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APPROXIMATION THEORY AND IMBEDDING PROBLEMS

by

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0. Denote by Q a compact metric space (a compactum) and let be $C(Q)$ the linear space of the real valued continuous functions on Q, endowed with the sup norm.

Some problems in the approximation theory in the space $C(Q)$ are closely related to topological properties of the compactum Q. The present note aims to pointed aut this relation by examples in various fields of the approximation theory. It has merely an expository character, containing the interpretations from the point of view of the approximation theory of some results of topological character. We remark a partial overlapping of the points 1 and 2 of our note and the note of YU. A. ŠAŠKIN [23].

In the point 1 the connection between the Weierstrass-Stone theorem and the imbedding of Q in Euclidean spaces is considered. The point 2 deals with the existence of Korvkin systems of functions and the imbedding of Q in topological spheres. The point 3 contains results concerning the topological characterization of Q in the case when $C(Q)$ contains subspaces of a given Chebyshevian rank. In the point 4 the existence of Chebyshev subspaces of a given Chebyshev space is considered.

In all what follows we will suppose that the compactum Q has finite topological dimension.

1. Dense subalgebras in C(Q) and imbedding of Q in Euclidean spaces

Let be A a set in $C(Q)$. Suppose that for all $x_1, x_2 \in Q$, such that $x_1 \neq \neq x_2$ there is an $f \in A$ such that $f(x_1) \neq f(x_2)$. Then we say that A is a *separating family of functions on* Q, or that A *separates* Q.

The set A in $C(Q)$ is said to be an *algebra*, if it is a linear subspace of $C(Q)$ with the property that if $f, g \in A$, then $fg \in A$.

We give here the following version of the WEIERSTRASS-STONE theorem ([7] (7.37), p. 98):

THEOREM *A subalgebra A of $C(Q)$ that separates points and vanishes identically at no point of Q is dense in $C(Q)$.*

We ask about a minimally generated subalgebra A in $C(Q)$, having the property that it is dense in $C(Q)$, i.e., about an algebra A with the minimal number of generators having the property in the above theorem. Denote this minimal number of generators by $w(Q)$. It is easy to show that $w(Q)$ is a topological invariant of Q and it is the minimal dimension of the Euclidean space in which Q may be imbedded. Thus we have (see [23]):

THEOREM 1. *The compactum Q may be imbedded in the Euclidean space $\mathbf{R}^n$ if and only if $w(Q) \leq n$.*

This immediate consequence of the Weierstrass-Stone theorem, gives by comparation with the NÖBELING-PONTRIAGIN'S imbedding theorem (see for ex. in [15]) the

Corollary 1. *If the compactum Q has the topological dimension m, then $w(Q) \leq 2m + 1$, and the equality holds if Q cannot be imbedded in the space $\mathbf{R}^{2m}$.*

It is clear that other results in the imbedding theory can be similarly interpreted in the terms of the invariant w, introduced here.

2. Korovkin spaces of minimal dimension in C(Q) and the imbedding of Q in topological spheres

Denote by $B(Q)$ the linear space of the real valued, bounded functions on Q, endowed with the sup norm.

The linear operator $L: C(Q) \to B(Q)$ is said to be *positive* if for each $f \in C(Q)$ with the property $f(x) \geq 0$ for any $x \in Q$, the element $g = Lf$ in $B(Q)$ has the same property, i.e., $g(x) \geq 0$ for any x in Q.

If the space A in $C(Q)$ has the property that for any sequence (L_i), $i = 1, 2, \ldots,$ of linear and positive operators L_i: $C(Q) \to B(Q)$, whose restrictions to A converges on A to the identical operator of A, it follows that this sequence converges on $C(Q)$ to the identical operator of this space, then we say that A is a *Korovkin space* or a *K-space*. A basis of a *K*-space is somentimes called a *Korovkin system*.

Denote by $m(Q)$ the minimal dimension of a K-space in $C(Q)$. From the Theorems 3 and 2″ in [20] of YU. A. ŠAŠKIN, it follows the

THEOREM 2. *The compactum Q may be imbedded in the topological sphere S^n if and only if $m(Q) \leq n + 2$.*

$m(Q)$ is a topological invariant of Q and we have by a comparation with the Nöbeling-Pontreagin imbedding theorem the

Corollary 2. *If the topological dimension of the compactum Q is m, then we have $m(Q) \leq 2m + 3$ and the equality holds if Q cannot be imbedded in the space $\mathbf{R}^{2m}$.*

By a comparation of theorems 1 and 2 we obtain the

Corollary 3. $m(Q) = w(Q) + 1$ *if Q is a topological sphere, and $m(Q) = w(Q) + 2$ otherwise.*

A different variant of Theorem 2 can be obtained by the application of the notion of Choquet boundary ([14]) of a subspace in the space $C(Q)$ (see [23]):

THEOREM 2′. *The compactum Q can be imbedded in S^n if and only if the minimal dimension of subspaces in $C(Q)$ which have Choquet boundaries all the compactum Q, is $n + 2$.*

A similar interpretation holds for the Corollary 2.

3. Topological characterization of the domain of definition of the Chebyshev spaces

The n-dimensional linear subspace F of the space $C(Q)$ is said to form a *Chebyshev space of the rank $n - k$* $(1 \leq k \leq n)$ if the set of elements of best approximation in F to any element $f \in C(Q)$ has the dimension $\leq$ $\leq n - k$.

The following theorem due to G. S. RUBINŠTEIN [16] is a generalization of the well known theorem of A. HAAR [6].

THEOREM *The n-dimensional linear subspace F of the space $C(Q)$ is a Chebyshev space of the rank $n - k$ if and only if each set of $n - k + 1$ linearly independent functions in F has at most $k - 1$ common zeros in Q.*

The subset M of the n-dimensional Euclidean space $\mathbf{R}^n$ is said to be *k-vectorial-idependent* $(1 \leq k \leq n)$, if each set of k distinst vectors of M is linearly independent.

By a simple algebraic reasoning, it may be seen (see [2]) that *an n-dimensional subspace F in $C(Q)$ spanned by the elements $\varphi_1, \ldots, \varphi_n$ is a Chebyshev space of the rank $n - k$ $(1 \leq k \leq n)$ if and only if the mapping $\Phi: Q \to \mathbf{R}^n$ defined by*

$$(*) \qquad \Phi: x \mapsto (\varphi_1(x), \ldots, \varphi_n(x)),$$

is an imbedding of the compactum Q in a k-vectorial-idenpendent set of $\mathbf{R}^n$.

This proposition which makes a connection between the notion of Chebyshev spaces in $C(Q)$ and that of k-vectorial-independent sets in $\mathbf{R}^n$, constitutes a first step in the topological characterization of the compactum Q which has the property that $C(Q)$ contains Chebyshev spaces of dimension n and of the rank $n - k$. In order to obtain more information of topological character about Q, it suffices, according to the above proposition to investigate the compact, k-vectorial-independent sets in $\mathbf{R}^n$. The results in the literature are formulated either directly, in terms of the Chebyshev spaces, or in the terms of the k-vectorial-independent sets in $\mathbf{R}^n$ (respectively, k-regular or k-independent sets, notions which are closely related to the notion of k-vectorial-independence). The first result in this direction is due to J. Mairhuber [10] and concerns the topological characterization of Q in the case when $C(Q)$ contains Chebyshev spaces of rank 0 and of dimension $\geqq 2$.

THEOREM 3. *The space $C(Q)$ contains Chebyshev spaces of dimension $n \geqq 2$ anf od the rank 0 if and only if Q may be imbedded (i) in S^1 for odd n, and (ii) in $I = [0, 1]$ for n even.*

Other proofs of this theorem were given and other aspects of the considered problem were investigated by J. SIECKLUKI [19], P. C. CURTIS [4], J. A. LUTTS [9], I. J. SCHOENBERG and C. T. YANG [18], C. B. DUNHAM [5], YU. A. ŠAŠKIN [21].

The problem of a similar characterization of the compactum Q in the case of the existence in $C(Q)$ of a subspace which is a Chebyshev space of the rank different from 0, as far as we know is open. It was conjectured (see in [25]) that the following theorem holds:

Imbedding conjecture *If $C(Q)$ contains a Chebyshev space of dimension n and of the rank $n - k$ $(1 \leqslant k \leqslant n)$, then Q can be imbedded in S^{n-k+1}.*

The conjecture is trivial for $k = 1, 2$ and contains Theorem 3 for $k = n$. A weakener imbedding theorem concerning k-vectorial-independent sets in $\mathbf{R}^n$ was obtained by K. BORSUK [3]. It can be formulated in the terms of the Chebyshev spaces as Follows:

THEOREM 4. *If in $C(Q)$ it exists a Chebyshev space of dimension n and of the rank $n - k$ $(2 \leqslant k \leqslant n)$, and if U is an open subset of Q which contains at least $k - 2$ distinct points, then $Q \setminus U$ can be imbedded in $\mathbf{R}^{n-k+1}$.*

A particular case of the conjecture was proved in our paper [13]. It can be formulated as follows:

THEOREM 5. *If $C(Q)$ contains a Chebyshev space of dimension n and of the rank $n - 3$ and if Q contains an $(n - 2)$-dimensional cell, then Q can be imbedded in S^{n-2}.*

This theorem can be formulated and formally proved for the case when $n - 3$ is changed in $n - k$ $(1 \leq k \leq n)$, but the existence in Q of a cell of dimension $n - k + 1$ restricts k to be ≤ 3 or $= n$ according to a result of S. S. RYŠKOV [17]. This result may be formulated in the terms of Chebyshev spaces as follows:

THEOREM 6. *If $C(Q)$ contains a Chebyshev space of dimension n and of the rank $n - k$ $(k > 1)$, and Q contains an m-dimensional cell, then,*

$$m \leqslant \frac{n - 1 - \left[\frac{k-1}{2}\right]}{\left[\frac{k}{2}\right]}$$

From this theorem it follows also, that for great n, and k „far'' from the endpoints of the sequence $2, 3, \ldots, n$, the Imbedding conjecture is weakener as the theorem of Nöbeling-Pontreagin. Therefore it follows that the Imbedding conjecture — even in the case if it is true — cannot

be considered to be a complete characterization of Q in case when $C(Q)$ contains Chebyshev spaces.

The imbedding theorem of V. G. BOLTEANSKIĬ [1] concerning the imbedding of a compactum in a k-vectorial-independent set of the Euclidean space can be formulated as follows:

THEOREM 7. *For each n-dimensional compactum Q there exist $(n+1) \times (k + 1)$ — dimensional subspaces in $C(Q)$ which are Chebyshev spaces of the rank $(n + 1)(k + 1) - k - 1$.*

From the same paper of Bolteanskiĭ it follows also that the set of $(n + 1)(k + 1)$ — dimensional Chebyshev spaces of the rank $(n+1) \times (k + 1) - k - 1$, is dense in the set of all $(n + 1)(k + 1)$-dimensional spaces in $C(Q)$, after introducing of a suitable topology in this set.

For other consequences of the theorems of RYŠKOV and of BOLTEANSKIĬ see [2], [11], [22].

4. Chebyshev subspaces of a given Chebyshev space

Let be given an n-dimensional Chebyshev space in $C(Q)$ of the rank $n - k$. We ask for the Chebyshev subspaces of it of the same rank. This problem is in a strong connection with the imbedding by projections of the k-vectorial-independent sets in R^n. We have the following theorem [12]:

THEOREM 8. *The n-dimensional Chebyshev space of the rank $n - k$ spanned by the functions $\varphi_1, \ldots, \varphi_n$ of the space $C(Q)$, has a subspace of dimension $n - s$ $(k \geqq s + 2)$ which is a Chebyshev space of the same rank, if and only if the compact set $\Phi(Q)$ (where Φ is the mapping defined by (*)), which is k-vectorial-independent set in $\mathbf{R}^n$, may be projected in a $(k - s)$-vectorial-independent set of an $(n - s)$-dimensional subspace $\mathbf{R}^{n-s}$ of $\mathbf{R}^n$, and this projection is one to one.*

In the same paper [12] a necessary and sufficient condition is given in order to a k-vectorial-independent set in $\mathbf{R}^n$ admits a projection as in Theorem 8. This result may be interpreted in the theory of Chebyshev spaces as follows:

THEOREM 9. *The n-dimensional Chebyshev space of the rank $n - k$ $(1 \leq k \leq n)$ spanned by the functions $\varphi_1, \ldots, \varphi_n$ of the space $C(Q)$ has an $(n - s)$-dimensional Chebyshev subspace $(k \geqq s + 2)$ of the same rank, if n d only if there exist s vectors $b_i = (b_i^1, \ldots, b_i^n)$, $i = 1, \ldots, s$ such that he matrix*

$$\left\| \begin{matrix} b_1^1 & b_1^2 & \ldots & b_1^n \\ \cdot & \cdot & \cdot & \cdot \\ b_s & b_s^2 & \ldots & b_s^n \\ \varphi_1(x_1) & \varphi_2(x_1) & \ldots & \varphi_n(x_1) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_1(x_{k-s}) & \varphi_2(x_{k-s}) & \ldots & \varphi_n(x_{k-s}) \end{matrix} \right\|$$

has the rank k, for each set of $k - s$ distinct points $x_1, \ldots, x_{k-s}$ in Q.

Suppose that $y_1, \ldots, y_s$ are some distinct points outside to Q and extend the functions φ_i to y_j by setting $\varphi_i(y_j) = b_j^i\ i = 1, \ldots, n, j = 1, \ldots s$. Then the condition of the theorem contains the possibility of the extension of the functions in the above space with the preserving of the property to form Chebyshev space of the rank $n - k$ for $s = 1$, and having a property somewhat weakener then that to be a Chebyshev space of the rank $n - k$, for $s \geqq 2$.

From this theorem it follows that some exemples of Chebyshev spaces, which were constructed in the papers of V. I. VOLKOV [24] and YU. A. ŠAŠKIN [20], do not have Chebyshev subspaces of a given dimension and of the same order. In the first of the above cited papers, V. I. Volkov has constructed a 3-dimensional Chebyshev space of the rank 0, which cannot be extended to any point with the preserving of the property to form Chebyshev space of rank 0, and therefore it contains no 2-dimensional Chebyshev subspace of the rank 0. This fact was firstly observed by other considerations by J. KIEFER and J. WOLFOWITZ [8]. A geometrical method of constructing Chebyshev spaces with this property was presented in the paper [11]*.

In the point 4.2 of the paper of YU. A. ŠAŠKIN [20], a Chebyshev space of dimension 4 and of the rank 1 is constructed, which cannot be extended to a point with preserving of the property to be Chebyshev space of order 1, and therefore, according Theorem 9, it cannot contain any Chebyshev subspace of the dimension 3 and of the rank 1.

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*) Constructions of Chebyshev spaces of dimension n whithout Chebyshev subspaces of dimension $n-1$ for $n > 3$, were announced by Roland Zielke at the Colloquy of constructive function theory, held in Cluj between 6—12 September 1973.

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