where $\mathcal{M}_q$ is the convex hull of M_q, $M = Z_m$, and so by Theorem 2.1 it follows that $p \in \mathcal{J}(Z_m ; F ; f)$.

Corollary 2.1. *Let K be a compact set containing at least $n + 1$ points and $f = (f_k)_1^n$ is a Tchebycheff system on K. Then the number m, in Theorem 2.2, satisfies the inequalities.*

$$n + 1 \leq m \leq 2n + 1.$$

Moreover, if F, $f = (f_k)_1^n$ are real valued functions on K, then $m = n + 1$.

Proof. If $m \leq n$, then since f is Tchebycheff system on K, we can find a polynomial $q \in \mathcal{P}(f)$ such that

$$q(f ; z) = F(z), \ \forall\, z \in Z_m,$$

i.e. q satisfies (1.2)–(1.3) on K, hence $p \notin \mathcal{J}(Z_m, F, f)$ therefore $p \notin \mathcal{J}(K ; F ; f)$, contradiction.

Definition 2.1. *We say that $M \subset K$ is a characteristic set of the juxtapolynomial $p \in \mathcal{J}(K ; F ; f)$ if $p \in \mathcal{J}(M ; F ; f)$ and for any $M_1 \subset M$, $M_1 \neq M$, $p \notin \mathcal{J}(M_1 ; F ; f)$.*

By Theorem 2.2 it follows that each juxtapolynomial $p \in \mathcal{J}(K ; F ; f)$, $p(f ; z) \neq F(z)$, $z \in K$ has at least one finite characteristic set M but it may be not unique.

From Definition 2.1 and Theorem 2.2 it follows:

Corollary 2.2. *Let Z_m be a characteristic set of $p \in \mathcal{J}(K ; F ; f)$ and $p(f ; z) \neq F(z)$, $z \in Z_m$. Then there exists the positive numbers δ_j such that (2.2) are satisfied.*

We saw in § 1 that each polynomial $p \in \mathcal{P}(f)$ of best approximation to F on K is a juxtapolynomial to F on K, too. We can characterize the juxtapolynomials to F on K which are at the same time a polynomial of best approximation to F on K.

THEOREM 2.3. *A juxtapolynomial $p \in \mathcal{J}(K ; F ; f)$, $p(f ; z) \neq F(z)$, $z \in K$ is a polynomial of best approximation to F on K if and only if on any its characteristic set $Z_m = \{z_j\}_1^m$ we have*

$$|F(z_j) - p(f ; z_j)| = \max |F(z) - p(f ; z)|, \ \forall\, j \in \{1, 2, \ldots, m\}. \tag{2.3}$$

Proof. If $p \in \mathcal{J}(K ; F ; f)$, $p(f ; z) \neq F(z)$, $z \in K$, then from Theorem 2.2 it follows that p possess at least a characteristic set $Z_m = \{z_j\}_1^m$ $(1 \leq m \leq 2n + 1)$ on which (2.2) are satisfied. Now if moreover on Z_m the equalities (2.3) are satisfied, then (2.2) may be written under the form

$$\sum_{j=1}^{m} \delta_j ex(i\theta_j) f_k(z_j) = 0, \ \forall k \in \{0, 1, \ldots, n\} \tag{2.4}$$

where $\theta_j = \arg [F(z_j) - p(f ; z_j)]$, $j = 1, 2, \ldots, m$. By the well-known REMEZ's theorem [15, p. 397] it follows, that p is a polynomial of best approximation to F on K.

Conversely, if $p \in \mathcal{P}(f)$ is a polynomial of best approximation to F on K in the uniform norm, then it is a juxtapolynom to F on K, too. Let $Z_m = \{z\}_1^m$ be a characteristic set of p. We shall show that

$$|F(z_j) - p(f ; z_j)| = \rho = \max_{z \in K} |F(z) - p(f ; z)|, \ \forall j \in \{1, 2, \ldots, m\}. \tag{2.5}$$

Let us suppose on the contrary that, for instance, $|p(z_m) - F(z_m)| < \rho$. We set $Z_{m-1} = Z_m - \{z_m\}$.

By Definition 2.1 of the characteristic set it follows that there exists a polynomial $q \in \mathcal{P}(f)$ such that

$$|F(z_j) - q(f ; z_j)| < |F(z_j) - p(f ; z_j)|, \ \forall\, j \in \{1, \ldots, m - 1\}.$$

But then for $t \in]0, 1[$ sufficiently small the polynomial

$$q_t = tp + (1 - t)q \in \mathcal{P}(f)$$

satisfies the inequality

$$|F(z_j) - q_t(f ; z_j)| < \rho, \ \forall j \in \{1, 2, \ldots, m\}$$

which contradicts the fact that p is a polynomial of best approximation to f on K.

§ 3. Matrix characterization of the juxtapolynomials

As we saw in Theorem 2.2, each juxtapolynomial to a continuous function on K is characterized by a finite subset $Z_m \subset K$, such that $p \in \mathcal{J}(Z_m ; F ; f)$. The problem is to characterize such a subset of K. In the case of polynomials of best approximation such a characterization was given by LEBEDEV N. A. and RYŽAKOV, I. JU. [6]. In the case of the generalized juxtapolynomials some similar characterizations were given by author [13].

Definition 3.1. *Let $p \in \mathcal{J}(K ; F ; f)$ be any juxtapolynomial to F on K. A subset $Z_m = \{z_j\}_1^m \subset K$ is called to be a $\mathcal{J}$ – set (juxta set) of p, if on Z_m the conditions of the Theorem 2.2, are satisfied.*

Let $M = (a_{ik})$ be a matrix of the type (m, n), where $a_{ik} \in \mathbf{C}$. We say that the vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ is orthogonal to M if

$$M\lambda' = 0$$

A vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ is called to be strictly different from zero (strictly positive) if

$$\forall k \in \{1, 2, \ldots, m\} \Rightarrow \lambda_k \neq 0 \ (\lambda_k > 0).$$

D e f i n i t i o n 3.2. *The matrix M is called H-matrix* (HP-matrix), *if there exists a vector λ strictly different from zero* (strictly positive) *orthogonal to M.*

Let $Z_m = \{z_j\}_1^m$ be an arbitrary subset of K. We denote by

$$M(f; Z_m) = \begin{pmatrix} f_1(z_1) & \ldots & f_1(z_m) \\ \cdot & \cdot & \cdot \\ f_n(z_1) & \ldots & f_n(z_m) \end{pmatrix}$$

$$M_c(f; Z_m) = \begin{vmatrix} f_1(z_1) & \ldots & f_1(z_m) \\ \cdot & \cdot & \cdot \\ f_n(z_1) & \ldots & f_n(z_m) \\ \bar{f}_1(z_1) & \ldots & \bar{f}_1(z_m) \\ \cdot & \cdot & \cdot \\ \bar{f}_n(z_1) & \ldots & \bar{f}_n(z_m) \end{vmatrix}$$

THEOREM 3.1. *A polynomial $\pi \in \mathcal{P}(f)$ for which*

$$W(z) = F(z) - \pi(f; z) \neq 0, \ z \in K$$

is a juxtapolynomial to F on K if and only if there exists a subset $Z_m = \{z_j\}_1^m \subset K (1 \leq m \leq 2n + 1)$ of distinct points, such that the matrix $M(f; Z_m)$ is a H — matrix and there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ strictly different from zero orthogonal to $M(f; Z_m)$ for which

$$\arg (\lambda_j W_j) = \arg (\lambda_1 W_1), \ \forall_j \in \{1, 2, \ldots, m\},$$

where $W_j = W(z_j)$.

Proof. Necessity. Let $\pi \in \mathcal{P}(f)$ be a juxtapolynomial to F on K, for which $W(z) \neq 0$, $z \in K$. Then by Theorem 2.2, there exist a subset $Z_m = \{z_j\}_1^m \subset K$ $(1 \leq m \leq 2n + 1)$ of distinct points, and the positive numbers δ_j, such that (2.2) are satisfied.

But the system (2.2) is equivalent to the following

$$\sum_{j=1}^{m} \frac{\delta_j}{|W_j|^2} \overline{W}_j f_k(z_j) = 0, \quad \forall k \in \{1, 2, \ldots, n\}. \tag{3.1}$$

Putting

$$\lambda_j = \frac{\delta_j}{|W_j|^2} \overline{W}_j \tag{3.2}$$

we are led to the further system:

$$\sum_{j=1}^{m} \lambda_j f_k(z_j) = 0, \quad \forall k \in \{1, 2, \ldots, n\}$$

where all $\lambda_j \neq 0$. Therefore the matrix $M(f; Z_m)$ is a H — matrix. From (3.2) we obtain

$$\lambda_j W_j > 0, \quad \forall j \in \{1, 2, \ldots, m\},$$

hence

$$\arg (\lambda_j W_j) = 0, \quad \forall j \in \{1, 2, \ldots, m\}.$$

S u f f i c i e n c y. Let us suppose that for $Z_m \subset K$ the matrix $M(f; Z_m)$ is a H— matrix, $W_i \neq 0$, $j = 1, 2, \ldots, m$, and there exists a vector λ strictly different from zero orthogonal to $M(f; Z_m)$ such that

$$\sum_{j=1}^{m} \lambda_j f_k(z_j) = 0, \quad \forall k \in \{1, 2, \ldots, n\}, \tag{3.3}$$

$$\arg (\lambda_j W_j) = \arg (\lambda_1 W_1), \ \forall j \in \{1, 2, \ldots, m\}.$$

Putting

$$\delta_j = \frac{|\lambda_j|}{|w_j|}, \quad \theta = \arg (\lambda_1 W_1)$$

we have

$$\lambda_j = \delta_j \overline{W}_j \operatorname{ex} (i\theta), \quad \forall j \in \{1, 2, \ldots, m\}$$

and replacing in (3.3) we obtain

$$\sum_{j=1}^{m} \delta_j \overline{W}_j e^{i\theta} f_k(z_j) = 0, \quad \forall k \in \{1, 2, \ldots, n\}$$

whence

$$\sum_{j=1}^{m} \delta_j \overline{W}_j f_k(z_j) = 0, \ \forall k \in \{1, 2, \ldots, n\}$$

and by Theorem 2.2 it follows that $\pi \in \mathcal{G}(K; F; f)$.

Corollary 3.1. *Let $Z_m = \{z_j\}_1^m$ $(1 \leq m \leq 2n+1)$ be a subset of distinct points of K. If the matrix $M(f; Z_m)$ is a H — matrix, then there exists a continuous function $F: K \to \mathbf{C}$, such that $\pi_0(f; z) \equiv 0$, $z \in K$, is a juxtapolynomial to F on K.*

Proof. Let $Z_m \subset K$ be an arbitrary subset of K such that $M(f; Z_m)$ is a H — matrix. Then $\lambda = (\lambda_1, \ldots, \lambda_m)$ being a vector strictly different from zero orthogonal to $M(f; Z_m)$ the relations (3.3) hold. We consider a continuous function $F: K \to C$ satisfying the conditions

$$F(z_j) = \frac{|\lambda_j|}{\lambda_j}, \ \forall j \in \{1, 2, \ldots, m\}$$

(Such a function, may be, for instance, Lagrange's interpolatory polynomial of degree $m-1$). Then the function F and the zero-polynomial π_0 satisfies the conditions of the Theorem 3.1, and therefore $\pi_0 \in \mathcal{G}(K; F; f)$.

Corollary 3.2. *Let Z_m $(1 \leq m \leq 2n+1)$ be an arbitrary subset of K and $p \in \mathcal{P}(f)$ a given generalized polynomial. If $M(f; Z_m)$ is a H — matrix, then there exists a continuous function $F: K \to C$ such that $p \in \mathcal{G}(K; F; f)$.*

Proof. If $M(f; Z_m)$ is a H — matrix then there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ strictly different from zero orthogonal to $M(f; Z_m)$. We consider a continuous function $F: K \to C$ satisfying the conditions:

$$F(z_j) = p(f; z_j) + \frac{|\lambda_j|}{\lambda_j}, \ \forall j \in \{1, 2, \ldots, m\} \tag{3.4}$$

From (3.4), we have

$$W_j = F(z_j) - p(f; z_j) = \frac{|\lambda_j|}{\lambda_j} \neq 0, \ \forall j \in \{1, 2, \ldots, m\}$$

and

$$\arg(\lambda_j W_j) = \arg(\lambda_1 W_1), \ \forall j \in \{1, 2, \ldots, m\}.$$

Therefore the polynomial $p \in \mathcal{P}(f)$ and the function F satisfy the conditions of Theorem 3.1, hence $p \in \mathcal{G}(K; F; f)$.

THEOREM 3.2. *Let $Z_m = \{z_j\}_1^m$ $(1 \leq m \leq 2n+1)$ be a subset of distinct points of K and $\pi \in \mathcal{G}(K; F; f)$ such that*

$$W_j = W(z_j) \neq 0, \ \forall z_j \in Z_m.$$

Z_m is a characteristic set for π if and only if the matrix $M_c(F_\pi; Z_m)$ is a HP — matrix of the rank $m-1$, where

$$F_\pi = (F_1, F_2, \ldots, F_n), \ F_k(z) = \overline{W}(z) f_k(z), \ k = 1, 2, \ldots, n.$$

Proof. Necessity. Let Z_m be a characteristic set of $\pi \in \mathcal{G}(K; F; f)$. Then by Corollary 2.2, there exist the positive numbers δ_j such that we have

$$\sum_{j=1}^{m} \delta_j \overline{W}_j f_k(z_j) = 0, \quad \forall k \in \{1, 2, \ldots, n\}. \tag{3.5}$$

But then we have also:

$$\sum_{j=1}^{m} \delta_j W_j \overline{f_k(z_j)} = 0, \quad \forall k \in \{1, 2, \ldots, n\}. \tag{3.6}$$

By (3.5), (3.6) it is clear that the positive vector $\delta = (\delta_1, \ldots, \delta_m)$ is orthogonal to the matrix $M_c(F_\pi; Z_m)$. Hence $M_c(F_\pi; Z_m)$ is a HP — matrix.

Obviously

$$r = \operatorname{rank} M_c(F_\pi; Z_m) \leq m - 1$$

If we suppose that $r < m-1$, then from a LEBEDEV and RYŽAKOV's result [6, p. 40] it follows that $M_c(F_\pi; Z_m)$ possesses a HP — simple component, i.e. there exists a subset $Z_p = \{z_{j_h}\}_1^p \subset Z_m$, $Z_p \neq Z_m$, such that $M_c(F_\pi; Z_p)$ is a HP — matrix, too. Therefore there exists a positive vector $d = (d_{j_1}, \ldots, d_{j_p})$ orthogonal to $M_c(F_\pi; Z_p)$. In special, we have

$$\sum_{h=1}^{p} d_{j_h} \overline{W}_{j_h} f_k(z_{j_h}) = 0, \quad \forall k \in \{1, 2, \ldots, n\}.$$

But from Theorem 2.2 it follows that $\pi \in \mathcal{G}(Z_p; F; f)$ which contradicts the assumption that Z_m is a characteristic set of π.

Sufficiency. Assume that $M_c(F_\pi, Z_m)$ is a HP — matrix of the rank $m-1$. Then there exists a positive vector $\delta = (\delta_1, \ldots, \delta_m)$ orthogonal to $M_c(F_\pi; Z_m)$, i.e. (3.5) and (3.6) are satisfied. Hence $\pi \in \mathcal{G}(Z_m; F; f)$.

If we suppose on the contrary that Z_m is not a characteristic set of π, then there exists a subset $Z_p = \{z_{j_h}\}_1^p \subset Z_m$, $Z_p \neq Z_m$ characteristic of π. But then from the necessity of Theorem 3.2 it follows that $M_c(F_\pi ; Z_p)$ is a HP — matrix and

$$\text{rank } M_c(F_\pi, Z_p) = p - 1 < m - 1,$$

contradiction. Therefore Z_m is a characteristic set of π.

D e f i n i t i o n 3.3. *The set* $Z_m = \{z_j\}_1^m$ *of distinct points of* K *is called* f — *juxtacharacteristic for* K, *if there exist a continuous function* F *on* K *and a juxtapolynomial* $p \in \mathcal{J}(K ; F ; f)$, *such that* Z_m *is a characteristic set of* p *and for which* $W(z) = F(z) - p(f ; z) \neq 0$, $z \in Z_m$.

We are going to give a theorem characterizing the f — juxtacharacteristic subset of K.

THEOREM 3.3. *Let* $Z_m = \{z_j\}_1^m$ $(1 \leq m \leq 2n + 1)$ *a subset of distinct points of* K. *This set* Z_m *is a* f — *juxtacharacteristic set for* K *if and only if*:

1° $M(f ; Z_m)$ *is a* H — *matrix.*

2° *There exists a vector strictly different from zero* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *orthogonal to* $M(f ; Z_m)$ *for which the* $(2n, m)$ *type matrix* $M(f ; \lambda ; Z_m) = (a_{ik})$, *where*

$$a_{ik} = \begin{cases} \dfrac{\lambda_k}{|\lambda_k|} f_i(z_k), & i = 1, 2, \ldots, n \\[2mm] \dfrac{|\lambda_k|}{\lambda_k} \bar{f}_i(z_k), & i = n + 1, \ldots, 2n \end{cases}$$

is a HP — *matrix of the rank* $m - 1$.

Proof. Necessity. Assume that $Z_m = \{z_j\}_1^m \subset K$ $(1 \leq m \leq 2n + 1)$ is a f — juxtacharacteristic set for K. Then by Definition 3.3 there exist a function F continuous on K and a juxtapolynomial $\pi \in \mathcal{J}(K ; F ; f)$ for which Z_m is a characteristic set and

$$\pi(f ; z_j) \neq F(z_j), \quad \forall j \in \{1, 2, \ldots, m\}.$$

By Theorem 3.2, $M_c(F_\pi ; Z_m)$ is a HP — matrix of the rank $m - 1$. But then $M(f ; Z_m)$ is a H — matrix.

Let us set

$$\lambda_j = W_j = F(z_j) - \pi(f ; z_j), \quad j = 1, 2, \ldots, m.$$

Let $\delta = (\delta_1, \ldots, \delta_m)$ be a positive vector orthogonal to $M_c(F_\pi ; Z_m)$. From (3.5), (3.6) it follows that the positive vector $d = (d_1, \ldots, d_m)$, where

$$d_j = \delta_j |W_j|$$

is orthogonal to $M(f ; \lambda ; Z_m)$.

Therefore $M(f ; \lambda ; Z_m)$ is a HP — matrix. The rank of $M(f ; \lambda ; Z_m)$ is $m - 1$, because this matrix is obtained from $M_c(F_\pi ; Z_m)$, multiplying its columns by different from zero factors.

S u f f i c i e n c y. If the vector λ satisfies the conditions of theorem then from Corollary 3.1, it follows that there exists a continuous function $F : K \to \mathrm{C}$ such that $\pi_0(f ; z) \equiv 0$, $z \in K$ is a juxtapolynomial to F on K. But in this case, by the definition of the matrix $M(f ; \lambda ; Z_m)$ and by the relations:

$$F(z_j) = \frac{|\lambda_j|}{\lambda_j}, \quad \forall j \in \{1, 2, \ldots, m\}$$

it follows

$$M_c(F_\pi ; Z_m) = M(F ; \lambda ; Z_m).$$

The conditions of Theorem 3.1 being satisfied, the subset Z_m is a characteristic set of the juxtapolynomial π_0 to F on K. Therefore Z_m is a f-juxtacharacteristic of K.

Let us next consider the pointsets of the real axis. Thus, let A be a compact real set and $F, f_k, k = 1, 2, \ldots, n$, continuous on A real valued functions. If $X_m = \{x_j\}_1^m \subset A$, then obviously

$$\text{rank } M_c(f ; X_m) = \text{rank } M(f ; X_m).$$

Hence, by Theorem 3.2, we have

C o r o l l a r y 3.3. *Let* $X_m = \{x_j\}_1^m$ $(1 \leq m \leq n + 1)$ *be a subset of distinct points of* A *and*

$$\pi(f ; x) = a_1 f_1(x) + \ldots + a_n f_n(x)$$

a juxtapolynomial to F *on* A *such that*

$$W_j = F(x_j) - \pi(f ; x_j) \neq 0, \quad \forall j \in \{1, 2, \ldots, m\}.$$

X_m *is a characteristic set of* $\pi \in \mathcal{J}(A ; F ; f)$ *if and only if* $M(F_\pi ; X_m)$ *is a* HP — *matrix of the rank* $m - 1$.

Similarly, by Theorem 3.3, we obtain

C o r o l l a r y 3.4. *A subset* $X_m = \{x_j\}_1^m \subset A (1 \leq m \leq n + 1)$ *is a f-juxtacharacteristic set of A if and only if the matrix* $M(f ; X_m)$ *is a H-matrix and*

$$\text{rank } M(f ; X_m) = m - 1.$$

R e m a r k 3.1. All results of this paragraph were established in the case where the juxtapolynomials $\pi \in \mathcal{P}(f)$ to F on K do not coincide with the function F on K. This restriction is not essential. It is possible to give similar results also in the case where π coincides with the function on certain number of points of K [12].

Let us suppose that the functions $F, f_k, k = 1, 2, \ldots, n$, are regular on a domain containing the compact K and let $\mathcal{P}(f; \mathcal{U}_s)$ be the class of polynomials $p \in \mathcal{P}(f)$ satisfying on $\mathcal{U}_s = \{u_i\}_1^s$, the conditions

$$p^{(k)}(f; u_j) = F^{(k)}(u_i), \; k = 0, 1, 2, \ldots, r_i, \; i = 1, 2, \ldots, s, \tag{3.7}$$

where $r_1, r_2, \ldots, r_s$ are integers satisfying the conditions

$$r = r_1 + r_2 + \ldots + r_s, \; 1 \leq r + s \leq n.$$

Definition 3.4. *A polynomial $p \in \mathcal{P}(f)$ is called $r + s$ — preinterpolatory juxtapolynomial to F on K if $p \in \mathcal{J}(K; F; f)$ and the conditions (3.7) are satisfied.*

We denote by $\mathcal{J}_{r+s}(K; \mathcal{U}_s; F; f)$ the set of all $r + s$ — preinterpolatory juxtapolynomials to F on K.

If $r + s = n$, then obviously, the set $\mathcal{J}_n(K; \mathcal{U}_s; F; f)$ contains only Lagrange — Hermitte's interpolatory polynomials to F on the knots $\mathcal{U}_s$. If $r = 0$, $s = n$ and $f = (f_k)$ is a Tchebycheff system on K, then $\mathcal{J}_n(K; \mathcal{U}_n; F; f)$ contains a single plynomial-Lagrange's interpolatory polynomial to F on $\mathcal{U}_n$.

It is clear that for $r + s = 0$ we obtain 0 — interpolatory juxtapolynomials to F on K which coincid with the juxtapolynomials to F on K.

Some of the results given in § 2 and § 3 for the juxtapolynomials to F on K, can be extended to the preinterpolatory justapolynomials [12]. Thus Theorem 2.2 has the following analogous.

THEOREM 3.4. *If $p \in \mathcal{J}_{r+s}(K; \mathcal{U}_s; F; f)$, $p(f; z) \neq F(z)$, $z \in K_1 = K - \mathcal{U}_s$, and if there exists an closed subset $C \subset K_1$, such that $p \in \mathcal{J}_{r+s}(C; \mathcal{U}_s; F; f)$, then there exist $m (1 \leq m \leq 2(n - r - s) + 1)$ points $z \in K_1$, the positive numbers δ_j with $\Sigma \delta_j = 1$, and the complex numbers A_{ih}, $h = 0, 1, \ldots r_j$, $i = 1, 2, \ldots, s$, such that*

$$\sum_{j=1}^{m} \delta_j \frac{f_k(z_j)}{W(z_j)} + \sum_{i=1}^{s} \sum_{h=0}^{r_j} A_{ih} f_k^{(h)}(u_i) = 0, \; \forall k \in \{1, 2, \ldots, n\} \tag{3.8}$$

If $p \in \mathcal{P}(f; \mathcal{U}_s)$ satisfies (3.8) then $p \in \mathcal{J}_{r+s}(Z_m; \mathcal{U}_s; F; f)$ where $Z_m = \{z_j\}_1^m$.

When

$$r_1 + r_2 + \ldots + r_s = 0$$

(3.8) becomes:

$$\sum_{j=1}^{m} \delta_j \frac{f_k(z_j)}{W(z_j)} + \sum_{i=1}^{s} A_i f_k(u_i) = 0, \; \forall k \in \{1, 2, \ldots, n\} \tag{3.9}$$

Therefore, a polynomial $p \in \mathcal{P}(f)$ wich satisfies the conditions

$$p(f; u_i) = F(u_i), \quad \forall j \in \{1, 2, \ldots, n\} \tag{3.10}$$

is a s — preinterpolatory polynomial on K if and only if there exists a subset $Z_m \subset K$ such that (3.9) are satisfied.

Let $q \in \mathbf{N}$ $(1 \leq q \leq m)$ be given. We say that the vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a q — vector strictly different from zero (q — vector positive) if

$$\lambda_k \neq 0, \; (\lambda_k > 0), \; \forall k \in \{1, 2, \ldots, q\}.$$

Definition 3.5. *A matrix M of type (n, m) is called Hq — matrix (HP_q — matrix) if there exists a q — vector strictly different from zero (q — vector positive) orthogonal to M.*

If we denote:

$$M(f; Z_m; \mathcal{U}_s) = \begin{pmatrix} f_1(z_1) & \ldots & f_1(z_m) f_1(u_1) & \ldots & f_1(u_s) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n(z_1) & \ldots & f_n(z_m) f(u_1) & \ldots & f_n(u_s) \end{pmatrix}$$

then by Theorem 3.4 we are led to:

THEOREM 3.5. *Under the conditions of Theorem 3.4, the polynomial $p \in \mathcal{P}(f; \mathcal{U}_s)$ is a s — preinterpolatory juxtapolynomial to F on K if and only if there exists a subset $Z_m \subset K$ $(1 \leq m \leq 2(n - s) + 1)$ of distinct points such that:*

1° the matrix $M(f; Z_m; \mathcal{U}_s)$ is a H_m — matrix;

2° there exists a m-vector $\Lambda = (\lambda, \gamma) = (\lambda_1, \ldots, \lambda_m, \gamma_1, \ldots, \gamma_s)$ strictly different from zero orthogonal to $M(f; Z_m; \mathcal{U}_s)$ such that

$$\arg(\lambda_j W_j) = \arg(\lambda_1 W_1), \quad \forall j \in \{1, 2, \ldots, n\}.$$

Proof. Necessity. Let $\pi \in \mathcal{J}_s(K; \mathcal{U}_s; F; f)$ be a s-preinterpolatory juxtapolynomial to F on K satisfying the conditions of the Theorem 3.4. Then by this theorem it follows that there exist a subset $Z_m \subset K (1 \leq m \leq 2(n - s) + 1)$, the numbers δ_j with $\Sigma \delta_j = 1$, and the complex numbers A_i, such that (3.9) holds.

Hence we have

$$\sum_{j=1}^{m} \lambda_j f_k(z_j) + \sum_{i=1}^{s} A_i f_k(u_i) = 0, \quad \forall k \in \{1, 2, \ldots, n\}, \tag{3.11}$$

where

$$\lambda_j = \delta_j / W_j.$$

Therefore, the vector $\Lambda = (\lambda, \gamma) = (\lambda_1, \ldots, \lambda_m, A_1, \ldots, A_s)$ is a m-vector strictly different from zero orthogonal to $M(f; Z_m; \mathfrak{U}_s)$. Because

$$\lambda_j W_j = \delta_j > 0, \quad \forall j \in \{1, 2, \ldots, m\}$$

the condition 2° is satisfied, too.

Sufficiency. Let $Z_m \subset K$ be a subset for which the conditions of Theorem 3.5 are satisfied. Let $\Lambda = (\lambda, \gamma)$ be a m-vector strictly different from zero orthogonal to $M(f; Z_m; \mathfrak{U}_s)$ satisfying 2°. Let us denote by

$$d_j = \frac{|\lambda_j|}{|W_j|} > 0, \quad \theta = \arg(\lambda_1, W_1),$$

whence

$$\lambda_j = d_j \overline{W}_j \operatorname{ex}(i\theta).$$

Then from (3.11) we obtain

$$\text{(3.12)} \quad \sum_{j=1}^{m} d_j \overline{W}_j f_k(z_j) \operatorname{ex}(i\theta) + \sum_{i=1}^{s} \gamma_i f_k(u_i) = 0, \quad \forall k \in \{1, 2, \ldots, n\}.$$

If we now define δ_j and A_i by the relations

$$\delta_j = \frac{d_j |W_j|^2}{\Sigma d_j |W_j|^2}, \quad A_j = \frac{\gamma_j}{\Sigma d_j |W_j|^2} ex(-i\theta)$$

(3.12) becomes:

$$\text{(3.13)} \quad \sum_{j=1}^{m} \delta_j \frac{f_k(z_j)}{W_j} + \sum_{i=1}^{s} A_i f_k(u_i) = 0, \quad \forall k \in \{1, 2, \ldots, n\},$$

where $\delta_j > 0$ and $\Sigma \delta_j = 1$. Therefore (3.9) is satisfied, and by Theorem 3.4, it follows that $\pi(f; z) = F(z) - W(z) \in \mathcal{G}_s(K; \mathfrak{U}_s, F; f)$.

Remark 3.2. Because each polynomial $\pi \in \mathcal{P}(f)$ of the best approximation to F on K is a juxtapolynomial to F on K [§ 1, 2°], too, our results contain as a particular case some similar results given by LEBEDEV, N. A. and RYŽAKOV, I. JU. [6] for the best approximation problem.

§ 4. Linear juxtaoperators

Using the previous results we are going to construct a linear operator defined over the set $C(K)$ of continuous functions on K and with values on the set of generalized juxtapolynomials to a continuous function on K.

Throughout this paragraph we assume that the compact set K consists of at least $n+1$ points and $f = (f_k)$ is a Tchebycheff system on K.

So if $f = (f_k)$ is Tchebycheff system on K then for each system of $n+1$ points z_j of K there exists an interpolatory polynomial $L(z_1, \ldots, z_n; f; F|z) \in \mathcal{P}(f)$ satisfying the conditions

$$L(z_1, \ldots, z_n; f; F|z_j) = F(z_j), \quad \forall j \in \{1, 2, \ldots, n\},$$

and it is unique. This polynomial can be expressed under the form

$$\text{(4.1)} \quad L(z_1, \ldots, z_n; f; F|z) = \sum_{j=1}^{n} \frac{U(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n; f)}{U(z_1, z_1, \ldots, z_n)} F(z_j),$$

where

$$U(x_1, x_2, \ldots, x_n; f) = \begin{vmatrix} f_1(x_1) & \ldots & f_n(x_1) \\ \ldots & \ldots & \ldots \\ f_1(x_n) & \ldots & f_n(x_n) \end{vmatrix}.$$

The main result of this paragraph is the following.

THEOREM 4.1. *Let K be a compact set in the complex plane containing at least $n+1$ points and $f = (f_k)$ a Tchebycheff system on K. The generalized polynomial $\pi \in \mathcal{P}(f)$, $\pi(f; z) \neq F(z)$, $z \in K$ is a juxtapolynomial to F on K, i.e. $\pi \in \mathcal{G}(K; F; f)$, if and only if there exists a subset $Z_m \subset K$ $(n+1 \leq m \leq 2n+1)$ and the positive constants d_j such that*

$$\text{(4.2)} \quad \pi(f; z) = \frac{\Sigma d_{j_1} \ldots d_{j_n} |U(z_{j_1}, \ldots, z_{j_n})|^2 L(z_{j_1}, \ldots, z_{j_n}; f; F|z)}{\Sigma d_{j_1}, \ldots, d_{j_n} |U(z_{j_1}, \ldots, z_{j_n})|^2},$$

where Σ is taken for all j_k from 1 to m.

Proof. Necessity. Assume that $\pi \in \mathcal{G}(K; F; f)$ and $\pi(f; z) \neq F(z)$, $z \in K$. Then by Theorem 2.2, there exist a set $Z_m = \{z_j\}_1^m \subset K$ $(n+1 \leq m \leq 2n+1)$ and a system of positive constants $\{\delta_j\}$ such that

$$\text{(4.3)} \quad \sum_{j=1}^{m} \delta_j \frac{f_k(z_j)}{F(z_j) - \pi(f; z_j)} = 0, \quad \forall k \in \{1, 2, \ldots, n\}.$$

But the equalities (4.3) can be written under the form

$$\text{(4.4)} \quad \sum_{j=1}^{m} d_j \pi(f; z_j) \overline{f}_k(z_j) = \sum_{j=1}^{m} d_j F(z_j) \overline{f}_k(z_j), \quad \forall k \in \{1, 2, \ldots, n\},$$

where

$$d_j = \frac{\delta_j}{|W(z_j)|^2}.$$

If

$$\pi(f; z) = a_1 f_1(z) + \ldots + a_n f_n(z)$$

then the system (4.4) is equivalent to the following

(4.5) $$a_1 L_{k1} + a_2 L_{k2} + \ldots + a_n L_{kn} = g_k, \quad k \in \{1, 2, \ldots, n\},$$

where

$$L_{ki} = \sum_{j=1}^{m} d_j \bar{f}_k(z_j) f_i(z_j),$$

$$g_k = \sum_{j=1}^{m} d_j F(z_j) \bar{f}_k(z_j).$$

The system (4.5) to which we add the equality

$$a_1 f_1(z) + \ldots + a_n f_n(z) = \pi(f; z)$$

permits to eliminate the unknown: $a_1, a_2, \ldots, a_n$, whence we get

$$\begin{vmatrix} \pi(f; z) & f_1(z) & \ldots & f_n(z) \\ g_1 & L_{11} & \ldots & L_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ g_n & L_{n1} & \ldots & L_{nn} \end{vmatrix} = 0.$$

Hence

(4.6) $$\pi(f; z) = - \begin{vmatrix} 0 & f(z) & \ldots & f(z) \\ g_1 & L_{11} & \ldots & L_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ g_n & L_{n1} & \ldots & L_{nn} \end{vmatrix} : \begin{vmatrix} L_{11} & \ldots & L_{1n} \\ \cdot & \cdot & \cdot \\ L_{n1} & \ldots & L_{nn} \end{vmatrix}$$

But, if we denote by

$$L_{ik} = \sum_{j_k=1}^{m} d_{j_k} \bar{f}_k(z_{j_k}) f_i(z_{j_k}),$$

then

(4.7) $$\det(L_{ik}) = \sum_{j_1, \ldots, j_n=1}^{m} d_{j_1} \ldots d_{j_n} f_1(z_{j_1}) \ldots f_n(z_{j_n}) U(z_{j_1}, \ldots, z_{j_n}; f).$$

Permuting in (4.7) the indexes $j_1, \ldots, j_n$ in all $n!$ possible manners and adding the resulting equalities, we are led to the further expression

$$\det(L_{i_k}) = \frac{1}{n!} \sum d_{j_1} \ldots d_{j_n} |U(z_{j_1}, \ldots, z_{j_n}; f)|^2$$

where Σ is taken for all indexes j_k from 1 to m.

Similarly the determinant of denominator of (4.7) can be put under the form

$$\frac{1}{n!} \Sigma d_{j_1}, \ldots, f_{j_n} U(z_{j_1}, \ldots, z_{j_n}; f) V(z, z_{j_1}, \ldots, z_{j_n}; f; F),$$

where

$$V(z, x_1, \ldots, x_n; f; F) = \begin{vmatrix} 0 & f_1(z) & \ldots & f_n(z) \\ F(x_1) & f_1(x_1) & \ldots & f_n(x_1) \\ \cdot & \cdot & \cdot & \cdot \\ F(x_n) & f_1(x_n) & \ldots & f_n(x_n) \end{vmatrix}$$

and $\bar{U}(x_1, \ldots, x_n; f)$ is the imaginary conjugate of $U(x_1, \ldots, x_n; f)$.

But in view of (4.1), we have

$$V(z, x_1, \ldots, x_n; f; F) = \sum_{k=1}^{n} (-1)^k F(x_k) U(z, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n; f) =$$

$$= - \sum_{k=1}^{n} U(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_n; f) F(x_k) =$$

$$= - U(x_1, \ldots, x_n; f) L(x_1, \ldots, x_n; f; F|z).$$

Replacing the expression of the determinants in (4.6), we obtain (4.2).

S u f f i c i e n c y. Now, assume that $\pi(f; z)$ is given by (4.2). First we remark that the numerator of (4.2) may be written under the form

$$\sum_{1}^{m} \frac{d_1 d_2 \ldots d_n}{d_{k_1} d_{k_2} \ldots d_{k_{m-n}}} |U(z_1, \ldots, z_{\hat{k}_1}, \ldots, z_{\hat{k}_2}, \ldots, z_{\hat{k}_{m-n}}, \ldots; f)|^2 \times$$

$$\times L(z_1, \ldots, z_{\hat{k}_1}, \ldots, z_{\hat{k}_2}, \ldots, z_{\hat{k}_{m-n}}, \ldots; f; F|z).$$

where „$\hat{k}$" points that the index k is missing.

Thus

(4.8)

$$F(z) - \pi(f\,;z) = \frac{1}{\Delta}\sum \frac{d_1 \ldots d_m}{d_{k_1}\ldots d_{k_{m-n}}}\,|U(z_1, \ldots, z_{\hat{k}_1}, \ldots, z_{\hat{k}_{m-n}}, \ldots; f)|^2\,|\times$$
$$\times\ [F(z) - L(z_1, \ldots, z_{\hat{k}_1}, \ldots, z_{\hat{k}_{m-n}}, \ldots, z_m; f; F|z)],$$

where

$$\Delta = \sum \frac{d_1 \ldots d_m}{d_{k_1} d_{k_2} \ldots d_{m-n}}\,|U(z_1, \ldots, z_{\hat{k}_1}, \ldots; f)|^2.$$

But we have

$$F(z) - L(x_1, x_2, \ldots, x_n; f; F|z) = \frac{D(z, x_1, \ldots, x_n; f; F)}{U(x_1, x_2, \ldots, x_n; f)},$$

where

(4.9)
$$D(z, x_1, x_2, \ldots, x_n; f; F) = \begin{vmatrix} F(z) & f_1(z) & \ldots & f_n(z) \\ F(x_1) & f_1(x_1) & \ldots & f_n(x_1) \\ \cdot & \cdot & \cdot & \cdot \\ F(x_n) & f_1(x_n) & \ldots & f_n(x_n) \end{vmatrix}$$

From (4.8) and (4.9) we obtain

$$F(z) - \pi(f\,;z) = \frac{1}{\Delta}\sum \frac{d_1 \ldots d_m}{d_{k_1} \ldots d_{k_{m-n}}}\,\overline{U}(z_1, \ldots, z_{\hat{k}_1}, \ldots; f)\times$$
$$\times D(z, z_1, \ldots, z_{\hat{k}_1}, \ldots; f; F),$$

whence

$$F(z_j) - \pi(f\,;z_j) = \frac{1}{\Delta}\sum \frac{d_1 \ldots d_m}{d_j d_{k_2} \ldots d_{k_{m-n}}}(-1)^{j-1}\,\overline{U}(z_1, \ldots, z_{\hat{j}}, \ldots, z_{\hat{k}_2}, \ldots; f)\times$$
$$\times\ D(z_1, \ldots, z_{\hat{k}_2}, \ldots; f; F).$$

Therefore

$$\sum_{j=1}^{m} d_j[F(z_j) - \pi(f\,;z_j)]\bar{f}_k(z_j) = \frac{1}{\Delta}\sum \frac{d_1 \ldots d_m}{d_{\hat{k}_2} \ldots d_{k_{m-n}}}\,D(z_1, \ldots, z_{\hat{k}_2}, \ldots; f; F)\times$$
$$\times \sum_{j=1}^{m}(-1)^{j-1}\,\overline{U}(z_1, \ldots, z_{\hat{j}}, \ldots, z_{\hat{k}_2}, \ldots)\bar{f}_k(z_j) =$$

$$= \frac{1}{\Delta}\sum \frac{d_1 \ldots d_m}{d_{k_2} \ldots d_{k_{m-n}}}\,D(z_1, \ldots, z_{\hat{k}_2}, \ldots; f; F)\ \times$$
$$\times \begin{vmatrix} \bar{f}_k(z_1) & \bar{f}_1(z_1) & \ldots & \bar{f}_n(z_1) \\ \cdot & \cdot & \cdot & \cdot \\ \bar{f}_k(z_{\hat{k}_2}) & \bar{f}_1(z_{\hat{k}_2}) & \ldots & \bar{f}_n(z_{\hat{k}_2}) \\ \cdot & \cdot & \cdot & \cdot \\ \bar{f}_k(z_{\hat{k}_{m-n}}) & \bar{f}_1(z_{\hat{k}_{m-n}}) & \ldots & \bar{f}_n(z_{\hat{k}_{m-n}}) \\ \cdot & \cdot & \cdot & \cdot \\ \bar{f}_k(z_m) & \bar{f}_1(z_m) & \ldots & \bar{f}_n(z_m) \end{vmatrix} = 0$$

Thus we are led to the further equations:

$$\sum_{j=1}^{m} \delta_j \frac{f_k(z_j)}{W(z_j)} = 0, \quad \forall k \in \{1, 2, \ldots, n\}$$

where

$$\delta_j = d_j|W(z_j)|^2.$$

and by Theorem 2.2 it follows that $\pi \in \mathcal{J}(K; F; f)$.

C o r o l l a r y 4.1. *Under the conditions of the Theorem 4.1, a polynomial* $\pi \in \mathcal{P}(f)$ *is a juxtapolynomial to* f *on* K *if and only if there exist a subset* $Z_m \subset K$ $(n + 1 \leq m \leq 2n + 1)$ *and the positive constants* $\Lambda_{j_1, \ldots, j_n}$ *with* $\Sigma\,\Lambda_{j_1, \ldots, j_n} = 1$ *such that*

(4.10)
$$\pi(f\,; z) = \Sigma\,\Lambda_{j_1, \ldots, j_n} L(z_j, \ldots, z_{j_n}; f; F|z).$$

Indeed (4.10) follows from (4.2) if we denote

$$\Lambda_{j_1, \ldots, j_n} = \frac{d_{j_{1_n}} \ldots d_{j_n}\,|U(z_{j_1} \ldots z_{j_n}; f)|^2}{\Sigma d_{j_1} \ldots d_{j_n}\,|U(z_{j_1} \ldots z_{j_n}; f)|^2}$$

because obviously $\Lambda_{j_1, \ldots, j_n} > 0$ and $\Sigma\,\Lambda_{j_1, \ldots, j_n} = 1$.

The operator of the right side of (4.10) which will be called *juxta-operator* is an interesting linear operator which assigns to each continuous function on K one of its generalized juxtapolynomials on K. This operator is linear because it is a linear combination of linear interpolatory operators L.

It is interesting to note that (4.10) gives us the possibility to construct easy a juxtapolynomial to a given function f continuous on a compact set K. It is sufficient to choose an arbitrary subset $Z_m \subset K$ $(n + 1 \leq m \leq$

$\leq 2n + 1)$ and by the arbitrary positive constants we construct the polynomial π as in (4.10).

When $m = n + 1$, then the juxtaoperator has a simpler form:

$$\pi(f; z) = \sum_{j=0}^{n} \Lambda_j L(z_0, \ldots, \hat{z_j}, \ldots, z_n; f; F|z). \tag{4.11}$$

By Corollary 2.1 it follows that this is the case when F and $f = (f_k)$ are real valued functions on K, i.e. we have.

Corollary 4.2. *If $f = (f_k)$ is a Tchebycheff system on K and F, $f = (f_k)$ are real valued functions on K, then a polynomial $\pi \in \mathcal{P}(f)$, $\pi(f; z) \neq F(z)$, $z \in K$ is a juxtapolynomial to f on K if and only if there exist $n + 1$ points $z_0, z_1, \ldots, z_n$ of K and positive constants $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ with $\Sigma \Lambda_j = 1$ such that (4.11) holds.*

By virtue of Theorem 2.3 and Theorem 4.1 we have

Corollary 4.3: *Let K be a compact set of the complex plane containing at least $n + 1$ points, F a continuous function on K and $f = (f_k)$ a Tchebycheff system on K. The generalized polynomial $\pi \in \mathcal{P}(f)$ is a polynomial of best approximation to F on K if and only if there exists a subset $Z_m \subset K(n + 1 \leq m \leq 2n + 1)$ and the positive constants $\Lambda_{j_1, \ldots, j_n}$ with $\Sigma \Lambda_{j_1, \ldots, j_n} = 1$ such that*

$$|F(z_j) - \pi(f; z_j)| = \rho = \max_{z \in K} |F(z) - \pi(f; z)|, \quad \forall j \in \{1, 2, \ldots, m\}, \tag{4.12}$$

$$\pi(f; z) = \Sigma \Lambda_{j_1, \ldots, j_n} L(z_{j_1}, \ldots, z_{j_n}; f; F|z). \tag{4.13}$$

This result was established by the author in 1964 [9].

When $m = n + 1$ the constants Λ_j can be expressed explicitly [9], and so we obtain

$$\rho = \frac{|D(z_0, z_1, \ldots, z_n; f; F)|}{\sum_{j=0}^{n} |U(z_0, \ldots, \hat{z_j}, \ldots, z_n; f)|}$$

$$\pi(f; z) = \frac{|U(z_0, \ldots, \hat{z_j}, \ldots, z_n; f)| \; L(z_0, \ldots, \hat{z_j}, \ldots, z_n; f; F|z)}{\sum_{j=0}^{n} |U(z_1, \ldots, \hat{z_j}, \ldots, z_n; f)|}$$

All our results were established under the assumption that the juxtapolynomial to F does not coincide with the function F on K. This assumption is not essential, it is possible to give a similar result also for the preinterpolatory juxtapolynomial to F on K. Thus we have [12]

THEOREM 4.2. *Let $\mathcal{U}_s \subset K(1 \leq s \leq n)$ be a subset of K, F continuous on K and $f = (f_k)$ a Tchebycheff system on K. A polynomial $q \in \mathcal{P}(f)$ satisfying the conditions*

$$q(f; u_i) = F(u_i), \quad \forall i \in \{1, 2, \ldots, s\}$$

$$q(f; z) \neq F(z), \quad z \in K - \mathcal{U}_s$$

is a s-preinterpolatory juxtapolynomial to F on K, i.e. $q \in \mathcal{J}_s(K; \mathcal{U}_s; F; f)$ if and only if there exist a subset $Z_m \subset K(n - s + 1 \leq m \leq 2n - 2s + 1)$ and the positive constants $\Lambda_{j_1, \ldots, j_{n-s}}$ with $\Sigma \Lambda_{j_1, \ldots, j_{n-s}} = 1$, such that

$$q(f; z) = \Sigma \Lambda_{j_1, \ldots, j_{n-s}} L(u_1, \ldots, u_s, z_{j_1}, \ldots, z_{j_{n-s}}; f; F|z). \tag{4.14}$$

§ 5. Extremal solutions of a differential equations

Let us consider the linear differential equation

$$L(y) = A_0(x)y^{(p)} + \ldots + A_p(x)y = F(x), \tag{5.1}$$

where $A_0, A_1, \ldots, A_n$, F are continuous functions on the compact real set K.

The problem which will be considered is to find an approximate solution of (5.1) in the set $\mathcal{P}(f)$, where f_k, $k = 1, 2, \ldots, n$, are assumed to be p-times differentiable liniearly independent (on K) functions.

If

$$y = a_1 f_1(x) + \ldots + a_n f_n(x), \tag{5.2}$$

then

$$L(y) = a_1 L(f_1) + \ldots + a_n L(f_n).$$

Let us denote by $L_k(x) = L[f_k]$, $k = 1, 2, \ldots, n$. Then the approximation problem is the following: to approximate the continuous function F on K by the generalized polynomials

$$p(L; x) = a_1 L_1(x) + \ldots + a_n L_n(x).$$

Definition 5.1. *A polynomial*

$$y_* = p^*(f; x) = a_1^* f_1(x) + \ldots + a_1^* f_n(x) \tag{5.3}$$

is called solution of best approximation of the equation (5.1) *on* K *in the class* $\mathcal{P}(f)$ *if*

$$(5.4)\qquad \inf_{p\in\mathcal{P}(f)} \max_{x\in K} \left|F(x) - \sum_{k=1}^{n} a_k L_k(x)\right| = \max_{x\in K} \left|F(x) - \sum_{k=1}^{n} a_k^* L_k(x)\right|.$$

From our assumption with respect to the functions F, A_j and f_k, it follows that each equation (5.1) has at least one solution of best approximation on K in the class $\mathcal{P}(f)$. Generally such an approximate solution may be not unique

Definition 5.2. *A polynomial* $y \in \mathcal{P}(f)$ *is called Cauchy's solution of best approximation of the equation* (5.1) *on* K *in the class* $\mathcal{P}(f)$, *if* y_* *beside* (5.4) *satisfies the Cauchy's condition*

$$(5.5)\qquad y_*^{(k)}(x_0) = y_*^{(k)}, \quad \forall k \in \{0, 1, \ldots, p-1\}.$$

Obviously, the problem is not trivial only if $p < n$, because otherwise, if $p \geq n$, either the system (5.5) is incompatible, or it permits to determine the coefficients $a_0, a_1, \ldots, a_n$ and, therefore the polynomial $y \in \mathcal{P}(f)$ which satisfies Cauchy's conditions (5.5) is completely determinated. Thus we assume that $p < n$.

Under our assumption it is known that such an approximation of Chauchy's solution always exists, [11] but it may not be unique.

It is clear that we can consider a similar approximation problem in the other norms. To include all these extremal solutions we shall consider the juxtasolutions of a linear differential equation (5.1).

Definition 5.3. *A polynomial* $y \in \mathcal{P}(f)$ *is called juxtasolution (Cauchy's juxtasolution) of the equation* (5.1) *on* K *if* $L(y)$ *is a juxtapolynomial* ($p-1$ — *preinterpolatory polynomial*) *to* f *on* K.

We denote by $\mathcal{J}(K; L; F; f)$ and $\mathcal{J}(x_0; K; L; F; f)$ the set of the juxtasolutions and respectively Cauchy's juxtasolutions of the equation (5.1) on K.

Obviously each solution of Cauchy's problem of the equation (5.1) of the form (5.2) (if such a solution exists) is respectively juxtasolution and Cauchy's juxtasolution of the equation (5.1) on every compact set $K \subseteq \mathbf{R}$.

By virtue of Definition 5.1, it follows that each best approximation solution of the equation (5.1) on K is also a juxtasolution of this equation on K.

Using the properties of the juxtapolynomial to a continuous functions on a compact K of the real axis, given in the previous paragraphs, we can state some properties of the juxtasolution respectively Cauchy's juxtasolutions of equation (5.1) on K.

First, by Theorem 2.2 we are led to the:

THEOREM 5.1. *If* $y \in \mathcal{J}(K; L; F; f)$, $L(y) \not\equiv F(x)$, $x \in K$, *then there exist* $m(1 \leq m \leq n+1)$ *distinct points* $x_j \in K$, *positive numers* δ_j *such that*

$$(5.6)\qquad \sum_{j=1}^{m} \delta_j \frac{L_k(x_j)}{F(x_j) - L(y(x_j))} = 0, \quad \forall k \in \{1, 2, \ldots, n\}.$$

If $y \in \mathcal{J}(K; L; F; f)$ *then* $y \in \mathcal{J}(X_m; L; F; f)$, *where* $X = \{x_j\}_1^m$

If furthermore $(L_k)_1^n$ is a Tchebycheff system on K, then the number m in Theorem 5.1 is equal to $n+1$.

Because Cauchy's conditions (5.5) are linear conditions with respect to the coefficients $a_1, a_2, \ldots, a_n$ by virtue of a general result established by the author for generalized restricted infrapolynomials [12] we have a similar characterisation for the Chauchy's juxtasolutions of a differential equations (5.1), too.

THEOREM 5.2. *If* $y \in \mathcal{J}_{p-1}(x_0, K; L; F; f)$, $y(x) \not\equiv F(x)$ $x \in K - \{x_0\}$ *and there exists a closed subset* $C \subset K - \{x_0\}$ *such that* $y \in \mathcal{J}_{p-1}(x_0, C; L; F; f)$ *then there exist* $m(1 \leq m \leq n-p+1)$ *distinct points* $x_j \in K - \{x_0\}$, *positive numbers* δ_j *with*, $\Sigma\delta_j = 1$ *and real numbers* γ_h *such that*

$$(5.7)\qquad \sum_{j=1}^{m} \delta_j \frac{L_k(x_j)}{F(x_j) - L(y(x_j))} + \sum_{h=0}^{p-1} \gamma_h f_k^{(h)}(x_0) = 0.$$

If $y \in \mathcal{J}_{p-1}(x_0, K; L; F; f)$ *then* $y \in \mathcal{J}_{p-1}(x_0, X_m; L; F; f)$.

If $L_1, L_2, \ldots, L_n$ is a Tchebycheff system on K, then from Corollary 4.2 regarding the structure of the juxtaoperator, it follows an interesting structure for the juxtasolutions of a linear differential equation.

THEOREM 5.3. *Let* L *and* $f = (f_k)_1^n$ *be such away that* $(L_k)_1^n$ *forms a Tchebycheff system on* K. *A polynomial* $y \in \mathcal{P}(f)$, $L(y) \not\equiv F(x)$, $x \in K$ *is a juxtasolution of the equation* (5.1) *on* K *if and only if there exist a subset* $X_{n+1} = \{x_j\}_1^{n+1}$, *and the positive constants* Λ_j *with* $\Sigma\Lambda_j = 1$, *such that*

$$(5.8)\qquad y(x) = \sum_{j=1}^{n+1} \Lambda_j L(x_1, \ldots, \check{x}_j, \ldots, x_{n+1}; L; F|x),$$

where $L(z_1, z_2, \ldots, z_n; L; F|x)$ *is the generalized Lagrange's interpolatory polynomial with respect to the sistem* $L_1, L_2, \ldots, L_n$ *on the knots* $z_1, z_2, \ldots, z_n$ *to the function* F.

In the case of the linear and homogeneous differential equation with constant coefficients, approximated by algebraic polynomial, it is possible to give an interesting interpretation of the well known Fejér's theorem about the location of the zeros of algebraic infrapolynomials [1].

Thus, let us consider the equation

$$L(y) = A_0 y^{(p)} + A_1 y^{(p-1)} + \dots + A_p y = 0, \tag{5.9}$$

where $A_0, A_1, \dots, A_p$ are constants and $p \leq n$.

Now we consider algebraic juxtasolutions of the equation (5.9) of the form

$$y = a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n. \tag{5.10}$$

Obviously, if $a_1, a_2, \dots, a_n$ are independent then the single juxtapolynomial of (5.9) is $y \equiv 0$, obtained for $a_1 = a_2 = \dots = a_n = 0$.

Thus it is natural to assume that $a_1 = 1$.

Furthermore, because in the case $A_p = A_{p-1} = \dots = A_{p-k} = 0$ $A_{p-k-1} \neq 0$, each polynomial of degree at most k is a solution i.e. a juxtasolution, of the equation (5.9), on every compact set K, we assume that $A_p \neq 0$. Thus the equation (5.9) can be written under de form

$$L(y) = A_0 y^{(p)} + A_1 y^{(p-1)} + \dots + A_{p-1} y' + y = 0. \tag{5.11}$$

We denote by $\mathcal{P}_n$ the set of all $n-th$ degree polynomials with leading coefficient one:

$$P_n(x) = x^n + a_1 x^{n-1} + \dots + a_n. \tag{5.12}$$

If $y = P_n(x)$, then

$$L(y) = x^n + c_1 x_{n-1} + \dots + c_n, \tag{5.13}$$

where

$$c_k = a_k + (n-k+1) A_{p-1} a_{k-1} + (n-k+1)(n-k+2) A_{p-2} a_{k-2} + \dots, \quad k = 1, 2, \dots, n.$$

D e f i n i t i o n 5.4. [8] *A polynomial $P_n \in \mathcal{P}_n$ is called infrapolynomial on K if there is no polynomial $Q_n \in \mathcal{P}_n$ satisfying the conditions*

$$1^\circ \ P_n(x) = 0, \ x \in K \Rightarrow Q_n(x) = 0$$

$$2^\circ \ P_n(x) \neq 0, \ x \in K \Rightarrow |Q_n(x)| < |P_n(x)|.$$

In [11] was shown the following assertion

THEOREM 5.4. *The polynomial $y = x^n + a_1 x^{n-1} + \dots + a_n$ is a juxtasolution of the equation (5.11) on K if and only if $L(y) = x^n + c_1 x^{n-1} + \dots + c_n$ is an infrapolynomial on K.*

From this theorem as a consequence we obtain the following result.

THEOREM 5.5. *Each algebric juxtasolution $y \in \mathcal{P}_n$ of the equation (5.11) on K is exact solution on n points x_j of the closed convex hull of K.*

Proof. If $y = x^n + a_1 x_{n-1} + \dots + a_n$ is a juxtasolution of the equation (5.11) on K, then by Theorem 5.4,

$$L(y) = P_n(x) = x^n + c_1 x^{n+1} + \dots + c_n$$

where c_k are given by (5.13), is an infrapolynomial of degree n on K. But then, by virtue of Fejér's theorem [1], all zeros of P_n must be in the closed convex hull of K, i.e. the points x_j for which

$$L[y(x_j)] = 0, \ \forall j \in \{1, 2, \dots, n\}.$$

belong to the closed convex hull of K.

Using a Marden's result [7] it is possible to give a converse theorem characterizing the algebric polynomials of degree n which are juxtasolutions of the equation (5.11) on K.

THEOREM 5.6. *Let K be a real compact set containing at least $n+1$ points and $[A, B]$ the smallest interval containing K. If the polynomial*

$$y = x^n + a_1 x^{n-1} + \dots + a_n$$

is exact solution of the equation (5.11) on n distinct points $x_n \in [A, B]$, $i = 1, 2, \dots, n$, which are separated by points of K, then y is a juxtasolution of the equation (5.11) on K.

Proof. We say that the points $x_1, x_2, \dots, x_n$ are separated by points of K if

$$[x_k, x_{k+1}] \cap K \neq \emptyset, \ \forall k \in \{1, 2, \dots, n-1\}.$$

Now, if $y = x^n + a_1 x^{n-1} + \dots + a_n$ has the property that there exists a system of distinct points $x_j \in [A, B]$ separated by points of K, such that

$$L[y(x_i)] = 0, \ \forall_i \in \{1, 2, \dots, n\}$$

then from the equality

$$L(y) = P_n(x) = x^n + c_1 x^{n-1} + \dots + c_n$$

it follows tht t all the zeros of P_n belong to $[A, B]$ and are separated by points of K. But in view of a MARDEN's result [7], then P_n is an infrapolynomial on K, an by Theorem 5.4, $y = P_n(x)$ is a juxtasolution of (5.11) on K.

Similar results can be established for Cauchy's juxtasolution, too using some of our results about restricted algebraic infrapolynomials [11].

§ 6. Extremal solution of a linear system.

To find an approximate solution of an incompatible linear system we use freqvently the last squares method. An other method, called the best approximation method, was given by REMEZ, E. I. [15]. Obviously, these two methods correspond to the two considered metrics. It is natural to study all such, „extremal" solution of an incompatible linear system, introducing the notion of the juxtasolution of such a system.

Let us consider a system

$$(6.1) \qquad y_i(z) = a_i z = b_i, \quad i = 1, 2, \ldots, m,$$

where

$$a_i = (a_{i1}, a_{i2}, \ldots, a_{in}), \ b^T = (b_1, b_2, \ldots, b_m) \ z^T = (z_1, z_2, \ldots, z_n)$$

are complex vectors. Some times this system will be denoted:

$$(6.2) \qquad Az = b.$$

Definition 6.1. *Let $z^T = (z_1, z_2, \ldots, z_n)$ be an approximate solution of* (6.1). *$u^T = (u_1, u_2, \ldots, u_n)$ is called undersolution of z (abbreviated $u \in \mathcal{U}(A, b; z)$) if the following conditions are satisfied:*

1° $Au \neq Az$;

2° *If* $j \in \{1, 2, \ldots, m\}$ and $a_j z = b \Rightarrow a_j u = b_j$,

3° *If* $j \in \{1, 2, \ldots, m\}$ and $a_j z \neq b \Rightarrow |a_j u - b_j| < |a_j z - b_j|$.

Definition 6.2. *An approximate solution z of* (6.1) *is called juxtasolution of the system* (6.1) *(abbreviated $z \in \mathcal{J}(A, b)$)*, if $\mathcal{U}(A; b; z) = \emptyset$.

In other words, $z \in C^n$ is juxtasolution of the system (6.1) if z has no undersolution of this system.

Obviously that each exact solution of the system (6.1) (if such a solution there exists) is a juxtasolution, too.

Among the most important juxtasolutions are those which minimize certain norms. We e going to give, for instance, two of the most frequarently met of them.

Definition 6.3. *An approximate solution $z^0 \in C^n$ of* (6.1) *is called the best approximation solution of the system* (6.1), *or a Tchebycheff point of the system* (6.1) *if.*

$$(6.3) \qquad \max_{(i)} |a_i z^0 - b_i| = \inf_{z \in C^n} \max_{(i)} |a_i z - b_i|.$$

Let us consider a system of positive numbers $\mu = (\mu_i)_1^m$ with $\Sigma \mu_i = 1$, and $p \geq 1$ a given real number.

Definition 6.4. *An approximate solution $z^* \in C^n$ of the system* (6.1) *is called to be a solution of best approximation in weighted mean of the order p of* (6.1) *if:*

$$(6.4) \qquad \left(\sum_{i=1}^m \mu_i |a_i z^* - b_i|^p\right)^{1/p} = \inf_{z \in C^n} \left(\sum_{i=1}^m \mu_i |a_i z - b|^p\right)^{1/p}.$$

We can verify immediately that we have the following:

THEOREM 6.1. *If z^0 and z^* are solutions of best approximation of* (6.1) *in uniform respectively in weighted mean of the order p, norms, then z^0, $z^* \in \mathcal{J}(A; b)$.*

By this theorem we have in particular

Corollary 6.1. *The solution z^* of* (6.1) *obtained by the last squares method is a juxtasolution of* (6.1).

Hence the set $\mathcal{J}(A, b)$ contains all the extremal solutions of the system (6.1) which intervene in the applications.

Because, the juxtasolution of a linear system are the special cases of the generalized juxtapolynomial to given function over a finite set, all the properties of the juxtapolynomials may be formulated for the juxtasolutions of a linear system, too.

Thus, let us denote by $A_{mj} = \{a_{ij}\}_{i=1}^m$, $j = 1, 2, \ldots, n$, $B_m = \{b_i\}_1^m$, and let $M_m = \{\zeta_i\}_1^m$, be arbitrary finite pointset in the complex plane. We consider the applications:

$$f_j : M_m \to A_{mj}, \ j = 1, 2, \ldots, n$$
$$F : M_m \to B_m$$

satisfying the conditions

$$(6.5) \qquad f_j(\zeta_i) = a_{ij}, \ F(\zeta_i) = b_i, \quad i = 1, 2, \ldots, m.$$

Then it is clear that the approximation problem of the solutions of the system (6.1) is equivalent to the approximation problem to the function F on the set M_m by the generalized polynomials.

$$(6.6) \qquad p(z; \zeta) = z_1 f_1(\zeta) + \ldots + z_n f_n(\zeta).$$

Remark 6.1. As functions f_k, F, we can take Lagrange's interpolatory polynomial of degree $m - 1$ on the knots $\zeta_1, \zeta_2, \ldots, \zeta_m$.

From the definition of the juxtapolynomial to a function on a compact set K, by virtue of Definition 6.2, it follows immediately

THEOREM 6.2. *Let $f_1, f_2, \ldots, f_n$, F, be the mappings satisfying* (6.5). *The approximate solution z^* of the system* (6.1) *is a juxtasolution of* (6.1) *if and only if $p(z^*, \zeta)$ is a generalized juxtapolynomial to f on M_m.*

As a consequence, from Theorems 2.2 and 6.2 we have immediately

THEOREM 6.3. *An approximate solution z^0 of the system* (6.1), *for which*

$$y_i(z_0) \neq b_i, \ \forall i \in \{1, 2, \ldots, m\}$$

is a juxtasolution of (6.1), *if and only if there exist $p(1 \leq p \leq 2n + 1)$ equations $y_{i_1}, y_{i_2}, \ldots, y_{i_p}$, and p positive numbers δ_j, such that*

$$\sum_{j=1}^{p} \delta_j \frac{a_{ijk}}{y_{ij}(z_0) - b_{ij}} = 0, \ \forall k \in \{1, 2, \ldots, n\}. \tag{6.7}$$

z^0 is a juxtasolution of the system

$$y_{i_1}(z) = b_{i_1}, \ldots, y_{i_p}(z) = b_{i_p},$$

too.

Definition 6.5. *We say that the linear forms, $y_1, y_2, \ldots, y_m$ are strictly linearly dependent of the order $m - 1$* [15, pag. 138] *if there exist the constants $c_i \neq 0$, $\forall i \in \{1, 2, \ldots, m\}$ such that*

$$c_1 y_1(z) + \ldots + c_m y_m(z) = 0$$

and there is no similar relation between the linear forms of a subsystem of $\{y_i\}_1^m$.

It is known that $y_1, y_2, \ldots, y_m$ are strictly linearly dependent of the order $m - 1$ if and only if rank $A = m - 1$ and if there exists a sub-matrix C of the type $(m, n - 1)$ whose all minors of the order $m - 1$ are different from zero

Definition 6.6. *Let r be an integer $(1 \leq r \leq m)$ and $\{i_j\}_{j=1}^r \subset \{i\}_1^m$. A subsystem*

$$A_r z = B_r,$$

where

$$A_r = \begin{pmatrix} a_{i_1 1} a_{i_1 2} \ldots a_{i_1 n} \\ \cdots\cdots\cdots \\ a_{i_r 1} a_{i_r 2} \ldots a_{i_r n} \end{pmatrix}, \quad B_r = \begin{pmatrix} b_{i_r} \\ \vdots \\ b_{i_r} \end{pmatrix} \tag{6.8}$$

is called juxtacharacteristic subsystem of (6.1) *and $z_0 \in \mathcal{J}(A, b)$ if*

a) $z^0 \in \mathcal{J}(A_r; B_r)$

b) *if $A_h z = B_h$ is a subsystem of $A_r z = B_r$ then $z_0 \notin \mathcal{J}(A_r; B_r)$. The matrix A_r is called juxtacharacteristic of z^0.*

Remark 6.2. By Theorem 6.3 it follows that each system (6.1) has at least one juxtacharacteristic subsystem of $z^0 \in \mathcal{J}(A, b)$ and the number p of the equations of a juxtacharacteristic subsystem of (6.1) satisfies the inequalities

$$1 \leq p \leq 2n + 1.$$

If moreover, the system (6.1) is real, then by Corollary 2.1, it follows

$$1 \leq p \leq n + 1$$

Definition 6.7. *A subsystem $A_r z = B_r$ of* (6.1) *is called elementary subsystem of* (6.1) *if the linear forms, $y_{i_1}, y_{i_2}, \ldots, y_{i_r}$ are strictly linearly dependent.*

Definition 6.8. *We say that the system* (6.1) *has* (T) *property* (*Tchebycheff's property*) *if the rank of every submatrix of A of the type (n, n) is equal to n.*

From Definitions 6.7—6.8, it follows

Corollary 6.2. *The system* (6.1) *has* (T) *property if and only if each its subsystem of $n + 1$ equations is an elementary subsystem of* (6.1).

Proof. The necessary and sufficient condition for a subsystem of $n + 1$ equations of (6.1) to be an elementary subsystem of (6.1) is that the rank of the matrix of this subsystem is n, and each its minor of the order n must be different from zero, because there is a single matrix of the type $(n + 1, n)$. Therefore, a system (6.1) has (T) property if and only if every its subsystem of $n + 1$ equations is an elementary subsystem of (6.1).

We can verify immediately that the system (6.1) has (T) property if and only if the system $f = (f_k)_1^n$ satisfying the conditions (6.5) is a Tchebycheff system on M_m. So by Corollary 2.1, we have

Corollary 6.3. *If the system* (6.1) *has* (T)*-property, then the number p in Theorem* 6.3. *satisfies the inequality*

$$m + 1 \leq p \leq 2n + 1$$

If moreover (6.1) *is real then $p = n + 1$.*

Using our results about matrix characterization of the juxtapolynomials to a given function on a compact set K, given in § 3, we can characterize the submatrix of A satisfying the Theorem 6.3.

A matrix A of theorem (6.7) will be called a *J-matrix* of (A, b) if the conditions of the Theorem 6.3 are satisfied.

THEOREM 6.4. *An approximate solution z^0 of the system $Az = b$, for which*

$$y_j(z_0) \neq b_j, \quad \forall j \in \{1, 2, \ldots, m\}$$

is its juxtasolution if and only if there exists a submatrix A_p of A $(1 \leq p \leq 2n + 1)$ which is a H-matrix and there exists a vector λ strictly different from zero orthogonal to A_p such that

$$\arg(\lambda_j Y_j) = \arg(\lambda_1 Y_1), \quad \forall j \in \{1, 2, \ldots, p\}, \tag{6.9}$$

where

$$Y_j = y_j(z^0) - b_j.$$

Proof. Necessity. Let us suppose that $z^0 \in \mathcal{J}(A, b)$ such that $y_i(z_0) \neq b_i$, $i = 1, 2, \ldots, m$. Then by Theorem 6.3 there exist a submatrix A_p of A $(1 \leq p \leq 2n + 1)$ and positive numbers δ_j such that (6.7) holds. But then if we put

$$\lambda_j = \frac{\delta_j}{|Y_j|^2} \overline{Y}_j. \tag{6.10}$$

(6.7) can be written under the from

$$\sum_{j=1}^{p} \lambda_j a_{ijk} = 0, \quad \forall k \in \{1, 2, \ldots, n\}. \tag{6.11}$$

Because $\lambda_j \neq 0$, $\forall j \in \{1, 2, \ldots, p\}$, (6.11) shows that A_p is a H-matrix. From (6.10) we obtain

$$\lambda_j Y_j > 0, \quad \forall j \in \{1, 2, \ldots, p\}$$

hence $\arg(\lambda_j Y_j) = 0$, $\forall j \in \{1, 2, \ldots, p\}$.

S u f f i c i e n c y. Let us suppose that for the matrix $A_p (1 \leq p \leq 2n + 1)$ and $z^0 \in \mathbb{C}^n$ the conditions of theorem hold, i.e. A is a H-matrix and there exists a vector λ strictly different from zero orthogonal to A_p such that (6.9) hold. Putting

$$\delta_j = \frac{|\lambda_j|}{|Y_j|}, \quad \theta = \arg(\lambda_1 Y_1)$$

i.e. $\lambda_j = \delta_j \overline{Y}_j \, ex(i\theta)$, and replacing in (6.11), we obtain

$$\sum_{j=1}^{p} \delta_j \overline{Y}_j \mathrm{ex}(i\theta) a_{ijk} = 0, \quad \forall k \in \{1, 2, \ldots, n\},$$

whence

$$\sum_{j=1}^{p} \delta_j \overline{Y}_j a_{ijk} = 0, \quad \forall k \in \{1, 2, \ldots, n\}$$

and by Theorem 6.3 it follows that $z^0 \in \mathcal{J}(A, b)$.

Similarly, by Theorems 3.2, 6.2, and 6.3 we can deduce the following

THEOREM 6.5. *Let A_p $(1 \leq p \leq 2n + 1)$ be a submatrix of A and $z^0 \in \mathcal{J}(A, b)$ such that*

$$Y_i = y_i(x^0) - b_i \neq 0, \quad \forall i \in \{1, 2, \ldots, m\}.$$

A_p is a characteristic juxtamatrix of $z^0 \in \mathcal{J}(A, b)$ if and only if the matrix

$$M_{z^0}(A_p, B_p) = \begin{pmatrix} A_p^0 & B_p \\ \overline{A}_p^0 & \overline{B}_p \end{pmatrix},$$

where

$$A_p^0 = \begin{pmatrix} \overline{Y}_{i_1} a_{i_1 1} & \ldots & \overline{Y}_{i_1} a_{i_1 n} \\ \cdot & \cdot \cdot \cdot & \cdot \\ \overline{Y}_{i_p} a_{i_p 1} & \ldots & \overline{Y}_{i_p} a_{i_p n} \end{pmatrix}$$

is a HP-matrix of the rank $p - 1$.

Finally, using the structure of the generalized juxtapolynomials in § 4, we are going to give an explicite form of the juxtasolution of a given linear system. For simplicity we suppose that (6.1) is a real system possessing (T)-property.

THEOREM 6.6. *Let*

$$Ax = b \tag{6.12}$$

be a real incompatible system possessing (T)-property. An approximate solution x^0 is a juxtasolution, i.e. $x^0 \in \mathcal{J}(A, b)$, if and only if there exist a subsystem of (6.12) (we can suppose: $Y_i(x) = 0$, $i = 1, 2, \ldots, n + 1$) and $n + 1$ positive numbers d_j such that

$$x_k^0 = \frac{\sum_{j=1}^{n+1} d_1 \ldots \hat{d_j} \ldots d_{n+1} D(1, \ldots, \hat{j}, \ldots, n + 1; A) D_k(1, \ldots, \hat{j}, \ldots, n + 1; A, b)}{\sum_{j=1}^{n+1} d_1, \ldots, \hat{d_j}, \ldots, d_{n+1} \; [D(1, \ldots, \hat{j}, \ldots, n + 1, A)]^2} \tag{6.13}$$

where

$$D(1, \ldots, \hat{j}, \ldots, n+1; A) = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\hat{j}1} & \cdots & a_{\hat{j}j} & \cdots & a_{\hat{j}n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+1,1} & \cdots & a_{n+1,j} & \cdots & a_{n+1,n} \end{vmatrix}$$

$$D_k(1, \ldots, \hat{j}, \ldots, n+1; A) = \begin{vmatrix} a_{11} & \cdots & a_{1,k-1} & b_1 & a_{1,k+1} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\hat{j}1} & \cdots & a_{\hat{j},k-1} & b_{\hat{j}} & a_{\hat{j},k+1} & \cdots & a_{\hat{j}n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+1,1} & \cdots & a_{n+1,k-1} & b_{n+1} & a_{n+1,k+1} & \cdots & a_{n+1,n} \end{vmatrix}$$

Proof. If the system (6.13) is real and has (T)-property then by Theorem 6.3 in view of Corollary 6.3, it follows that $x^0 \in \mathcal{J}(A; b)$ if and only if there exists a subsystem of (6.12) of $n+1$ equations (assume that these are the first $n+1$ equations) and $n+1$ positive numbers d_j such that

$$\sum_{j=1}^{n+1} d_j Y_j(x^0) a_{jk} = 0, \quad \forall k \in \{1, 2, \ldots, n\}. \tag{6.14}$$

Putting

$$\sum_{j=1}^{n+1} d_j a_{ji} a_{jk} = D_{ki}$$

$$\sum_{j=1}^{n+1} d_j b_j a_{jk} = D_k$$

the system (6.14) becomes

$$D_{i1} x_1^0 + D_{i2} x_2^0 + \ldots + D_{in} x_n^0 = D_i, \quad i = 1, 2, \ldots, n \tag{6.15}$$

which permits to determine $x_1^0, x_2^0, \ldots, x_n^0$. Using the same method as in [9], the solution of the system (6.15) can be put under the form (6.13).

If we denote by

$$\Lambda_j = \frac{d_1 \ldots d_{\hat{j}} \ldots d_{n+1} [D(1, \ldots, \hat{j}, \ldots, n+1; A)]^2}{\sum_{j=1}^{n+1} d_1 \ldots d_{\hat{j}} \ldots d_{n+1} [D(1, \ldots, \hat{j}, \ldots, n+1; A)]^2},$$

then (6.12) can be written under the form

$$x_k^0 = \sum_{j=1}^{n+1} \Lambda_j \frac{D_k(1, \ldots, \hat{j}, \ldots, n+1, A; b)}{D(1, \ldots, \hat{j}, \ldots, n+1; A)} \tag{6.16}$$

So we have the following interesting result:

C o r o l l a r y 6.4. *Let* $Ax = b$ ***be an incompatible linear real system possesing*** (T)-*property. Then each juxtasolution of this system is a convex combination of the Cramerian solutions of the elementary subsystem of* (6.12).

We saw that every solution of best approximation of (6.1) in uniform norm or last squares norm are juxtasolution of (6.1), too. By virtue of the form (6.13) of the juxtasolutions of a real system (6.12) we conclude that for certain choise of the constants d_j in (6.11) we can obtain both of mentioned solution of best approximation of (6.12).

a) First we begin with last squares approximate solution. Thus for obtain the solution of last squares method of the system (6.12) it is necessary to minimize the function

$$F(x) = \sum_{i=1}^{m} Y_i^2(x)$$

.e. it is necessary to solve the system

$$\frac{\partial F}{\partial x_k} = 2 \sum_{j=1}^{n} Y_i(x) a_{ik} = 0, \quad k = 1, 2, \ldots, n. \tag{6.17}$$

But (6.17) coincides with (6.14) for $m = n + 1$ and $d_1 = d_2 = \ldots = d_{n+1} = 1$.

Therefore we have

C o r o l l a r y 6.5. *A juxtasolution of a real system* (6.11) *possessing* (T)-*property is a last squares solution of this system if and only if in* (6.13) $d_1 = d_2 = \ldots = d_{n+1} = 1$, *i.e.*

$$x_k^0 = \frac{\sum_{j=1}^{n+1} D(1, \ldots, \hat{j}, \ldots, n+1; A) D_k(1, \ldots, \hat{j}, \ldots, n+1; A; b)}{\sum_{j=1}^{n+1} D^2(1, \ldots, \hat{j}, \ldots, n+1, A)}. \tag{6.18}$$

b) To caracterize the juxtasolutions of a system (6.12) which is at the same time the best approximate solution of (6.12) in uniform norm, we use Theorem 2.3 in view of which, a juxtasolution of a system is at the same time a best approximate solutin of this system if and only if for each juxtacharacteristic subsystem of (6.12):

$$Y_{i_j}(x) = 0, \quad j = 1, 2, \ldots, n+1$$

we have

$$|Y_{i_1}(x_0)| = |Y_{i_2}(x_0)| = \ldots = |Y_{i_{n+1}}(x_0)|. \tag{6.19}$$

Using the conditions (6.19) it is possible to express the constants Λ_j in (6.16) and so, the solution of best approximation of the system (6.12) can be expressed explicitly, if we know a juxtacharacteristic subsystem of (6.12).

Remark 6.3. All our results can be easily extended to the case where the system (6.1) is an infinite incompatible system i.e. when the matrix $A = (a_{ik})$ is of the type (∞, n).

REFERENCES

[1] Fejér, L., *Über die Lage der Nullstellen gewisser Art entspringen.* Math. Ann., **85**, 42–48 (1922).

[2] Fekete, M., *On the structure of extremal polynomials.* Proc. Nat. Acad. Sci. U.S.A., **37**, *2*, 95–103 (1951).

[3] Gordon, Yehoram, *Properties of generalized juxtapolynomials.* Israel Journal of Math., **3**, 177–188 (1966).

[4] Grebenjuk, D. G., *Polynomi nailučsego približenia, koeffitzienty kotoryh udovletvoriaiut lineinym zavisimostiam.* Taskent, 1960.

[5] Kolumban, I., *Über die Charakterisierung der Lösungen eines optimalen Approximationsproblems.* Mathematica (Cluj), **10** (**33**), 285–291 (1968).

[6] Lebedev, N. A., Ryžakov, I. Ju., *O tchebysevskom približenii funkcii nepreryznyh na kompaktnyh množestvah kompleksnoi ploskosti.* Vestnik Leningr. Univ., **13**, 39–50 (1969).

[7] Marden, M., *Location of the zeros of infrapolynomials.* Amer. Math. Monthly, **70**, *4*, 361–371 (1963).

[8] Maruşciac, I., *Sur certains infrapolynomes conditionnés.* Mathematica (Cluj), **4**, (**27**) *1*, 33–52 (1962), **7**, (**30**), *2*, 283–285 (1965).

[9] — , *Une forme explicite du polynome de meilleure approximation d'une fonction dans le domaine complexe.* Mathematica (Cluj), **6** (**29**), *2*, 257–263 (1964).

[10] — , *Obobscionnye infrapolinomy.* Mathematica (Cluj), **7** (**30**), *2*, 263–282 (1965).

[11] — , *Infrapolinomy i ekstremalnye resenia differencialnyh uravnenii.* Mathematica (Cluj), **9** (**32**), *1*, 101–110 (1967).

[12] — , *On the structure of restricted generalized infrapolynomials.* Mathematica (Cluj), **12** (**35**), *1*, 111–125 (1970).

[13] — , *Caracterisation matricielle des juxtapolynomes généralisés.* Birkhäuser-Verlag Numerische Meth. der Approximationstheorie, Band 1, v. 16, 1971, 77–92.

[14] Motzkin, T. S., and Walsh, J. L., *Underpolynomials and infrapolynomials.* Illinois J. Math. **1**, 406–426 (1957).

[15] Remez, E. I., *Basis of the numerical methods in Tchebycheff's approximation,* Kiev, 1969.

[16] Smirnov, V. I., Lebedev, N. A., *Konstructivnaja teorija funkcii kompleksnogo peremennogo.* Moskva, 1964.

Received 8. V. 1971.