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ON THE INTERVAL EQUATION $AX + B = CX + D$

by

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Let **I** be the set of real closed intervals $A = [a_1, a_2]$. In **I** are defined the binary operations:

$$A \circ B = \{u \circ v : u \in A \;\&\; v \in B\} \qquad \circ \in \{+, -, \cdot, /\}$$

where $O \notin B$ for division. In the factorization problem [2] is treated the interval equation

$$AX = B.$$

The solution of the interval equation

$$AX + B = C$$

is given in [1].

The object of this paper is the study of the interval equation

$$AX + B = CX + D \tag{1}$$

in some particular cases.

Notations. $a, b, c, \ldots$ are positive numbers, $s, t, \ldots$ are positive numbers < 1 and $a_1, a_2, \ldots$ are real numbers. The interval $[0, 1]$ is denoted by U. If r is a real number then we write $r = [r, r]$, hence $\mathbf{R} \subset \mathbf{I}$. W also define the binary interval relations:

$$r_< = \{(A, B) : a_2 < b_1\} \quad r_> = \{(A, B) : a_1 > b_2\}$$

$$r_\subset = \{(A, B) : b_1 < a_1 < a_2 < b_2\}, \quad r_\supset = \{(A, B) : a_1 < b_1 < b_2 < a_2\}$$

$$r_\dashv = \{(A, B) : a_1 < b_1 < a_2 < b_2\}, \quad r_\vdash = \{(A, B) : b_1 < a_1 < b_2 < a_2\}$$

where $A = [a_1, a_2]$ and $B = [b_1, b_2]$. We denote $A < B$ iff $(A, B) \in r_<$ $A \dashv B$ iff $(A, B) \in r_\dashv$ etc.

With the above notations for an interval, the representations

$$[a_1, a_1 + a] \text{ or } a_1 + aU$$

are valid.

The classification of the equations of type (1). We have 6 cases depending on the form of intervals A and C:

$$(a)\ 0 < A,\ 0 < C \quad (b)\ 0 < A,\ 0 \sqsubset C \quad (c)\ 0 < A,\ C < 0$$

$$(d)\ A < 0,\ C < 0 \quad (e)\ 0 \sqsubset A,\ C < 0 \quad (f)\ 0 \sqsubset A,\ 0 \sqsubset C.$$

For the case (a) the following subcases are considered:

$$(a.1)\quad 0 < A < C, \qquad (a.2)\quad 0 < A \dashv C, \qquad (a.3)\quad 0 < C \supset A.$$

Depending on the form of the unknown interval the following subcases of (a) follow:

$(a, +)$ if X is a positive interval,

$(a, -)$ if X is a negative interval,

$(a, -+)$ if X is a mixed interval.

For the subcase $(a.1)$, $(a.1, +)$ is the case: X positive interval, $(a.1, -) - X$ is negative, $(a.1, -+) - X$ is mixed. Similarly are defined subcases $(a.2, +)$, $(a.2, -)$, $(a.2, -+)$, $(a.3, +)$, $(a.3, -) + (a.3, -+)$.

From case (a) we give in our paper only the study of subcase $(a.1)$. Cases $(b)-(e)$ are omitted in the present paper.

For case $(f, -+)$ (the other cases (f) are also omitted) first we give the comparison of the quantities:

$$(2)\qquad \frac{a_1}{a_2},\ \frac{c_1}{c_2},\ \frac{a_2}{a_1},\ \frac{c_2}{c_1}.$$

Since

$$\frac{a_1}{a_2} < \frac{a_2}{a_1} \Leftrightarrow a_2 > 0 \ \&\ a_1 < -a_2 \Leftrightarrow \frac{a_1c_1}{a_2} > \frac{a_2c_1}{a_1}$$

and

$$\frac{a_1}{a_2} > \frac{a_2}{a_1} \Leftrightarrow a_1 < 0 \ \&\ a_2 > -a_1 \Leftrightarrow \frac{a_1c_1}{a_2} < \frac{a_2c_1}{a_1},$$

one of the following cases occurs:

$(f, -+, 1)\quad a_2 > 0 \ \&\ a_1 < -a_2 \ \&\ c_1 < 0 \ \&\ c_2 < \frac{a_2c_1}{a_1}$

$(f, -+, 2)\quad a_2 > 0 \ \&\ a_1 < -a_2 \ \&\ c_1 < 0 \ \&\ \frac{a_2c_1}{a_1} < c_2 < -c_1$

$(f, -+, 3)\quad a_2 > 0 \ \&\ a_1 < -a_2 \ \&\ c_1 < 0 \ \&\ -c_1 < c_2 < \frac{a_1c_1}{a_2}$

$(f, -+, 4)\quad a_2 > 0 \ \&\ a_1 < -a_2 \ \&\ c_1 < 0 \ \&\ c_2 > \frac{a_1c_1}{a_2}$

$(f, -+, 5)\quad a_1 < 0 \ \&\ a_2 > -a_1 \ \&\ c_1 < 0 \ \&\ c_2 < \frac{a_1c_1}{a_2}$

$(f, -+, 6)\quad a_1 < 0 \ \&\ a_2 > -a_1 \ \&\ c_1 < 0 \ \&\ \frac{a_1c_1}{a_2} < c_2 < -c_1$

$(f, -+, 7)\quad a_1 < 0 \ \&\ a_2 > -a_1 \ \&\ c_1 < 0 \ \&\ -c_1 < c_2 < \frac{a_2c_1}{a_1}$

$(f, -+, 8)\quad a_1 < 0 \ \&\ a_2 > -a_1 \ \&\ c_1 < 0 \ \&\ c_2 > \frac{a_2c_1}{a_1}.$

For quantities (2) in the above cases follows that:

$(f, -+, 1)_1\quad \frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{a_2}{a_1} < \frac{c_2}{c_1}$, $(f, -+, 2)_1\quad \frac{a_1}{a_2} < \frac{c_1}{c_2} < \frac{c_2}{c_1} < \frac{a_2}{a_1}$,

$(f, -+, 3)_1 \quad \frac{a_1}{a_2} < \frac{c_2}{c_1} < \frac{c_2}{c_2} < \frac{a_2}{a_1}, \quad (f, -+, 4)_1 \quad \frac{c_2}{c_1} < \frac{a_1}{a_2} < \frac{a_2}{a_1} < \frac{c_1}{c_2},$

$(f, -+, 5)_1 \quad \frac{c_1}{c_2} < \frac{a_2}{a_1} < \frac{a_1}{a_2} < \frac{c_2}{c_1}, \quad (f, -+, 6)_1 \quad \frac{a_2}{a_1} < \frac{c_1}{c_2} < \frac{c_2}{c_1} < \frac{a_1}{a_2},$

$(f, -+, 7)_1 \quad \frac{a_2}{a_1} < \frac{c_2}{c_1} < \frac{c_1}{c_2} < \frac{a_1}{a_2}, \quad (f, -+, 8)_1 \quad \frac{c_2}{c_1} < \frac{a_2}{a_1} < \frac{a_1}{a_2} < \frac{c_1}{c_2}.$

In this way the following lemma is proved:

L e m m a 1. *If $A = [a_1, a_2]$ and $C = [c_1, c_2]$ are mixed intervals and*

$$\alpha = \left[\min\left(\frac{a_1}{a_2}, \frac{a_2}{a_1}\right), \max\left(\frac{a_1}{a_2}, \frac{a_2}{a_1}\right)\right]$$

$$\gamma = \left[\min\left(\frac{c_1}{c_2}, \frac{c_2}{c_1}\right), \max\left(\frac{c_1}{c_2}, \frac{c_2}{c_1}\right)\right]$$

then

$$\{(\alpha, \gamma) : (\alpha, \gamma) \in \mathbf{I}^2\} = r_{\subset} \cup r_{\supset}.$$

In the above cases the following parametrical forms (for the intervals A and C) hold:

$$(f, -+, 1)_2 \quad A = [-a-b, a], \; C = \left[-c, \frac{acs}{a+b}\right]$$

$$(f, -+, 2)_2 \quad A = [-a-b, a], \; C = \left[-c, \frac{(a+bs)c}{a+b}\right]$$

$$(f, -+, 3)_2 \quad A = [-a-b, a], \; C = \left[-c, c + \frac{bcs}{a}\right]$$

$$(f, -+, 4)_2 \quad A = [-a-b, a], \; C = \left[-c, c + d + \frac{bc}{a}\right]$$

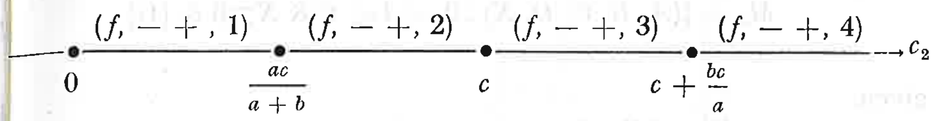
$$(f, -+, 5)_2 \quad A = [-a, a+b], \; C = \left[-c, \frac{acs}{a+b}\right]$$

$$(f, -+, 6)_2 \quad A = [-a, a+b], \; C = \left[-c, \frac{(a+bs)c}{a+b}\right]$$

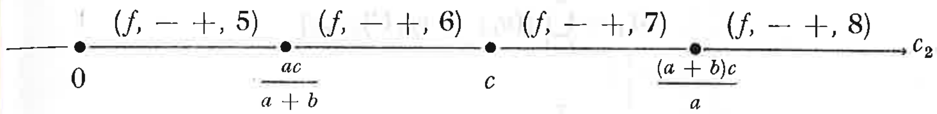
$$(f, -+, 7)_2 \quad A = [-a, a+b], \; C = \left[-c, c + \frac{bcs}{a}\right]$$

$$(f, -+, 8)_2 \quad A = [-a, a+b], \; C = \left[-c, c + d + \frac{bc}{a}\right].$$

In cases $(f, -+, 1) - (f, -+, 4)$, for fixed $a_1 = -a-b$, $a_2 = a$, $c_1 = -c$ the following representation holds:



and for cases $(f, -+, 5) - (f, -+, 8)$, for fixed $a_1 = -a$, $a_2 = a + b$, $c_1 = -c$ the representation



From the symmetrical form of equation (1) follow the equivalences $(f, -+, 1) \Leftrightarrow (f, -+, 2)$, $(f, -+, 3) \Leftrightarrow (f, -+, 5)$, $(f, -+, 4) \Leftrightarrow (f, -+, 6)$ and $(f, -+, 7) \Leftrightarrow (f, -+, 8)$, therefore the following 4 subcases of $(f, -+)$ are to be considered: $(f, -+, 1)$, $(f, -+, 3)$, $(f, -+, 4)$ and $(f, -+, 7)$. In each of thesecases we have 5 subcases tobe considered, depending on the interval in which $\frac{x_2}{x_1}$ is contained. For example in the case $(f, -+, 4)$ we have one on the situations

$$\frac{x_2}{x_1} < \frac{c_2}{c_1}, \; \frac{c_2}{c_1} < \frac{x_2}{x_1} < \frac{a_1}{a_2}, \; \frac{a_1}{a_2} < \frac{x_2}{x_1} < \frac{a_2}{a_1}, \; \frac{a_2}{a_1} < \frac{x_2}{x_1} < \frac{c_1}{c_2} \text{ or } \frac{x_2}{x_1} > \frac{c_1}{c_2}$$

and we have the cases

$(f, -+, 4{,}1)$, $(f, -+, 4{,}2)$, $(f, -+, 4{,}3)$, $(f, -+, 4{,}4)$ respectively $(f, -+ 4{,}5)$.

From case (f) we give in this paper only the study of subcases $(f, -+, 4{,}3)$ and $(f, -+, 4{,}5)$.

§ 1. The study of the equation (1) in case (a. 1)

In case $(a.1)$ the study of the sets

$$M_1 = \{(A, B, C, D, X) : 0 < A < C \;\&\; X > 0 \;\&\; (1)\}$$
$$M_2 = \{(A, B, C, D, X) : 0 < A < C \;\&\; X < 0 \;\&\; (1)\}$$

and

$$M_3 = \{(A, B, C, D, X) : 0 < A < C \;\&\; X \supset 0 \;\&\; (1)\}$$

is given.

THEOREM 1. *The following representations of sets* M_1, M_2 *and* M_3 *are possible :*

$$M_1 = \bigcup_{j=1}^{5} \{(u_{j,k} + v_{j,k} U)_{k \in \overline{1,5}}\}$$

$$M_2 = \bigcup_{j=6}^{7} \{(u_{j,k} + v_{j,k} U)_{k \in \overline{1,5}}\}$$

$$M_3 = \{(u_{8,k} + v_{8,k} U)_{k \in \overline{1,5}}\}$$

where quantities $u_{j,k}$ *and* $v_{j,k}$ *are given by the following tables :*

$$(u_{j,k}) = \begin{pmatrix} a & a_1 + e & a + b + c & a_1 & \dfrac{e}{b+c} \\ a & a_1 + e + f & a + b + c & a_1 & \dfrac{e+f}{b+c} \\ a & a_1 + e + \dfrac{(c+d)es}{b} & a + b + c + d & a_1 & \dfrac{e(b+c)ds}{b(b+c+d)} \\ a & a_1 + e + f & a + c + ds & a_1 & \dfrac{e+f}{c+ds} \\ a & a_1 + e + f + \dfrac{(c+d)e}{b} & a + b + c + d & a_1 & \dfrac{e}{f} \\ a & a_1 & a + b + c & a_1 + e & -\dfrac{e}{c+d} \\ a & a_1 & a + b + c & a_1 + e + f & -\dfrac{(e+f)f}{(b+c)es + (bs+c)f} \\ a & a_1 & a + b + c & a_1 + e & -\dfrac{e}{c+d} \end{pmatrix}$$

and

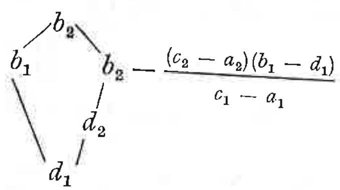
$$(v_{j,k}) = \begin{pmatrix} b & f + g + \dfrac{(c+d)e}{b+c} & d & e + f & \dfrac{g}{c+d} \\ b & g + \dfrac{(c+d)(e+f)}{b+c} & d & e & \dfrac{f+g}{c+d} \\ b + d & \dfrac{(c+d)e\,(b(1-st) + (c+d)s(1-t))}{b(b+c+d)} & d & e & \dfrac{es(1-t)}{b} \\ ds & \dfrac{(c+d)e}{c+ds} + \dfrac{c(1-t) + d(1-st)}{c+ds} \cdot f & d & e & \dfrac{(1-t)f}{c+d} \\ b + d & \dfrac{(c+d)\,(b(e+f) + e(c+d))\,s}{b(b+c+d)} \cdot f & d & e & \dfrac{es}{b} + \dfrac{fs}{b+c+d} + \dfrac{bf}{(c+d)(b+c+d)} \\ b & e + f & d & f + \dfrac{(b+c)es}{c+d} & \dfrac{e(1-s)}{c+d} \\ b & e & bs + \dfrac{(b+c)es}{f} & \dfrac{((b+c)(e+bf)f(1-s)t}{(bs+c)f + (b+c)es} & \dfrac{f(bf+be+ce)(1-s)(1-t)}{(b+c)((bs+c)f+(b+c)es)} \\ b & e + f + g & d & f & \dfrac{e+g}{c+d} \end{pmatrix}$$

Proof. For set M_1 we have the representation:

$$M_1 = \{(a_1 + (a_2 - a_1)U, b_1 + (b_2 - b_1)U, c_1 + (c_2 - c_1)U, d_1 + (d_2 - d_1)U, x_1 + (x_2 - x_1)U):$$

$$0 < a_1 < a_2 < c_1 < c_2 \ \& \ d_1 < b_1 < b_2 \ \& \ d_1 < d_2 < b_2 - \frac{(c_2 - a_2)(b_1 - d_1)}{c_1 - a_1} \ \& \ \& \ x_1 = \frac{d_1 - b_1}{a_1 - c_1} \ \& \ x_2 = \frac{d_2 - b_2}{a_2 - c_2}\Big\}.$$

For parameters b_1, b_2, d_1, d_2 we have the partially ordered graph



hence, one of the following subcases are possible:

(a.1.1) $$d_1 < b_1 < d_2 < b_2 - \frac{(c_2 - a_2)(b_1 - d_1)}{c_1 - a_1} < b_2$$

(a.1.2) $$d_1 < d_2 < b_1 < b_2 - \frac{(c_2 - a_2)(b_1 - d_1)}{c_1 - a_1} < b_2$$

and

(a.1.3) $$d_1 < d_2 < b_2 - \frac{(c_2 - a_2)(b_1 - d_1)}{c_1 - a_1} < b_1 < b_2.$$

Case (a.1.1) is equivalent with the system of relations:

$$(a.1.1)_1 \begin{cases} 0 < a_1 < a_2 < c_1 < c_2 \ \& \ d_1 < b_1 < d_2 \ \& \ b_2 > d_2 + \dfrac{(c_2 - a_2)(b_1 - d_1)}{c_1 - a_1} \ \& \\ \& \ x_1 = \dfrac{d_1 - b_1}{a_1 - c_1} \ \& \ x_2 = \dfrac{d_2 - b_2}{a_2 - c_2}, \end{cases}$$

hence for the interval-coefficients A, B, C, D the parametrical representation is valid:

$$A = u_{1,1} + v_{1,1}U, \quad B = u_{1,2} + v_{1,2}U, \quad C = u_{1,3} + v_{1,3}U, \quad D = u_{1,4} + v_{1,4}U$$

where

$$(u_{1,1}, u_{1,2}, u_{1,3}, u_{1,4}) = (a, a_1 + e, a + b + c, a_1)$$

and

$$(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}) = \left(b, f + g + \frac{(c + d)e}{b + c}, d, e + f\right).$$

Similarly, the case (a.1.2) is equivalent with the system of relations:

$$(a.1.2)_1 \quad 0 < a_1 < a_2 < c_1 < c_2 \ \& \ d_1 < d_2 < b_1 \ \& \ b_2 < b_1 + \frac{(c_2 - a_2)(b_1 - d_1)}{c_1 - a_1} \ \& \ \& \ x_1 = \frac{d_1 - b_1}{a_1 - c_1} \ \& \ x_2 = \frac{d_2 - b_2}{a_2 - c_2}$$

hence

$$A = u_{2,1} + v_{2,1}U, \quad B = u_{2,2} + v_{2,2}U, \quad C = u_{2,3} + v_{2,3}U, \quad D = u_{2,4} + v_{2,4}U$$

where

$$(u_{2,1}, u_{2,2}, u_{2,3}, u_{2,4}) = (a, a_1 + e + f, a + b + c, a_1).$$

For the study of case (a.1.3) first we give the representation of the interval-coefficients:

$$(a.1.3)_1 \quad A = a + bU, \ B = a_1 + e + f + gU, \ C = a + b + c + dU, \ D = a_1 + eU.$$

The condition

$$d_2 < b_2 - \frac{(c_2 - a_2)(b_1 - d_1)}{c_1 - a_1} < b_1$$

i.e.

$$(a.1.3)_2 \quad \frac{(c + d)(e + f)}{b + c} - f < g < \frac{(c + d)(b_1 + f)}{b + c}$$

is necessary for case (a.1.3) and the equivalence:

$$(a.1.3) \Leftrightarrow (a.1.3)_1 \ \& \ (a.1.3)_2$$

follows

From equivalence

$$(a.1.3) \Leftrightarrow (a.1.3.1) \vee (a.1.3.2) \vee (a.1.3.3)$$

where

$$(a.1.3.1) \Leftrightarrow (a.1.3)_1 \;\&\; b > d \;\&\; f < \frac{(c+d)e}{b+d} \;\&\; \frac{(c+d)(e+f)}{b+c} - f < g < \frac{(c+d)(e+f)}{b+c}$$

$$(a.1.3.2) \Leftrightarrow (a.1.3)_1 \;\&\; b < d \;\&\; \frac{(c+d)(e+f)}{b+c} - f < g < \frac{(c+d)(e+f)}{b+c}$$

and

$$(a.1.3.3) \Leftrightarrow (a.1.3)_1 \;\&\; b > d \;\&\; f > \frac{(c+d)e}{b+d} \;\&\; 0 < g < \frac{(c+d)(e+f)}{b+c},$$

holds that in $(a.1.3)_1$ we can make the substitutions

$$(a.1.3.1)_2 \; b \to b+d,\; f \to \frac{(c+d)es}{g},\; g \to \frac{(c+d)e(b(1-st)+(c+d)s(1-t))}{b(b+c+d)},$$

$$(a.1.3.2)_2 \; b \to ds,\; g \to \frac{(c+d)(e+f)}{ds+c} - ft$$

respectively

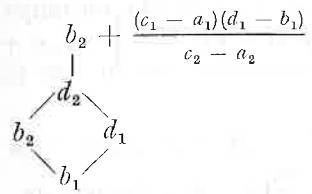
$$(a.1.3.3)_2 \; b \to b+d,\; f \to f + \frac{(c+d)e}{b},\; g \to \frac{(c+d)((e+f)f+(c+d)e)s}{b(b+c+d)}.$$

With the above substitutions, the 3^e, 4^e and 5^e line of $(u_{j,k})$ and $(v_{j,k})$ are obtained.
The 5^e column (i.e. the components of X) for the first 5 line can be easily computed.

For set M_2 we have the representation

$$M_2 = \Big\{(a_1 + (a_2 - a_1)U,\; b_1 + (b_2 - b_1)U,\; c_1 + (c_2 - c_1)U,\; d_1 + (d_2 - d_1)U,\; x_1 + (x_2 - x_1)U):$$
$$d_1 < d_2 \;\&\; b_2 < d_2 < b_2 + \frac{(c_1 - a_1)(d_1 - b_1)}{c_2 - a_2} \;\&\; x_1 = -\frac{d_1 - b_1}{c_2 - a_2} \;\&\; x_2 = \frac{d_2 - b_2}{c_1 - a_1}\Big\}$$

and in this way the partially ordered graph follows



therefore one of the cases is possible:

$$(a.1.4) \quad b_1 < d_1 < b_2 < d_2 < b_2 + \frac{(c_1 - a_1)(d_1 - b_1)}{c_2 - a_2}, \quad 0 < a_1 < a_2 < c_1 < c_2$$

or

$$(a.1.5) \quad b_1 < b_2 < d_1 < d_2 < b_2 + \frac{(c_1 - a_1)(d_1 - b_1)}{c_2 - a_2}, \quad 0 < a_1 < a_2 < c_1 < c_2.$$

By a simple computation we obtain the 6^e and 7^e line of $(u_{j,k})$ and $(v_{j,k})$.
For set M_3 we have the parametrical representation:

$$M_3 = \Big\{\Big(a + bU,\; a_1 + (e+f+g)U,\; a + b + c + dU, a_1 + e + fU,\; -\frac{e}{c+d} + \frac{e+g}{c+d}U\Big)\Big\}.$$

The proof is similar with the above considerations and may be left to the reader.

Thus the Theorem 1 is proved.

§ 2. The study of the interval equation (1) in case (f, — +, 4,3)

In case $(f, -+, 4{,}3)$ we have the system of conditions:

$$(f, -+, 4{,}3)_1 \begin{cases} a_2 > 0,\; a_1 < -a_2,\; c_1 < 0,\; c_2 > \frac{a_1 c_1}{a_2},\; \frac{a_1}{a_2} < \frac{x_2}{x_1} < \frac{a_2}{a_1}, \\ X = \frac{a_1(d_2 - b_2) - a_2(d_1 - b_1)}{a_2 c_2 - a_1 c_1},\; \frac{c_2(d_2 - b_2) - c_1(d_1 - b_1)}{a_2 c_2 - a_1 c_1} \\ \frac{a_1}{a_2} < \frac{c_2(d_2 - b_2) - c_1(d_1 - b_1)}{a_1(d_2 - b_2) - a_2(d_1 - b_1)}\,\frac{a_2}{a_1}, \end{cases}$$

therefore we can obtain the parametrical form (only necessary)

$$A = [-a-b, a],\ B = [b_1, b_1 + e],\ C = \left[-c,\ c + d + \frac{bc}{a}\right],\ D = [d_1, d_1 + f]$$

where

$$1 - \frac{b}{a} < \frac{\left(c + d + \frac{bc}{a}\right)(d_1 - b_1 + f - e) + c(d_1 - b_1)}{-(a+b)(d_1 - b_1 + f - e) - a(d_1 - b_1)} < -\frac{a}{a+b},$$

hence

$$\frac{a}{a+b} < \frac{(d_1 - b_1)\left(2c + d + \frac{bc}{a}\right) + (f - e)\left(c + d + \frac{bc}{a}\right)}{(d_1 - b_1)(2a + b) + (f - e)(a + b)} < 1 + \frac{b}{a}.$$

Therefore we consider one of the cases:

$$d_1 > b_1 + \frac{(a+b)(e-f)}{2a+b} \quad \text{or} \quad d_1 < b_1 + \frac{(a+b)(e-f)}{2a+b}$$

with the parametrical forms

$$d_1 = b_1 + g + \frac{(a+b)(e-f)}{2a+b} \quad \text{respectively} \quad d_1 = b_1 - g + \frac{(a+b)(e-f)}{2a+b}.$$

In this way we have one of the following situations:

$$0 < \frac{adf}{(2a+b)^2 g} < 1 + \frac{b}{a} + \frac{ade}{(2a+b)^2 g} - \frac{d}{2a+b} - \frac{c}{a}$$

$$\frac{a}{a+b} + \frac{ade}{(2a+b)^2 g} - \frac{d}{2a+b} - \frac{c}{a} < \frac{adf}{(2a+b)^2 g} < 1 + \frac{b}{a} + \frac{ade}{(2a+b)^2 g} - \frac{d}{2a+b} - \frac{c}{a}$$

$$0 < \frac{adf}{(2a+b)^2 g} < -\frac{a}{a+b} + \frac{ade}{(2a+b)^2 g} + \frac{d}{2a+b} + \frac{c}{a} < -1 - \frac{b}{a} + \frac{ade}{(2a+b)^2 g} +$$

$$+ \frac{d}{2a+b} + \frac{c}{a} < \frac{adf}{(2a+b)^2 g} + \frac{a}{a+b} + \frac{ade}{(2a+b)^2 g} + \frac{d}{2a+b} + \frac{c}{a}$$

which give some parametrical forms for the interval coefficients and also the solution of the equation (1). For some details see for example case (f, — +, 4, 5, 1, 2).

§ 3. The study of equation (1) in case (f, —+, 4, 5)

In case (f, — +, 4, 5) one of the following subcases holds:

$$(f, -+, 4, 5, 1)\begin{cases} a_2 > 0,\ a_1 - < a_2,\ b_1 \in \mathbf{R},\ d_1 > b_1,\ \frac{a_2^2}{a_1} < c_1 < 0, \\ c_2 > a_2,\ 1 + \frac{a_1 - c_1}{c_2 - a_2} > 0, \\ b_2 > b_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2},\ d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - c_2} \end{cases}$$

$$(f, -+, 4, 5, 2)\begin{cases} a_2 > 0,\ a_1 < -a_2,\ b_1 \in \mathbf{R},\ d_1 > b_1,\ \frac{a_2^2}{a_1} < c_1 < 0, \\ c_2 > a_2,\ 1 + \frac{a_1 - c_1}{c_2 - a_2} < 0, \\ b_2 > b_1,\ d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - c_2} \end{cases}$$

$$(f, -+, 4, 5, 3)\begin{cases} a_2 > 0,\ a_1 < -a_2,\ b_1 \in \mathbf{R},\ d_1 > b_1,\ c_1 < \frac{a_2^2}{a_1},\ c_2 > \frac{a_1 c_1}{a_2}, \\ 1 + \frac{a_1 - c_1}{c_2 - a_2} > 0, \\ b_2 > d_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2},\ d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - c_2} \end{cases}$$

$$(f, -+, 4, 5, 4)\begin{cases} a_2 > 0,\ a_1 < -a_2,\ b_1 \in \mathbf{R},\ d_1 > b_1,\ c_1 < \frac{a_2^2}{a_1},\ c_2 > \frac{a_1 c_1}{a_2}, \\ 1 + \frac{a_1 - c_1}{c_2 - a_2} < 0, \\ b_2 > b_1,\ d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - c_2} \end{cases}$$

$$(f, -+, 4, 5, 5)\begin{cases} a_2 > 0,\ a_1 < -a_2,\ b_1 \in \mathbf{R},\ d_1 < b_1,\ \frac{a_2^2}{a_1} < c_1 < 0,\ \frac{a_1 c_1}{a_2} < c_2 < a_2, \\ 1 + \frac{a_1 - c_1}{c_2 - a_2} < 0, \\ b_2 > b_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2},\ d_2 = \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - c_2} \end{cases}$$

$$(f, -+, 4, 5, 6)\begin{cases} a_2 > 0,\ a_1 < -a_2,\ b_1 \in \mathbf{R},\ d_1 < b_1,\ \frac{a_2^2}{a_1} < c_1 < 0, \\ \frac{a_1 c_1}{a_2} < c_2 < a_2,\ 1 + \frac{a_1 - c_1}{c_2 - a_2} > 0, \\ b_2 > b_1,\ d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - c_2}. \end{cases}$$

For the proof of the above decomposition in the subcases of case $(f, -+, 4, 5)$ we use the system of conditions which gives $(f, -+, 4, 5)$ and the following remarks:

a) In case $(f, -+, 4, 5)$ the equation (1) is equivalent with the system:

$$a_2x_1 + b_1 = c_2x_1 + d_1, \quad a_1x_1 + b_2 = c_1x_1 + d_2,$$

hence the solution of (1) (if exists) is $X = [x_1, x_2]$ where $x_1 = \frac{d_1 - b_1}{a_2 - c_2}$ with the condition $d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - b_2}$, since also $x_1 = \frac{d_2 - b_2}{a_1 - c_1}$ and $0 < x_1 < \frac{c_1(d_1 - b_1)}{c_2(a_2 - c_2)}$.

b) From $d_2 > d_1$ for b_2 follows

$$b_2 > b_1 \ \& \ b_2 > d_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2}$$

hence one of the cases is possible:

$$b_1 > d_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2}\left(\text{i.e. } (d_1 - b_1)\left(1 + \frac{a_1 - c_1}{c_2 - a_2}\right) < 0\right) \text{ if } b_1 < b_2$$

or

$$b_1 < d_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2}\left(\text{i.e. } (d_1 - b_1)\left(1 + \frac{a_1 - c_1}{c_2 - a_2}\right) > 0\right) \text{ if}$$

$$b_2 > d_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2},$$

c) Since $x_1 < 0$, we also have condition: $(d_1 - b_1)(a_2 - c_2) < 0$.

Combining the above remarks, we have the subcases $(f, -+, 4, 5, 1)$ — $(f, -+, 4, 5, 6)$. To obtain the parametrical forms of the interval coefficients A, B, C, D we also have the following results:

$$1 + \frac{a_1 - c_1}{c_2 - a_2} > 0 \Leftrightarrow (c_1 > a_1, c_2 > c_1 + a_2 - a_1) \vee (c_1 < a_1, c_2 > a_2) \vee$$

$$\vee (c_1 > a_1, c_2 < a_2) \vee (c_1 < a_1, c_2 < c_1 + a_2 < -a_1)$$

$$1 + \frac{a_1 - c_1}{c_2 - a_2} < 0 \Leftrightarrow (c_1 < a_1, c_1 + a_2 - a_1 < c_2 < a_2) \vee$$

$$\vee (c_1 > a_1, a_2 < c_2 < c_1 + a_2 - a_1).$$

The new subcases (in detail) we give only for case $(f, -+, 4, 5, 1)$:

$$a_2 > 0, \ a_1 < -a_2, \ b_1 \in \mathbf{R}, \ d_1 > b_1, \ \frac{a_2^2}{a_1} < c_1 < a_1, \ c_2 < a_2,$$

$(f, -+, 4, 5, 1, 1)$

$$b_2 > d_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2}, \ d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - c_2}$$

$$a_2 > 0, \ a_1 < -a_2, \ b_1 \in \mathbf{R}, \ d_1 > b_1, \ \frac{a_2^2}{a_1} < c_1 < 0, \ c_2 > c_1 + a_2 - a_1,$$

$f, -+, 4, 5, 1, 2)$

$$b_2 > d_1 + \frac{(a_1 - c_1)(d_1 - b_1)}{c_2 - a_2}, \ d_2 = b_2 + \frac{(a_1 - c_1)(d_1 - b_1)}{a_2 - b_2}.$$

For case $(f, -+, 4, 5, 1, 2)$ we also give the general parametrical form of the interval coefficients:

$$A = [-a - b, a], \ B = \left[a_1 + c + e, \ \frac{c(a^2s - (a + b)^2)}{(a + b)(a + b + d) - a^2s}\right],$$

$(f, -+, 4, 5, 1, 2)_1$

$$C = \left[-\frac{a^2s}{a + b}, \ 2a + b + d - \frac{a^2s}{a + b}\right], \ D = [a_1 + c, \ a_1 + c + e]$$

and in case $(f, -+, 4, 5, 1, 2)$ we also give the solutions of (1):

$$(f, -+, 4, 5, 1, 2)_2 \quad X = X_t = \left[-\frac{(a + b)c}{(a + b)(a + b + d) - a^2s}, \right.$$

$$\left. \frac{a^2(a + b)cst}{((a + b)(2a + b + d) - a^2s)((a + b)(a + b + d) - a^2s}\right],$$

where $0 < t < 1$.

From the above relations (parametrical forms of A, B, C, D, X) it follows:

$$(f, -+, 4, 5, 1, 2)_3 \quad AX + B = CX + D =$$

$$= \left[a_1 - \frac{a(a + b)c}{(a + b)(a + b + d) - a^2s}, \ a_1 + c + e + \frac{a^2sc}{(a + b)(a + b + d) - a^2s}\right]$$

§ 4. Some observations

Above we have given only some very particular cases for the solution of the interval equation (1). Cases: (1) has 0, 1 or an infinity of solutions, are considered in some examples above. The solution of (1) in another cases can be similarly considered.

We give in this finally part of the present paper the following relation, which gives the parametrical representation of a new case:

$$\{(A, B, C, D, X) \mid 0 < C > A,\ 0 < X,\ AX + B = CX + D\} =$$

$$= \Big\{\Big(a + b + cU,\ a_1 + eU,\ a_1 + (b + c + d)U,$$

$$a_1 + \frac{bes}{b+d} + e(1-s)tU,\ \frac{es}{b+d} + \frac{e(1-s)(1-t)}{d}U\Big) \Big| a, b, c, d, e \in \mathbf{R}^+,$$

$$a_1 \in \mathbf{R},\ 0 < s,\ t < 1\Big\}.$$

A very particular case of the above general case is the equation

$$[2, 3]X + [1, 9] = [1, 4]X + [3, 6]$$

with the solution

$$X = [2, 3]$$

In this case:

$$a_1 = a = b = c = d = 1,\ e = 8,\ s = \frac{1}{2} \text{ and } t = \frac{3}{4}\cdot$$

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