REVUE D'ANALYSE NUMÉRIQUE ET DE LA THÉORIE DE L'APPROXIMATION, Tome 2, 1973, pp. 49–53

SUPPORTING SPHERES FOR FAMILIES OF SETS IN PRODUCT SPACES

by

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Let M be a given set in the n-dimensional Euclidean space R^n. We shall denote by conv M the convex hull of M, i.e. the intersection of all convex sets in R^n which contain the set M. In the following we need the notion of convexly connected sets introduced by O. HANNER and H. RADSTRÖM in [5].

Definition 1. *A set M in R^n is called convexly connected, if there is no hyperplane H such that $H \cap M = \emptyset$ and M contains points in both the open halfspaces determined by H.*

Remark. The notion of convexly connected sets is used in [8] also with an other meaning.

It is easy to verify that a connected set is also convexly connected and that the union of convexly connected sets having a point in common is convexly connected.

Definition. 2. *Let M be a set in R^n. A maximal convexly connected subset of M will be called a convexly connected component of M.*

O. Hanner and H. Radström have shown that there is a unique decomposition of a set M into convexly connected components. They have also proved the following result ([5], corollary 1): If $M \subset R^n$ is compact and has almost n convexly connected components, then for each point p in conv M there exist n (or fewer) points $a_1, a_2, \ldots, a_n$ of M such that p belongs to conv $\{a_1, a_2, \ldots a_n\}$.

H. KRAMER and A. B. NÉMETH have given in [6] the following two definitions:

Definition 3. *There will be said that the family F of sets in a metric space E has a supporting sphere, if there is a sphere S in E having* *common points with each member of the family F and the interior of S contains no point of any member of F.*

Definition 4. *A family F of sets in* R^n *is said to be independent if for any* $n+1$ *pairwise distinct members* $K_1, K_2, \ldots, K_{n+1}$ *of F any set of points* $p_1, p_2, \ldots, p_{n+1}$, *where* $p_i \in K_i$, $i = 1, 2, \ldots, n+1$, *determines a simplex of dimension n, or equivalently the vectors* $p_2 - p_1, p_3 - p_1, \ldots, p_{n+1} - p_1$ *are linearly independent. An equivalent condition is:* $p_1, p_2, \ldots, p_{n+1}$ *are in general position* (see [3], p. 12 definition 5).

A family F of sets is independent if and only if there is no hyperplane which intersects $n+1$ (or more) members of F.

THEOREM 1. *Let* $F = \{K_i : i \in I\}$ *be a family of independent compact convexly connected sets in* R^n. *Then*

$$F_{conv} = \{\text{conv } K_i : i \in I\}$$

is a family of independent compact sets.

Proof. The sets K_i are compact by the theorem of Mazur (see for instance [7], theorem 3.2.18). Suppose now that the family F_{conv} is not independent. Then there exist $n+1$ members of F_{conv}, say conv K_i, $i = 1, 2, \ldots, n+1$, which are not independent. Let be H a hyperplane for which $H \cap \text{conv } K_i \neq \emptyset$, $i = 1, 2, \ldots, n+1$. Let be $a_i \in H \cap \text{conv} K_i$, $i = 1, 2, \ldots, n+1$. Because K_i is convexly connected and a_i is in conv K_i, by the theorem of O. Hanner and H. Radström there are n points $a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}$ in K_i such that

$$(1) \qquad a_i \in \text{conv } \{a_j^{(i)} : j = 1, 2, \ldots, n\}.$$

We have also $a_i \in H$. Let us now suppose $H \cap K_i = \emptyset$. The hyperplane H determines in R^n two open halfspaces H' and H''. By (1) and by our supposition $H \cap K_i = \emptyset$ follows then $H' \cap K_i \neq \emptyset$ and $H'' \cap K_i \neq \emptyset$.

This contradicts the property of K_i to be convexly connected. Therefore $H \cap K_i \neq \emptyset$, $i = 1, 2, \ldots, n+1$. This is in contradiction with the independence of the family F. That completes the proof of our Theorem.

Remark. The hypothesis that the sets K_i are convexly connected is essential. Let be a_i, $i = 1, 2, \ldots, n$, n linearly independent points in R^n and let be H the hyperplane determined by these n points. Choose 2 points b_1 and b_2 on opposite sides relative to H. Consider now $K_i = \{a_i\}$, $i = 1,2, \ldots, n$ and $K_{n+1} = \{b_1, b_2\}$. Of course $\{K_1, K_2, \ldots, K_{n+1}\}$ is a family of independent sets, but this cannot be said about the family $\{\text{conv } K_1, \ldots, \text{conv } K_{n+1}\}$.

In the proof of theorem 1 it is not necessary to use the theorem of Hanner-Radström. The proof can be made also with the well-known theorem of Caratheodory on the convex hull of a compact set (see [1]).

Lemma 1. *Let be* $K_1, K_2, \ldots, K_{n+1}$ *independent convex sets in* R^n, *and let be* $a_i, b_i \in K_i$, $i = 1, 2, \ldots, n+1$. *Then*
conv $\{a_1, a_2, \ldots, a_{n+1}\} \cap$ conv $\{b_1, b_2, \ldots, b_{n+1}\} \neq \emptyset$.

Proof. Let us suppose

$$\text{conv } \{a_1, a_2, \ldots, a_{n+1}\} \cap \text{conv } \{b_1, b_2, \ldots, b_{n+1}\} = \emptyset.$$

By the theorem on the separation of polyhedra (see [7], p. 68 theorem 2.12.9) the two simplexes conv $\{a_1, a_2, \ldots, a_{n+1}\}$ and conv $\{b_1, b_2, \ldots, b_{n+1}\}$ can be separated by a hyperplane H. From the convexity of K_i, from $a_i, b_i \in K_i$ and from the fact that a_i and b_i are on opposite sides relative to H, follows $H \cap K_i \neq \emptyset$, in contradiction with the independence of the family $\{K_1, \ldots, K_{n+1}\}$ This completes the proof of the lemma.

Lemma 2. *Let be* C_i, $i = 1, 2, \ldots, n+1$ *independent compact convexly connected sets in* R^n. *If* $a_i, b_i \in \text{conv } C_i$, $i = 1, 2, \ldots, n+1$ *then:* conv $\{a_1, a_2, \ldots, a_{n+1}\} \cap$ conv $\{b_1, b_2, \ldots, b_{n+1}\} \neq \emptyset$.

By theorem 1 follows the independence of the sets conv C_i, $i = 1, 2, \ldots, n+1$. Then lemma 2 follows immediately from lemma 1.

The following theorem was proved in [6] by H. KRAMER and A. B. NÉMETH:

THEOREM 2. *Let be* $K_1, K_2, \ldots, K_{n+1}$ *independent convex and compact sets in* R^n. *Then this family admits one and only one supporting sphere.*

Let now E_1, E_2 be two metric spaces with the distances d_1, d_2 on E_1 respectively E_2. In $E_1 \times E_2$ can be defined two distances d' and d'' in the following way: For any pair of points $x = (x_1, x_2)$, $y = (y_1, y_2)$ let:

$$d'(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

and

$$d''(x, y) = ((d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2)^{1/2}.$$

Denote by E' respectively E'' the corresponding metric spaces.

THEOREM 3. *Let* $F_1 = \{A_i : i \in I\}$ *and* $F_2 = \{B_j : j \in J\}$ *be families of compact sets in the metric space* E_1 *respectively* E_2. *The family of compact sets* $F_{12} = \{A_i \times B_j : i \in I, j \in J\}$ *in* E' (respectively E'') *has a supporting sphere if and only if the family* F_1 *has a supporting sphere in* E_1 *and the family* F_2 *has a supporting sphere in* E_2.

Proof. We shall prove our theorem only in the case when F_{12} is considered as a family of compact sets in E''. The other case may be treated in the same way. The compactness of the members of F_{12} in E'' (respectively in E') follows from ([2] p. 72, theorem 3.20.16).

Necessity. Suppose that F_{12} has a supporting sphere in E'', i.e. there is a point (a_1, a_2) in $E_1 \times E_2$ such that

$$d''((a_1, a_2), A_i \times B_j) = d''((a_1, a_2), A_k \times B_h)$$

for each $i, k \in I$ and $j, h \in J$. We have then in particular

$$d''((a_1, a_2), A_i \times B_j) = d''((a_1, a_2), A_k \times B_j) \tag{2}$$

for a fixed j in J and for all i, k in I. As

$$d''((a_1, a_2), A_i \times B_j) = \min\{d''((a_1, a_2), (x_1, x_2)) : (x_1, x_2) \in A_i \times B_j\} = \min\{(d_1^2(a_1, x_1) + d_2^2(a_2, x_2))^{1/2} : (x_1, x_2) \in A_i \times B_j\} = ((\min\{d_1(a_1, x_1) : x_1 \in A_i\})^2 + (\min\{d_2(a_2, x_2) : \in B_j\})^2)^{1/2} = (d_1^2(a_1, A_i) + d_2^2(a_2, B_j))^{1/2}.$$

We can then write (2) under the form :

$$d_1^2(a_1, A_i) + d_2^2(a_2, B_j) = d_1^2(a_1, A_k) + d_2^2(a_2, B_j).$$

Hence $d_1(a_1, A_i) = d_1(a_1, A_k)$ for all i, k in I, i.e. the family F_1 has a supporting sphere with the center a_1. The same reasoning can be applied to show the existence of a supporting sphere for the family F_2 with the center a_2.

Sufficiency. Suppose that F_i has a supporting sphere in E_i with the center a_i, $i = 1, 2$. Then we have

$$d_1(a_1, A_i) = d_1(a_1, A_k) \quad \text{for all } i, k \text{ in } I$$

and

$$d_2(a_2, B_j) = d_2(a_2, B_h) \quad \text{for all } j, h \text{ in } J.$$

Hence $d''((a_1, a_2), A_i \times B_j) = d''((a_1, a_2), A_k \times B_h)$ for all i, k in I and j, h in J, i.e. the family F_{12} has a supporting sphere in E''.

THEOREM 4. *Let $F_m = \{C_1, C_2, \ldots, C_{m+1}\}$ be a family of independent compact convexly connected sets in R^m and $F_n = \{D_1, D_2, \ldots, D_{n+1}\}$ a family of independent compact and convexly connected sets in R^n. Then the family*

$$F = \{\text{conv } C_i \times \text{conv } D_j : i = 1, 2, \ldots, m+1 ; j = 1, 2, \ldots, n+1\}$$

has one and only one supporting sphere in R^{m+n}.

Proof. By theorem 1 follows the independence in the space R^m of the family $F_{m,\text{conv}} = \{\text{conv } C_i : i = 1, 2, \ldots, m+1\}$ and the independence in the space R^n of the family $F_{n,\text{conv}} = \{\text{conv } D_j : j = 1, \ldots, n+1\}$. By theorem 2 $F_{m,\text{conv}}$ has a unique supporting sphere in R^m and $F_{n,\text{conv}}$ has a unique supporting sphere in R^n. The existence and the unicity of a supporting sphere for the family F follows then from theorem 4.

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Received 29. XI. 1972.

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