REVUE D'ANALYSE NUMÉRIQUE ET DE LA THÉORIE DE L'APPROXIMATION, Tome 2, 1973, pp. 37—48

ITERATIVE SYSTEMS, A TOPOLOGICAL AND CATEGORIAL APPROACH

by

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Introduction

An iterative system [2] is a pair (A, R), where A is a set (of states) and R is a binary relation in A. A complex iterative system [3] is a pair $(A, \{R_i : i \in I\})$, where $\{R_i : i \in I\}$ is a family of binary relations in the set A. We construct, in an analogous way as in [6] and [7], a closure operation and a topology in the iterative system. The category of iterative systems is studied and we show that it has limits and colimits. Some relations between the category of iterative systems, the category of complex iterative systems, the category of closure spaces, the category of topological spaces and the category of partially ordered sets will be given, using a theorem of J. BENABOU [1] and some analogous results from [7] and [5]. The isomorphism theorems for complex iterative systems will be proved in the same manner as in [11].

1. Relations

Let A be a set and let R and P be two binary relations in A, *i.e.* R, $P \subset A \times A$. The following notations will be used:

$$RP = \{(x, y) : \exists z \in A((x, z) \in R \text{ and } (z, y) \in P)$$

$$R + P = \{(x, y) : \text{either } (x, y) \in R \text{ or } (x, y) \in P\}$$

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

$$R^{\circ} = I = \{(x, x) : x \in A\}$$

$$R^{n} = RR^{n-1} \text{ for any integer } n \geqslant 1$$

$R^+ \quad = \Sigma\{R^i : i \geqslant 1\}$ (plus closure of R)

$R^* \quad = \Sigma\{R^i : i \geqslant 0\}$ (star closure of R)

$R^\wedge \quad = R + I = \{(x, y) : (x, y) \in R$ or $x = y\}$

$R^{\cdot\cdot} \quad = R - I = \{(x, y) : (x, y) \in R$ and $x \neq y\}$

$R(x) \quad = \{y : (x, y) \in R\}$ (the fibre of R at x)

$R(Y) \quad = \bigcup\{R(x) : x \in Y\}$ for any $Y \subset A$

$\mathrm{Dom}\, R = \{x : R(x) \neq \emptyset\}$

$E = R^* \cap (R^*)^{-1}$

$R_Y = R \cap (Y \times Y)$ (the restriction of R to $Y \subset A$)

$R \subset P$ *iff* $((x, y) \in R$ implies $(x, y) \in P$ for any $x, y \in A)$.

The following results are well-known:

P r o p o s i t i o n 1.1. *$R^\wedge$ is the smallest reflexive relation containing $R \cdot R^+$ is the smallest transitive relation containing R and R^* is the smallest reflexive and transitive relation containing R. Moreover:*

$$(R^+)^\wedge = (R^\wedge)^* = R^*,$$
$$(R^{\cdot\cdot})^* = R^* = (R^\wedge)^*,$$
$$(R^*)^{\cdot\cdot} = R^+,$$
$$(R^*)R = R(R^*) = R^+,$$

E is an equivalence and the relation induced by R^* in the quotient set A/E is an order (i.e. $(A/E, R^*)$ is a partially ordered set).

D e f i n i t i o n 1.1. *The pair (A, R) will be called an iterative system, A will be called the set of states and R — the transition realation. For a subset $Y \subset A$ the set $t(Y) = Y - \mathrm{Dom}\, R_Y^{\cdot\cdot}$ will be called the set of exit states of Y; the set $i(Y) = Y - \mathrm{Dom}\, (R^{\cdot\cdot})_Y^{-1}$ — the set of entrance states of Y. Particularly, the set $t(A)$ will be called the set of terminal states and $i(A)$ — the set of initial states of the iterative system (A, R).*

A relational machine [8] is an iterative system (A, R) such that $i(A) \cap t(A) = \emptyset$. A Pawlak machine is an iterative system (A, R) such that for any $x \in A$, $R(x)$ has at most one element (R is a single-valued relation).

2. Some topological aspects

Let $c_R : \exp A \to \exp A$ be the map defined by $c_R(X) = R^\wedge(X)$ for any $X \subset A$ ($\exp A = \{Y : Y \subset A\}$).

It is clear that $c_R(X) = X \cup R(X) = X \cup R^{\cdot\cdot}(X)$ and $c_R(\emptyset) = \emptyset$, $c_R(X \cup Y) = C_R(X) \cup c_R(Y)$. Consequently:

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P r o p o s i t i o n 2.1 *(A, c_R) is a closure space.*

D e f i n i t i o n 2.1. *The map $c_R = \{X \to R^\wedge(X)\}$ will be called the quasidiscrete closure generated by the relation R in the set A.*

P r o p o s i t i o n 2.2. *For any $X \subset A$, $c_R(X) = \bigcup\{c_R(x) : x \in X\}$. Moreover, every familly of subsets of A is closure preserving. Every closed set of (A, c_R) is of the form $R^*(Z)$ with $Z \subset A$.*

Proof. The first and the second part follow from the definition. For the third, let $X \subset A$ be a closed set of (A, c_R). $X = c_R(X) = X \cup R(X)$ implies $R^i(X) = R^i(X) \cup R^{i+1}(X)$ and then $X = \bigcup\{R^i(X) : i \in N\}$ (N is the set of nonnegative integers), hence $X = R^*(X)$. Conversely, for any $Y \subset A$, $R^*(Y)$ is a closed set of (A, c_R), since $R^*(Y) \cup RR^*(Y) = R^*(Y) \cup \bigcup R^+(Y) = R^*(Y)$.

C o r o l l a r y 2.1. *The familly of all closed sets of (A, c_R) is $\{R^*(Y) : Y \subset A\}$; at the same time the familly of all closed sets may be defines by the following two conditions:*

1) *A set of terminal states is closed,*

2) *If Z is a closed set and $R^{\cdot\cdot}(Y) \subset Z$ then $Z \cup Y$ is closed.*

Proof. $Z \subset t(A)$ implies $R^{\cdot\cdot}(Z) = \emptyset$ and then $c_R(Z) = Z$, also $c_R(Z \cup Y) = Z \cup Y \cup R(Z \cup Y) = Z \cup Y \cup R(Y) = Z \cup Y$ since Z is a closed set and $R^{\cdot\cdot}(Y) \subset Z$, that is $R(Y) \subset Z \cup Y$. As well, it is clear that the conditions 1) and 2) determine the familly $\{R^*(Y) : Y \subset A\}$.

P r o p o s i t i o n 2.3. *$i(c_R(X)) = i(X)$, $i(R^*(X)) \subset X$ and $t(R^*(X) \subset \subset t(A)$ for any $X \subset A$. Moreover, $x \in c_R(A - \{x\})$ for any noninitial state x and consequently $i(A)$ is the set of all isolated points of (A, c_R).*

Proof. For any binary relation P in A, $\mathrm{Dom}\, P_X = X \cap P^{-1}(A)$, therefore $i(c_R(X)) = X \cup R^{\cdot\cdot}(X) - ((X \cup R^{\cdot\cdot}(X)) \cap R^{\cdot\cdot}(A)) = X \cup R^{\cdot\cdot}(X) - (\mathrm{Dom}\,(R^{\cdot\cdot})_X^{-1} \cup R^{\cdot\cdot}(X)) = X - \mathrm{Dom}\,(R^{\cdot\cdot})_X^{-1} = i(X)$, hence $i(c_R(c_R(X))) = i(X)$ and by repetition $i(R^*(X)) = i(X) \subset X$. Simultaneously, $t(R^*(X)) = R^*(X) - ((R^{\cdot\cdot})^{-1}(A) \cap R^*(X)) = R^*(X) - \mathrm{Dom}\, R^{\cdot\cdot} = t(A) \cap R^*(X)$ since $(R^{\cdot\cdot})^{-1}(A) \cap R^*(X) = \{x : x \in R^*(X)$ and $R^{\cdot\cdot}(x) \neq \emptyset\} = \mathrm{Dom}\, R^{\cdot\cdot} \cap R^*(X)$. For the second part of the proposition, let x be a noninitial state of (A, R). Then there is a state y such that $(y, x) \in R^{\cdot\cdot}$ and in conclusion $x \in c_R(y) \subset c_R(A - \{x\})$ namely every $x \in A - i(A)$ is an accumulation point of (A, c_R) and therefore $i(A)$ is the set of all isolated points of (A, c_R).

Let u_R be the topological modification of the closure c_R, *i.e.* the closure defined by $u_R(X) = \bigcap\{Y : X \subset Y = c_R(Y),\ Y \subset A\}$

P r o p o s i t i o n 2.4. *The topological modification u_R of the closure operation c_R is the closure operation generate by the star closure of R (briefly $u_R = c_{R^*}$).*

Proof. R^* is a reflexive and transitive relation and then the map $c_{R^*} = X \to R^*(X)$ is a topological closure operation for A, $((R^*)^\wedge = (R^+)^\wedge = R^*)$. Indeed, $c_{R^*}(c_{R^*}(X)) = c_{R^*}(X)$ as $R^*R^* \subset R^*$. Also the familly of all closed sets of (A, c_{R^*}) is $\{R^*(Z) : Z \subset A\}$ then coincides with the familly of all closed sets of (A, c_R). But $R^*(X)$ is the smallest closed set of (A, c_R) containing X, for any $X \subset A$, therefore c_{R^*} is the topological modification of c_R and will be denoted by u_R.

P r o p o s i t i o n 2.5. *For every subset X of A there is a subset I of the initial states such that $X \subset u_R(I)$. Particularly, $i(A)$ is dense and every subset Y of noninitial states is a nowhere dense set in (A, u_R).*

Proof. $I = i(A) \cap (R^*)^{-1}(X)$ and then $R^*(I) \supset X$. Particularly, $A \subset R^*(i(A))$ that is $i(A)$-dense set in (A, u_R). If $Y \subset A - i(A)$ then $u_R(Y) \cap i(A) = \emptyset$ therefore $u_R(A - u_R(Y)) = A$.

3. Some categorial aspects

D e f i n i t i o n 3.1 *Let (A, R) and (B, P) be two iterative systems and $f: A \to B$ a mapping of the set A into the set B. $f: (A, R) \to (B, P)$ is called a morphism of iterative systems* (briefly is-morphism) *iff $(x, y) \in R$ implies $(f(x), f(y)) \in P$ for any $x, y \in A$.*

It is easy to verify that the composition of two is-morphisms is an is-morphism. Also the identical mapping $e: (A, R) \to (A, R)$ is an is-morphism. Consequently:

P r o p o s i t i o n 3.1 *All iterative systems with morphisms of iterative systems form a category denoted by* Is.

P r o p o s i t i o n 3.2 *If $f: (A, R) \to (B, P)$ is an is-morphism then $f: (A, c_R) \to (B, c_P)$ and $f: (A, u_R) \to (B, u_P)$ are continuous mappings.*

Proof. For any integer $i > 1$, $(x, y) \in R^i$ implies the existence of a sequence $a_0, a_1, \ldots, a_i$ with $a_0 = x$, $a_i = y$ and $(a_k, a_{k+1}) \in R$ for $k = 0, 1, \ldots, i - 1$. Then $f(a_0) = f(x)$, $f(a_i) = f(y)$ and $(f(a_k), f(a_{k+1})) \in P$ for $k = 0, 1, \ldots, i - 1$, namely $(f(x), f(y)) \in P^i$. Hence $(x, y) \in R^*$ implies $(f(x), f(y)) \in P^*$ for any $x, y \in A$ and therefore $f(u_R(X)) = f(R^*(X)) \subset P^*(f(X)) = u_P(f(X))$, for any $X \subset A$. Besides, $(x, y) \in R^\wedge$ is equivalent with $(x, y) \in R$ or $x = y$, then $(f(x), f(y)) \in P$ or $f(x) = f(y)$, namely $(f(x), f(y)) \in P^\wedge$, for any $x, y \in A$. Consequently, $f(c_R(X)) = f(R^\wedge(X)) \subset P^\wedge(f(X)) = c_P(f(X))$, for any $X \subset A$.

P r o p o s i t i o n 3.3 *In the category* Is *a morphism f is a monomorphism* (epimorphism) *if and only if it is an injective* (surjective) *mapping; f is an isomorphism if and only if it is a bijective* is-*morphism and $(f(x), f(y)) \in P$ implies $(x, y) \in R$ for any $x, y \in A$, i.e. iff f and f^{-1} are* is-*morphisms.*

P r o p o s i t i o n 3.4 *Two iterative systems are isomorphic if and only if the induced closure spaces are homeomorphic.*

Proof. (A, R) and (B, P) are isomorphic iff there is an isomorphism $f: (A, R) \to (B, P)$. Then f and f^{-1} are is-morphisms hence continuous mappings, i.e. f is a homeomorphism. Conversely, let f be a homeomorphism of the closure space $(A\ c_R)$, onto the closure space (B, c_P), then $f(c_R(X)) = c_P(f(X))$ and therefore $f(R(X)) \subset f(X) \cup P(f(X))$ for any $X \subset A$. But $(x, y) \in R$ implies $y \in R(x)$ and then $f(y) \in f(x) \cup P(f(x))$ that is $(f(x), f(y)) \in P$ since f is bijective. Consequently, f is an is-morphism. Similarly, f^{-1} is an is-morphism and then f is an isomorphism in the category Is.

If Ens is the category of sets let F: Is $\to$ Ens be the forgetful functor (i.e. the functor wich assigns to each iterative system the underlying set of states, and to each is-morphism the underlying mapping of the underlying sets).

P r o p o s i t i o n 3.5 *The forgetful functor F: Is $\to$ Ens is faithful and it has an adjoint functor F'' and a coadjoint functor $''F$. Moreover, for any set A, $''F(A) = (A, \emptyset)$ and $F''(A) = (A, A \times A)$.*

Proof. For any $f, g: (A, R) \to (B, P)$ $f = g$ if and only if $F(f) = F(g)$ and then F is a faithful functor ($f = F(f): A \to B$). After that, for any iterative system (A, R) and for any set B, the set of is-morphisms from $(B, \emptyset)$ into (A, R) coincides with the set of mappings of B into A since $\bar{f}(\emptyset) \subset R$ for any $f: B \to A$, then $''F$ is a coadjoint of F. Similarly, $\bar{f}(R) \subset B \times B$ for any $f: A \to B$, and then the set of mappings of A into B is the same that the set of is-morphisms of (A, R) into $(B, B \times B)$, consequently F'' is an adjoint of F. ($\bar{f}$ denotes the mapping induced by $f: A \to B$ from the square of A into square of B, that is $\bar{f}((x, y)) = (f(x), f(y))$ for any $x, y \in A$.)

P r o p o s i t i o n 3.6 *The category of iterative systems is complete and cocomplete, i.e. Is has limits and colimits.*

Proof. The things that we have to prove are that the category Is has products, coproducts, difference kernels and difference cokernels. Let $\{(A_i, R_i): i \in I\}$ be a familly of iterative systems. We define an iterative system (A, R) with $A = \Pi A_i$ (in Ens) and $R = \cap \bar{p}_i^{-1}(R_i)$, where $p_i: A \to A_i$ for $i \in I$ is the canonic projection (in Ens) and $\bar{p}_i: A \times A \to A_i \times A_i$ is the mapping naturaly induced by p_i. In other words, if $\{x_i: i \in I\}$ and $\{y_i: i \in I\}$ are two elements of A ($x_i \in A_i$ and $y_i \in A_i$ for $i \in I$) then $(\{x_i: i \in I\}, \{y_i: i \in I\}) \in R$ iff for every $i \in I$ $(x_i, y_i) \in R_i$. It is clear that $\bar{p}_i(R) = R_i$ and therefore $p_i: (A, R) \to (A_i, R_i)$ is an is-morphism for any $i \in I$. Also, it is easy to verify that $< (A, R), \{p_i: i \in I\} >$ is the product of the familly $\{(A_i, R_i): i \in I\}$. In fact, if $s_i: (B, P) \to (A_i, R_i)$ is an is-morphism for any $i \in I$ then there is a mapping $s: B \to A$ such that $s_i = p_i s$ for any $i \in I$, since $< A, \{p_i: i \in I\} >$ is the product of the familly $\{A_i: i \in I\}$ (in Ens). But for any $x, y \in B$, $(x, y) \in P$ implies $(s_i(x), s_i(y)) \in R$ then $(p_i s(x), p_i s(y)) \in R_i$, for any $i \in I$, namely $(s(x), s(y)) \in R$. Consequently, $s: (B, P) \to (A, R)$ is an is-morphism.

For two is-morphisms $f, g: (A, R) \to (B, P)$ we define an iterative system (K, Q) with $K = \{x: x \in A,\ f(x) = g(x)\}$ and $Q = R_K$. Let $h: (K, Q) \to (A, R)$ be the inclusion mapping. Then $\bar{h}(Q) \subset R$ and h is an is-morphism. If $j: (C, S) \to (A, R)$ is another is-morphism such that $fj = gj$ then there is a mapping $k: C \to K$, $j = hk$ because $< K, h >$ is the difference kernel of f, g (in Ens); if $(x, y) \in S$ for $x, y \in C$ then $(j(x), j(y)) = (hk(x), hk(y)) \in R$ and consequently, $(k(x), k(y)) \in Q$, since h is an inclusion, i.e. k is an is-morphism, and $< (K, Q), h >$ is the difference kernel for f, g in Is.

Coproducts and difference cokernels may be construct in the same manner. Consequently, the category Is has products, coproducts, difference kernels and difference cokernels and then Is has any limits and colimits. Particularly, we can construct the intersection of a familly of iterative systems and the pullbacks, images and inverse images etc. in a similar way.

4. Some functorial aspects

Let Cl be the category of closure spaces and Top — the category of topological spaces (with continuous mappings). It is known that Cl and Top have limits and colimits.

The association of a closure space (A, c_R) to an iterative system (A, R) defines a functor $C: \mathrm{Is} \to \mathrm{Cl}$; the association of the topological space (A, u_R) to the iterative system (A, R) defines a functor $U: \mathrm{It} \to \mathrm{Top}$. The proposition 3.2 provides that $C(f)$ and $U(f)$ are morphisms in Cl respective in Top, for any is-morphism $f: (A, R) \to (B, P)$.

The association of a reflexive and transitive relation R^* to a relation R defines a functor O of the category Is into the category of ordered sets (denoted by Ord). Indeed, the proposition 1.1 provides that $(A/E, R^*)$ is an ordered set for any iterative system (A, R). Also, from the proof of the proposition 3.2, $\bar{f}(R^*) \subset P^*$ and $\bar{f}(R^{*-1}) \subset P^{*-1}$ for any is-morphism $f: (A, R) \to (B, P)$. Accordingly, f is an isotone mapping of $(\underline{A}, R^*)$ into $(\underline{B}, P^*)$, where $\underline{A} = A/(R^* \cap R^{*-1})$ and $\underline{B} = B/(P^* \cap P^{*-1})$.

P r o p o s i t i o n 4.1 *The functors C, U, O defined above are faithful and limits preserving functors.*

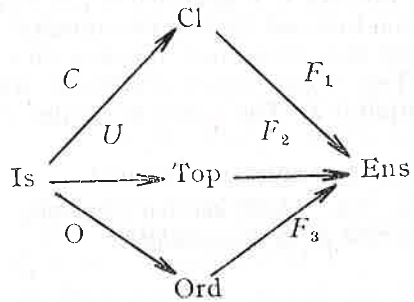
Proof. The proof is similar for any one of functors. We prove for example for the first. Let (A, R) and (B, P) be two iterative systems and (A, c_R), (B, c_P) — the corresponding closure spaces. There is a bijective mapping of the set of is-morphisms $\mathrm{Is}((A, R), (B, P))$ onto the set of continuous mappings $\mathrm{Cl}((A, c_R), (B, c_P))$ and then the functor C is faithful. Now, we prove that C preserves products and difference kernels. Let $f, g: (A, R) \to (B, P)$ be two is-morphisms and let (K, Q) be the difference kernel of f, g in Is. It is easy to prove that (K, c_Q) with inclusion $h: (K, c_Q) \to (A, c_R)$ is the difference kernel of f, g in Cl. Similarly, for the product. If (A, R) is the product of the familly $\{(A_i, R_i): i \in I\}$ then (A, c_R) is the product (in Cl) of the familly $\{(A_i, c_{R_i}): i \in I\}$ with the same projections $p_i: A \to A_i$. Indeed, $p_i: (A, c_R) \to (A_i, c_{R_i})$ is a continuous mapping since $p_i: (A, R) \to (A_i, R_i)$ is an is-morphism, for any $i \in I$. Besides, if (A', c') is a closure space and, for any i, there is a continuous mapping $s_i: (A', c') \to (A_i, c_{R_i})$, then there is a mapping $s: A' \to A$ (since A is the product in Ens) such that $s_i = p_i s$ for any $i \in I$. But $\bar{p}_i(R) = R_i$ implies $p_i(c_R(X)) = c_{R_i}(p_i(X))$, and $s_i(c'(X)) \subset c_{R_i}(s_i(X))$ implies $p_i s(c'(X)) \subset \subset c_{R_i}(p_i s(X))$ and then $s(c'(X)) \subset c_R(s(X))$ (since $\{p_i: i \in I\}$ are canonic projections in Ens) for any set $X \subset A$. Then s is a continuous mapping and therefore (A, c_R) is the product of the familly $\{(A_i, c_{R_i})\}$ in the category of the closure spaces. Consequently, the functor C preserves the difference kernels and the products and then C preserves the limits.

P r o p o s i t i o n 4.2. *There are three functors:*

$$''C: \mathrm{Cl} \to \mathrm{Is}, \quad ''U: \mathrm{Top} \to \mathrm{Is}, \quad ''O: \mathrm{Ord} \to \mathrm{Is}$$

such that C is an adjoint of $''C$, U is an adjoint of $''U$ and O is an adjoint of $''O$.

Proof. We have the diagram:



where F_1, F_2, F_3 are corresponding forgetful functors and $F_1 C = F_2 U = F$. All this functors are faithful and there are $''F_1$ — a coadjoint of F_1, $''F_2$ — a coadjoint of F_2 and $''F_3$ — a coadjoint of F_3. Since the category Is has limits and the functors C, U, O are limits preserving functors, it follows from theorem 4 of (1) that C, U, O have adjoint functors, respectively, $''C$, $''U$, $''O$.

We can construct the functors $''C$, $''U$, $''O$ directly. Let (A, c) be a closure space. We define a relation $R \subset A \times A$ by $(x, y) \in R$ iff $y \in c(x)$ for any $x, y \in A$. It is clear that R is a reflexive relation. R is transitive iff the closure c is topological, and R is an order iff (A, c) is a T_0-topological space (i.e. $x \in c(y)$ and $y \in c(x)$ imply $x = y$). This construction defines a functor $C': Cl \to \mathrm{Is}$ and a functor $U': \mathrm{Top} \to \mathrm{Is}$ (even a functor $T: T_0 \to \to \mathrm{Ord}$ where T_0 is the subcategory of T_0-topological spaces). Indeed, if $f: (A, c) \to (A', c')$ is a continuous mapping and $x, y \in A$ then $x \in c(y)$ implies $f(x) \subset f(c(y) \subset c'(f(y))$ and therefore f is an is-morphism of underlying iterative systems. Moreover, if $C'(A, c) = (A, R_c)$ then c_{R_c} is the quasidiscrete modification of the closure c. Now, let (A, R) be an iterative system and (B, c) a closure space such that $C(A, R) = (A, c_R)$ and $C'(B, c) = (B, R_c)$. Let f be an is-morphism of (B, R_c) into (A, R), that is for any $x, y \in B$, $y \in c(x)$ implies $(f(x), f(y)) \in R$, then $f(y) \in R(f(x)) \subset \subset c_R(f(c))$ and therefore $f: (B, c) \to (A, c_R)$ is a continuous mapping. Conversely, if $g: (B, c) \to (A, c_R)$ is a continuous mapping, (i.e. $f(c(X)) \subset \subset c_R(f(X))$ for any $X \subset B$) then $y \in c(x)$ implies $f(y) \in c_R(f(x)) = f(x) \cup \cup R(f(x))$ and then $(f(x), f(y)) \in R$ or $f(x) = f(y)$, namely $f: (B, R_c) \to \to (A, R)$ is an is-morphism. This bijective correspondence between the morphisms of $C'(B, c)$ into (A, R) and the morphisms of (B, c) into $C(A, R)$

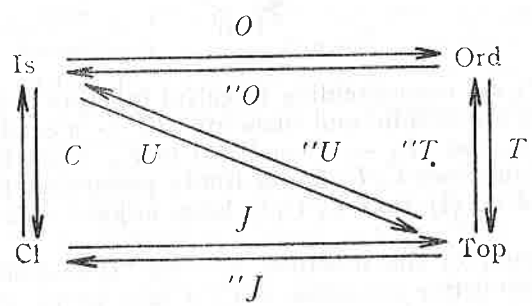
prove that C' is a coadjoint functor of C (that is equivalent with: C is an adjoint for the functor C') and we can replace C' by $''C$. The proof is the same if we exchange C, C' and Cl by U, U' and Top. Consequently:

P r o p o s i t i o n 4.3. *For any closure* (topological) *space* (A, c) *the relation* R *defined by* $(x, y) \in R$ *iff* $y \in c(x)$, $\forall x, y \in A$ *induces a functor of the category of the closure* (*topological*) *spaces into the category of iterative systems, which is a coadjoint of* $C(U)$.

Let now (A, u) be a topological space. We define an equivalence by $x \sim y$ iff $u(x) = u(y)$ for any $x, y \in A$. Then $(A/\sim, \underline{u})$, with $\underline{u}$ — the topological closure operation induced by u in the quotient set, is a T_0 — topological space and the relation R defined above is an order. In this way, we obtain a functor $''T$: Top → Ord which is faithful, limits preserving functor and which has an adjoint T. The proof is similar to that of propositions 4.1 and 4.2.

Combining the above results we obtain:

P r o p o s i t i o n 4.4. *There are ten functors, adjoints in pairs, so that the following diagram to be commutative*



Proof. The functor $''J$: Top → Cl is the inclusion (a topological space is a closure one) and has an adjoint functor J — the topological modification of the closure space. The commutativity of the diagram follows from the construction of the functors. Particularly, $JC = U = TO$ is just the proposition 2.4.

5. Complex interative systems

D e f i n i t i o n 5.1. *Let* I *be a set and let* $\{R : i \in I\}$ *be a family of binary relations in a set* A. *The pair* $(A, \{R_i : i \in I\})$ *will be called a complex iterative system.*

Let $(A, \{R_i : i \in I\})$ and $(B, \{P_i : I \in I\})$ be two complex iterative systems. A mapping $f : A \to B$ is called a morphism of complex iterative systems (briefly a cis-morphism) *iff* $(x, y) \in R_i$ implies $(f(x), f(y)) \in P_i$ for any $x, y \in A$ and for any $i \in I$.

P r o p o s i t i o n 5.1. *All complex iterative systems with morphisms of complex iterative systems form a category denoted by* Cis.

P r o p o s i t i o n 5.2 *The association of an iterative system* (A, R) *to a complex iterative system* $(A, \{R_i : i \in I\})$, *where* $R = \Sigma\{R_i : i \in I\}$, *defines a functor* S : Cis → Is. S *is a faithful functor.*

Proof. Let $f : (A, \{R_i : i \to I\}) \to (B, \{P_i : i \in I\})$ be a cis-morphism Then, for any $x, y \in A$, $(x, y) \in R_i$ implies $(f(x), f(y)) \in P_i \subset \Sigma\{P_i : i \in I\}$ therefore, $(x, y) \in \Sigma\{R_i : i \in I\}$ implies $(f(x), f(y)) \in \Sigma\{P_i : i \in I\}$. Consequently, $S(f) = f : (A, \Sigma\{R_i : i \in I\}) \to (B, \Sigma\{P_i : i \in I\})$ is a morphism in Is. It is clear that S is faithful.

P r o p o s i t i o n 5.3 *The mapping* $(A, \{R_i : i \in I\}) \to (A, \bigcap\{R_i : i \in I\})$ *defines a faithul functor* V : Cis → Is.

Proof. Let $f : (A, \{R_i : i \in I\}) \to (B, \{P_i : i \in I\})$ be a cis-morphism. Then, for any $i \in I$, $(x, y) \in R_i$ implies $(f(x), f(y)) \in P_i$ and therefore $(x, y) \in \bigcap\{R_i : i \in I\}$ implies $(f(x), f(y)) \in \bigcap\{P_i : i \in I\}$. Consequently, $V(f) = f : (A, \bigcap R_i\{ : i \in I\}) \to (B, \bigcap\{P_i : i \in I\})$ is a morphism in the category Is.

P r o p o s i t i o n 5.4 *The mapping* $Y : (A, R) \to (A, \{R_i : i \in I\})$, *where* $R_i = R$ *for any* $i \in I$, *defines a faithful functor of* Is *into* Cis. *This functor is an adjoint of* S *and a coadjoint of* V.

Proof. First part is clear. For the second, let (A, R) be an iterative system and let $(B, \{P_i : i \in I\})$ be a complex iterative system. For any $f : A \to B$, f is a cis-morphism of $Y(A, R)$ into $(B, \{P_i : i \in I\})$ if and only if f is an is-morphism of (A, R) into $V(B, \{P_i : i \in I\})$ and consequently, the sets of morphisms $[(A, R), (B, \bigcap\{P_i : i \in I\})]$ and $[Y(A, R), (B, \{P_i : i \in I\})]$ are isomorphic. Then the bifunctors $(-, V(-))$, $(Y(-), -)$ are naturally equivalent. Similarly, there is an isomorphism between the set of cis-morphisms of $(B, \{P_i : i \in I\})$ into $Y(A, R)$ and the set of is-morphisms of $S(B, \{P_i : i \in I\})$ into (A, R), and then the bifunctors $(-, Y(-))$, $(S(-), -)$ are naturally equivalent.

Y is an embedding functor and then the category Is can be considered a subcategory of the category Cis. Proposition 5.4 proved that the category Is is a reflective and coreflective subcategory of the category Cis.

P r o p o s i t i o n 5.5 *The category of complex iterative systems is complete and cocomplete.*

Proof. In the same manner as in proposition 3.6., we have to prove that the category Cis has products and difference kernels. Let $\{(A_j, \{R_{ji} : i \in I) : j \in J\}$ be a familly of complex iterative systems. For any $i \in I$, we define $R_i = \bigcup\{\bar{p}_j^{-1}(R_{ji}) : j \in J\}$, where $p_j : \Pi A_j \to A_j$ is the canonic projection (in Ens) and $\bar{p}_j : \Pi A_j \times \Pi A_j \to A_j \times A_j$ is naturally induced by p_j, $j \in J$. The complex iterative system $(\Pi A_j, \{R_i : i \in I\})$ with the canonic projections $\{p_j : j \in J\}$ is a product of the familly $\{(A_j, \{R_{ji} : i \in I\}) : j \in J\}$. Properly $\bar{p}_j(R_i) = R_{ji}$, for any $j \in J$ and then p_j is a cis-morphism, for any $j \in J$. Also, *if* $\{s_j : j \in J\}$ is a familly of cis-morphisms of a complex iterative system $(B, \{P_i : i \in I\})$ into the familly $\{(A_j, \{R_{ji} : i \in I\}) : j \in J\}$, there is a mapping $s : B \to \Pi A_j$ such that $s_j = p_j s$ for any $j \in J$ (since $< \Pi A_j, \{p_j : j \in J\} >$ is a product in Ens). But for any $i \in I$, $x, y \in B$, $(x, y) \in P_i$ implies $(s_j(x), s_j(y)) \in R_{ji}$ for any $j \in J$, then $(p_j s(x), p_j s(y)) \in R_{ji}$, namely $(s(x), s(y)) \in R_i$. Consequently $s: (B, \{P_i : i \in I\}) \to (\Pi A_j, \{R_i : i \in I\})$ is a cis-morphism.

For two cis-morphisms $f, g: (A, \{R_i : i \in I\}) \to (B, \{P_i : i \in I\})$ we define a complex iterative system $(K, \{Q_i : i \in I\})$ with $K = \mathrm{Ker}_{\mathrm{Ens}}(f, g)$ and $Q_i = R_{iK}$, for any $i \in I$. Let $h: (K, \{Q_i : i \in I\}) \to (A, \{R_i : i \in I\})$ be the inclusion mapping, i.e. a cis-morphism. Let $t: (C, \{S_i : i \in I\}) \to (A, \{R_i : i \in I\})$ be a cis-morphism such that $ft = gt$. Then there is a mapping $k: C \to K$, $t = hk$ (from the definition of K), and, for any $i \in I$, $(x, y) \in S_i$ implies $(t(x), t(y)) = (hk(x), hk(y)) \in R_i$ therefore $(k(x), k(y)) \in R_{iK} = Q_i$, since h is an inclusion. In conclusion k is a cis-morphism and $< (K, \{Q_i : i \in I\}), h >$ is the difference kernel for f, g in the category Cis.

The existence of colimits follows from theorem 5 of (1) since the category Is is cocomplete and the functor V: Cis $\to$ Is has an adjoint Y. Naturally, we can construct directly coproducts and difference cokernels.

6. Isomorphism theorems

D e f i n i t i o n 6.1 *A* cis-morphism $f: (A, \{R_i : i \in I\}) \to (B, \{P_i : i \in I\})$ *will be called a strong factorization* (11) *or a strong cis-morphism if, for any* $i \in I$ *and for any* $x, y \in A$, $(f(x), f(y)) \in P_{if(A)}$ *implies* $(x, y) \in R_i$.

P r o p o s i t i o n 6.1 *All complex iterative systems with strong* cis-*morphisms form a category denoted by* Ciss. Ciss *is a subcategory of* Cis. *Moreover,* Ciss *is a balanced category, i.e. every bijection is an isomorfism.*

D e f i n i t i o n 6.2 *A relation* $K \subset A \times A$ *will be called left-permitted in the complex iterative system* $(A, \{R_i : i \in I\})$ *if, for any* $i \in I$, $KR_i \subset R_i$ *An equivalence relation will be called a congruence in* $(A, \{R_i : i \in I\})$ *if it is a left-permitted and a right-permitted relation.*

It is clear that the identical relation I is a congruence in every complex iterative system with the same set of states.

D e f i n i t i o n 6.3 Let $f: (A, \{R_i : i \in I\}) \to (B, \{P_i : i \in I\})$ *be a* cis-*morphism. The binary relation defined in* A *by* $K(f) = \{(x, y): f(x) = f(y)\}$ *will be called the kernel of the* cis-*morphism* f.

P r o p o s i t i o n 6.2 *For any* cis-*morphism* f, $K(f)$ *is an equivalence relation.* $K(f)$ *is a congruence iff* f *is a strong* cis-*morphism.*

The proof is easy.

Let $(A, \{R_i : i \in I\})$ be a complex iterative system and let E be an equivalence in A. If R'_i is the restriction of R_i to the quotient set A/E, for any $i \in I$, then the complex iterative system $(A/E, \{R'_i : i \in I\})$ will be called the quotient complex iterative system relating to E.

P r o p o s i t i o n 6.3 *The canonical mapping* $f: A \to A/E$ *is a* cis-*epimorphism in the quotient complex iterative system, for any equivalence* E; f *is a strong* cis-*epimorphism if and only if* E *is a congruence in* $(A, \{R_i : i \in I\})$.

Proof. First part is easy. For the second, let $\bar{x}, \bar{y}$ be two classes modulo E, and let be $(\bar{x}, \bar{y}) \in R'_i$. Then, for any $x \in \bar{x}$ and $y \in \bar{y}$, $(x, y) \in R_i$ iff E is a left-permitted and right-permitted relation.

P r o p o s i t i o n 6.4 (I-st isomorphism theorem). *Let* $f: (A, \{R_i : i \in I\}) \to (B, \{P_i : i \in I\})$ *be a* cis-*morphism, and let* $K(f)$ *be the kernel of* f. *If* $(A/K(f), \{R'_i : i \in I\})$ *is the quotient complex iterative system and* $(f(A), \{R_{if(A)} : i \in I\})$ *is the* f-*image complex iterative system then we have the commutative diagram :*

$$\begin{array}{ccc} (A, \{R_i : i \in I\}) & \xrightarrow{f} & (B, \{P_i : i \in I\}) \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle g} \\ (A/K(f), \{R'_i : i \in I\}) & \xrightarrow{\bar{f}} & (f(A), \{R_{if(A)} : i \in I\}) \end{array}$$

where h is the canonical cis-morphism, g is the inclusion of the f-image — that is a strong cis-monomorphism, and $\bar{f}$ is the bijection induced naturally by f. If f is a strong cis-morphism then $\bar{f}$ is an isomorphism in Ciss, and h is also in Ciss. Consequently, if f is in category Ciss then all above diagram is in Ciss.

Proof. We have the decomposition $f = g\bar{f}h$ in Ens. If f is a cis-morphism then g, $\bar{f}$ and h are cis-morphisms. The propositions 6.1, 6.2 and 6.3 complete the proof.

P r o p o s i t i o n 6.5 (II-ed isomorphism theorem). *Let* h *be an inclusion of* $(B, \{P_i : i \in I\})$ *into* $(A, \}R_i : i \in I\})$ *and let* E *be an equivalence in* A. *If* A/E *is considered as a subset of* A, *let* $(B', \{P''_i : i \in I\})$ *be the complex iterative system defined by* $B' = B \cap (A/E)$, $P''_i = P_{iB'}$, $i \in I$. *There is a* cis-*bijection* $f: (B/E, \{P'_i : i \in I\}) \to (B', \{P''_i : i \in I\})$. *If* E *is a congruence then* f *is a strong* cis-*isomorphism.*

Proof. Fundamentally $B' = \{\bar{x} : \bar{x} \in A/E,\ \bar{x} \cap B \neq \emptyset\}$ and $\bar{x} \in B/E_B$ iff $\bar{x} \in B'$. This bijection (in Ens) is a cis-morphism. As a matter of fact, we can apply the proposition 6.4 to the cis-morphism th, where $t: (A, \{R_i : i \in I\}) \to (A/E, \{R'_i : i \in I\})$ is the canonical cis-epimorphism. Then there is a cis-bijection $f = \overline{th}: (B/K(th), \{P'_i : i \in I\}) \to (th(B), \{R_{ith(B)} : i \in I\})$ and $K(th) = E_B$, $th(B) = B'$.

P r o p o s i t i o n 6.6 (III-rd isomorphism theorem). *Let* E *and* E' *be two equivalence relations in* A *and let* $(A, [R_i : i \in I\})$ *be a complex iterative system. If* $E' \subset E$ *then we have the following commutative diagram :*

$$\begin{array}{ccccc} (A, \{R_i : i \in I\}) & \xrightarrow{f'} & (E/A', \{R'_i : i \in I\}) & \xrightarrow{h} & ((A/E)/K(g), \{R''_i : i \in I\}) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle k} & \\ & & (A/E, \{R'_i : i \in I\}) & & \end{array}$$

where f, f' and h are canonical cis-epimorphisms and k is an isomorphism. If E is a congruence then the above diagram is in Ciss, i.e. all morphisms are strong.

Proof. If $E' \subset E$ and f', f are canonical cis-epimorphisms (in the corresponding quotient systems) then there is a mapping g (in Ens) such that $gf' = f$. It is easy to verify that g is a cis-epimorphism. Proposition 6.4 applied to the cis-epimorphism g prove that k is a cis-isomorphism. Now, if E is a congruence then f is a strong cis-epimorphism; $E' \subset E$ will be, too, a congruence and then f' and g are strong cis-epimorphisms. At the same time h and k will be strong cis-morphisms since $K(g)$ is a congruence in A/E'.

Remark. The final results can be transposed for the simple iterative systems. Also, the most of above results are independent of the kind of relations, and they may be formulated for n-ary relations.

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Received 1. XII. 1972.

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