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SOME APPROXIMATION THEOREMS

by

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1. We denote by $C_0[a, b]$, $a \leqslant 0 \leqslant b$, the subspace of $C[a, b]$ formed with all functions $f \in C[a, b]$ for which $f(0) = 0$. Let $B[a, b]$ be the space of all functions $f: [a, b] \to R$ which are bounded on $[a, b]$. Both spaces are considered to be normed by means of $\|\cdot\| = \sup_{t \in [a,b]} |\cdot(t)|$.

In this paper we establish some results about the uniform approximation of the elements f from $C_0[a, b]$ by $L_n f$, where L_n are linear operators $C_0[a, b] \to B[a, b]$.

2. The following notations and definitions are used: for $j = 1, 2, \ldots$ let $e_j \in C_0[a, b]$ be defined as

$$e_j(t) = t^j, \quad t \in [a, b].$$

Definition 1. *A function $f \in C_0[a, b]$ is called starshaped of the order k, on $[a, b]$, if for each system of $k+1$ distinct points*

$$x_1, x_2, \ldots, x_{k+1}, x_j \in [a, b] \setminus \{0\}, j = 1, 2, \ldots, k+1,$$

the following inequality is valid

$$[0, x_1, x_2, \ldots, x_{k+1}; f] \geqslant 0, \tag{1}$$

where $[t_1, t_2, \ldots, t_{k+2}; h]$ is the divided difference of a function h at the points $t_1, t_2, \ldots, t_{k+2}$. Let us denote by $\mathcal{S}_k[a, b]$ the set of all starshaped functions of order k.

Definition 2. *We say that* $f\colon [a, b] \to R$ *belongs to the class* $B_k[a, b]$ *if:*

1. $f \in C_0[a, b]$

2. *there exist two numbers* m_f, M_f *such that*

$$m_f \leqslant [0, x_1, x_2, \ldots, x_{k+1}; f] \leqslant M_f \tag{2}$$

for any $k+1$ *distinct points from* $[a, b] \setminus \{0\}$.

Lemma 1. *Let* $L_n\colon C_0[a, b] \to B[a, b]$, $n = 1, 2, \ldots$, *be a sequence of linear operators which map any starshaped function of order* k *into a non-negative function. Then for each* $f \in B_k[a, b]$ *and* $n = 1, 2, \ldots$

$$m_f(L_n e_{k+1})(x) \leqslant (L_n f)(x) \leqslant M_f(L_n e_{k+1})(x), \ x \in [a, b]. \tag{3}$$

Proof. Let $f \in B_k[a, b]$ and $m_f, M_f \in R$ such that (2) is verified. If we define $g_j \in C_0[a, b]$, $j = 1, 2$, by

$$\begin{aligned} g_1(t) &= f(t) - m_f t^{k+1}, \ t \in [a, b] \\ g_2(t) &= M_f t^{k+1} - f(t), \ t \in [a, b] \end{aligned} \tag{4}$$

then we have

$$[0, x_1, x_2, \ldots, x_{k+1}; g_j] \geqslant 0, \ j = 1, 2,$$

for any distinct points $x_1, x_2, \ldots, x_{k+1}$ from $[a, b] \setminus \{0\}$. Thus g_1 and g_2 are starshaped of order k on $[a, b]$.

According to our hypothesis

$$(L_n g_j)(x) \geqslant 0, \ j = 1, 2, \ n = 1, 2, \ldots, \ x \in [a, b].$$

On the other hand these two inequalities are equivalent with

$$(L_n f)(x) \geqslant m_f(L_n e_{k+1})(x)$$

and

$$(L_n f)(x) \leqslant M_f(L_n e_{k+1})(x)$$

Lemma 2. *If* $L_n\colon C_0[a, b] \to B[a, b]$, $n = 1, 2, \ldots$, *are such that*

$$L_n g \geqslant g, \ \text{on} \ [a, b]$$

for each $g \in \mathcal{S}_k[a, b]$, *then on* $[a, b]$ *the following inequalities are true*

$$m_f(L_n e_{k+1} - e_{k+1}) \leqslant L_n f - f \leqslant M_f(L_n e_{k+1} - e_{k+1}). \tag{5}$$

Proof. Let $f \in B_k[a, b]$ and $g_1, g_2 \in C_0[a, b]$ be defined as in (4). Then $g_j \in \mathcal{S}_k[a, b]$, that is

$$L_n g_j \geqslant g_j, \ j = 1, 2, \ n = 1, 2, \ldots$$

which are equivalent with

$$M_f L_n e_{k+1} - L_n f \geqslant g_2$$

and

$$L_n f - m_f L_n e_{k+1} \geqslant g_1.$$

By using (4) we obtain

$$L_n f - f \leqslant M_f(L_n e_{k+1} - e_{k+1})$$

and

$$L_n f - f \geqslant m_f(L_n e_{k+1} - e_{k+1})$$

which completes the proof.

3. Further we try to find a subset $H \subset C_0[a, b]$ so that the pointwise convergence of certain sequences of linear operators on H to the zero operator (or to the identity operator) imply the pointwise convergence to the same operator, on the whole space $C_0[a, b]$.

THEOREM 3. *Let* $L_n\colon C_0[a, b] \to B[a, b]$, $n = 1, 2, \ldots$, *a sequence of linear operators such that*

a. $g \in \mathcal{S}_k[a, b]$ *implies* $L_n g \geqslant 0$, on $[a, b]$, $n = 1, 2, \ldots$

b. $\lim\limits_{n \to \infty} \| L_n e_{k+1} \| = 0$

c. $(L_n)_{n=1}^{\infty}$ *is uniformly bounded.*

Then

$$\lim_{n \to \infty} \| L_n f \| = 0 \ \textit{for every} \ f \in C_0[a, b].$$

Proof. Taking into account that $e_j \in B_k[a, b]$, there exist two sequences $(m_j)_{j=1}^{\infty}$, $(M_j)_{j=1}^{\infty}$, with the properties

$$m_j \leqslant [0, x_1, x_2, \ldots, x_{k+1}; e^j] \leqslant M_j, \ j = 1, 2, \ldots$$

for any distinct points $x_1, x_2, \ldots, x_{k+1}$ from $[a, b] \setminus \{0\}$.

Since $e_{k+1} \in \mathcal{S}_k[a, b]$, we have

$$L_n e_{k+1} \geqslant 0 \text{ on } [a, b], \ n = 1, 2, \ldots.$$

Let $c_j = \max\{|m_j|, |M_j|\}$, $j = 1, 2, \ldots$. Then the inequalities (3) imply

$$|L_n e_j| \leqslant c_j |L_n e_{k+1}| \leqslant c_j ||L_n e_{k+1}||.$$

Therefore

$$(6) \qquad ||L_n e_j|| \leqslant c_j ||L_n e_{k+1}||$$

and

$$\lim_{n \to \infty} ||L_n e_j|| = 0, \ j = 1, 2, \ldots.$$

Now we use the fact that the set of polynomials vanishing at $t = 0$ is dense in $C_0[a, b]$. The boundedness of the sequence $(L_n)_{n=1}^{\infty}$ implies the pointwise convergence on $C_0[a, b]$ to the zero operator.

THEOREM 4. *If* $L_n : C_0[a, b] \to B[a, b]$, $n = 1, 2, \ldots$, *is a sequence of linear operators with the properties*

a. $g \in \mathcal{S}_k[a, b]$ *implies* $L_n g \geqslant g$, $n = 1, 2, \ldots$

b. $\lim_{n \to \infty} ||L_n e_{k+1} - e_{k+1}|| = 0$

c. *there exist a positive number* M *such that for every* $n = 1, 2, \ldots$

$$||L_n|| \leqslant M$$

then for every $f \in C_0[a, b]$

$$\lim_{n \to \infty} ||L_n f - f|| = 0.$$

The *Proof* is similar with the above reason. Lemma 2 enables us to write instead of (6)

$$(6') \qquad ||L_n e_j - e_j|| \leqslant c_j ||L_n e_{k+1} - e_{k+1}||$$

which asserts that for every $f \in C_0[a, b]$ the sequence $(L_n f)_{n=1}^{\infty}$ converges uniformly to the function f.

As a special case of the theorem 3 we obtain a result which was established in [3, Theorem II]. This may be formulated in the following way:

Let $W_n : C_0[a, b] \to B[a, b]$, $n = 1, 2, \ldots$, defined by

$$(W_n f)(x) = \int_a^b f(t)\, dw_n(t, x)$$

where

a. $w_n(t, .) \in C_0[a, b]$, $t \in [a, b]$

b. for an arbitrary $x \in [a, b]$, $w_n(., x)$ is a signed measure.

Corollary. *If:*

(i) $$W_n e_1 = 0, \ n = 1, 2, \ldots, \ \lim_{n \to \infty} ||W_n e_2|| = 0.$$

(*ii*) $$(W_n)_{n=1}^{\infty} \text{ is uniformly bounded}$$

(iii) $$\int_u^b t\, dw_n(t, x) \geqslant 0, \ u \in]0, b], \ x \in [a, b], \ n = 1, 2, \ldots$$

and

(iv) $$\int_a^u t\, dw_n(t, x) \leqslant 0, \ u \in [a, 0[, \ x \in [a, b], \ n = 1, 2, \ldots$$

then for every $f \in C_0[a, b]$

$$\lim_{n \to \infty} ||W_n f|| = 0.$$

Proof. Ideed, by using Theorem 2.6 from [1] one obtains that for every $g \in \mathcal{S}_1[a, b]$ and $n = 1, 2, \ldots$

$$W_n g \geqslant 0 \text{ on } [a, b].$$

By applying the theorem 3 from this paper, with $k = 1$, we conclude that $\lim_{n \to \infty} ||W_n f|| = 0$, $f \in C_0[a, b]$.

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