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ON EXISTENCE OF DIVIDED DIFFERENCES IN LINEAR SPACES

BY

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In a large number of problems, the notion of divided difference of the mappings in different linear spaces is very important. In the investigation of the divided differences many results are known ([6], [3], [1]), but the question of their existence is still an open problem. In our paper we state certain theorems which solve that question.

Let X and Y be real vector spaces and P a mapping defined on X with values in Y.

Definition 1. *A linear mapping $\tilde{P}_{u,v}$ defined on X with values in Y is called algebraic divided difference of P in the pair of different points u, v of X, iff*

$$\tilde{P}_{u,v}(u - v) = P(u) - P(v). \tag{1}$$

Proposition 1. *For every mapping P defined on X with values in Y, there exists an algebraic divided difference in every pair of different points u, v of X.*

Proof. For arbitrarily selected different u and v of X, let

$$X_1 = \{x \in X \mid x = \alpha(u - v),\ \alpha \in \mathbf{R}\}.$$

We define on the linear subspace X_1 of X the linear mapping

$$\overline{P}_{u,v}[\alpha(u - v)] = \alpha P(u) - \alpha P(v).$$

We know that there is a linear projector Π of X on X_1 (see, e.g. [5] p. 71), i.e.

$$\Pi(X) = X_1, \text{ and } \Pi(x) = x \text{ for every } x \text{ in } X_1.$$

If we put

$$\tilde{P}_{u,v} = \bar{P}_{u,v} \circ \Pi,$$

we have a linear mapping defined on X with values in Y, and by definitions of $\tilde{P}_{u,v}$, Π and $\bar{P}_{u,v}$

$$\tilde{P}_{u,v}(u - v) = P(u) - P(v).$$

Example. Let $X = s$, the set of the sequences of real numbers with usual linear structure, Y a linear space and P a mapping defined on s with values in Y. If

$$u = \{u^1, u^2, \ldots, u^n, \ldots\} \text{ and } v = \{v^1, v^2, \ldots, v^n, \ldots\},\ u \neq v,$$

then

$$X_1 = \{x_\alpha \in s \mid x_\alpha = (\alpha(u^1 - v^1), \alpha(u^2 - v^2), \ldots, \alpha(u^n - v^n), \ldots, \alpha \in R)\}$$

and

$$P_{u,v}[\alpha(u - v)] = \alpha P(u) - \alpha P(v).$$

Because $u \neq v$, we can consider that for i_0, $u^{i_0} - v^{i_0} \neq 0$. Then we have as a basis of X, the set of linearly independent vectors $e_1, e_2, \ldots e_{i_0-1}$, $x_1, e_{i_0+1}, \ldots$, where

$$x_1 = \{u^1 - v^1, u^2 - v^2, \ldots, u^n - v^n, \ldots\}$$

and e_k is the sequence formed with O, excepting k-th element. Now we can define the linear projector Π of X on X_1 in a following manner:

$$\Pi(x_1) = x_1, \Pi(e_k) = 0, \ (k = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots).$$

Then for

$$x = \{x^1, x^2, \ldots, x^n, \ldots\}$$

we have

$$\Pi(x) = \frac{x^{i_0}}{u^{i_0} - v^{i_0}} x_1 = \frac{x^{i_0}}{u^{i_0} - v^{i_0}} (u - v),$$

and

$$\tilde{P}_{u,v}x = \bar{P}_{u,v} \circ \Pi x = \frac{x^{i_0}}{u^{i_0} - v^{i_0}} [P(u) - P(v)],$$

which is an algebraic divided difference of P in u, v.

Definition 2. *An algebraic divided difference $\tilde{P}_{u,v}$ of a mapping P defined on a topological vector space X, with values in a topological vector space Y, is called divided difference $P_{u,v}$ of P in the different points u and v of X, iff $\tilde{P}_{u,v}$ is a continuous mapping of X in Y.*

THEOREM 2. *Every mapping P of a separated locally convex space X in a topological vector space Y, has a divided difference in all different points u and v of X.*

Proof. Let X, and $\bar{P}_{u,v}$ be as in the preceeding theorem. X_1 having the dimension 1, there exists the linear and continuous projector of X on X_1 [5, p. 96—97], and hence the mapping

$$P_{u,v} = \bar{P}_{u,v} \circ \Pi$$

is a linear and continuous mapping verifying the condition (1).

Example. Let $X = l_2$ the set of the sequences of the real numbers with $\sum_{n=1}^{\infty} (x^n)^2 < +\infty$, having the usual linear structure and norm, P mapping l_2 in the linear normed space Y.

Defining X_1 and $\bar{P}_{u,v}$ in the same way as in the first example, we can consider as vector generating the subspace X_1, the element

$$e = \frac{1}{\sqrt{\sum_{n=1}^{\infty} (u^n - v^n)^2}} (u - v).$$

Then for

$$x = \{x^1, x^2, \ldots, x^n, \ldots\}, \text{ we have}$$

$$\Pi x = (x, e)e = \frac{\sum_{n=1}^{\infty} x^n(u^n - v^n)}{\sum_{n=1}^{\infty} (u^n - v^n)^2} (u - v),$$

and hence

$$P_{u,v}x = \frac{\sum_{n=1}^{\infty} x^n(u^n - v^n)}{\sum_{n=1}^{\infty} (u^n - v^n)^2} \cdot [P(u) - P(v)].$$

THEOREM 3. *Every mapping P of a linear normed space X in a linear normed space Y has a divided difference $P_{u,v}$, with the norm*

$$||P_{u,v}|| = \frac{||P(u) - P(v)||}{||u - v||}.$$

Proof. Let X_1 and $P_{u,v}$ be as in the precedings, and let be the real valued function f_1 defined on the X_1 by $f_1(\alpha(u - v)) = \alpha$. Considering the norm-preserving extension f of f_1 to X [see e.g. [7] p. 106] and

$$\text{Ker } (f) = \{x \in X \mid f(x) = 0\},$$

every point y of X can be expressed by $y = x + f(y)(u - v)$, where $x \in \text{Ker } (f)$. We define

$$P_{u,v}y = f(y)\ \bar{P}_{u,v}(u - v),$$

for every y in X, and so we obtain that $P_{u,v}$ is a divided difference of P, and

$$|| P_{u,v} || = \frac{||P(u) - P(v)||}{||u - v||}.$$

Remarks. 1. In this way we do not obtain the uniqueness of the divided differences, which is natural, because there are some known examples when the divided difference isn't unique.

2. This paper gives only some principial answer to the question of the existence of the divided differences, but the effective construction of divided differences in every practical case must be solved in other ways.

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R E F E R E N C E S

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