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ON THE DIVIDED DIFFERENCES

by

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In the present paper some properties and examples for divided differences of the mappings in the linear normed spaces are given.

We recall the definition of the divided differences of a mapping P of the linear normed space X into the linear normed space Y.

Definition. *The linear and continuous mapping* $P_{u,v}$ *(i.e.* $P_{u,v} \in \mathfrak{L}(X, Y)$*) defined on* X *with values in* Y *is a divided difference of* P *in the distinct points* u, v *of* X, *if*

$$P_{u,v}(u - v) = P(u) - P(v). \tag{1}$$

We note that the study of the existence of divided differences is given in our previous paper [2].

In the general case, the divided differences of a mapping of the linear normed space X into the linear normed space Y are not unique. For illustration we give some propositions and examples.

Proposition 1. *Let be* B *a bilinear mapping, i.e.* $B \in \mathfrak{L}(Y, \mathfrak{L}(Y, Z))$, P *and* Q *mappings of* X *into* Y *and* $P_{u,v}$, *and* $Q_{u,v}$ *divided differences of* P *respectively* Q. *Then the linear mapping*

$$R_{u,v} = B[P_{u,v}\,;\, Q(u)] + B[P(v)\,;\, Q_{u,v}]$$

is a divided difference of the mapping $R = B(P\,;\, Q)$ *of* X *into* $\mathfrak{L}(Y, Z)$. *The mapping*

$$\tilde{R}_{u,v} = B[P(u)\,;\, Q_{u,v}] + B[P_{u,v}\,;\, Q(v)]$$

is also a divided difference of R, [1].

Proof. Evidently $R_{u,v} \in \mathfrak{L}(X, Z)$ and we have

$$R_{u,v}(u - v) = B[P_{u,v}(u - v)\,;\, Q(u)] + B[P(v)\,;\, Q_{u,v}(u - v)] =$$
$$= B[P(u)\,;\, Q(u)] - B[P(v)\,;\, Q(u)] + B[P(v)\,;\, Q(u)] -$$
$$- B[P(v)\,;\, Q(v)] = B[P(u)\,;\, Q(u)] - B[P(v)\,;\, Q(v)].$$

The proof of the condition (1) for the $\tilde{R}_{u,v}$ is analogous.

We can generalize this Proposition, for a multilinear mapping M.

Proposition 2. *If $P^{(i)}_{u,v}$, for $i = 1, 2, \ldots, n$, are divided differences of the mappings $P^{(i)} \in \mathfrak{L}(X, Y)$ for $i = 1, 2, \ldots, n$ then*

$$T_{u,v} = M[P^{(1)}_{u,v}\,;\, P^{(2}(u)\,;\, \ldots\,;\, P^{(n)}(u)] + M[P^{(1)}(v)\,;\, P^{(2)}_{u,v}\,;\, P^{(3)}(u)\,;$$
$$\ldots\,;\, P^{(n)}(u)] + \ldots + M[P^{(1}(v)\,;\, \ldots\,;\, P^{(n-1)}(v)\,;\, P^{(n)}_{u,v}]$$

and

$$\tilde{T}_{u,v} = M[P^{(1)}_{u,v}\,;\, P^{(2)}(v);\ldots\,;\, P^{(n)}(v)] + M[P^{(1)}(u);\, P^{(2)}_{u,v};\, P^{(3)}(u)\,;$$
$$\ldots,;\, P^{(n)}(v)] + \ldots + M[P^{\,\prime}\,(u);\, P^{(2)}(u);\ldots\,;\, P^{(n-1)}(u)\,;\, P^{(n)}_{u,v}]$$

are divided differences of $M = [P^{(1)}, \ldots, P^{(n)}]$, [1].

Proof. The Proposition is proved as the preceding Proposition.

Example 1. Let $X \subset R^n$, $Y = R$, $P(x) = f(x^1, x^2, \ldots, x^n)$. Following Ulm's notation [3] we put

$$u = (u^1, u^2, \ldots, u^n),$$
$$v = (v^1, v^2, \ldots, v^n),$$
$$f(\ldots\,;\, u^i v^i\,;\, \ldots) = \frac{f(\ldots, u^i, \ldots) - f(\ldots, v^i, \ldots)}{u^i - v^i} \qquad (i = 1, 2, \ldots, n).$$

For a point $h \in X$, $h = (h^1, h^2, \ldots, h^n)$ we have as a divided difference the mapping $P_{u,v} \in \mathfrak{L}(X, Y)$ defined by inner product

$$P_{u,v}h = \langle d, h \rangle$$

where

$$d = (f(u^1v^1, u^2, \ldots, u^n), f(v^1, u^2v^2, u^3, \ldots, u^n), \ldots, f(v^1, \ldots, v^{n-1}, u^n v^n)).$$

It is evidently that $P_{u,v} \in \mathfrak{L}(X, Y)$, and

$$P_{u,v}(u - v) = f(u^1, u^2, \ldots, u^n) - f(v^1, v^2, \ldots, v^n) = P(u) - P(v).$$

For the mapping P we have an other divided difference

$$\tilde{P}_{u,v} = (f(u^1 v^1, v^2, \ldots, v^n), f(u^1, u^2 v^2, v^3, \ldots, v^n), \ldots, f(u^1, \ldots, u^{n-1}, u^n v^n)).$$

We can define a divided difference fo the second order in the points u, v, w of P as divided difference of the mapping $P_{u,x}$ of X intro $\mathfrak{L}(X, Y)$, in the points v, w of X, or using an other divided difference of the first order, or fixing an other point (v, or w), or fixing a point in $P_{u,v}$ in the second place. In the chosen case, we have

$$P_{u,x} = (f(u^1x^1, u^2, \ldots, u^n), f(x^1, u^2v^2, u^3, \ldots, u^n), \ldots$$
$$\ldots, f(x^1, \ldots, x^{n-1}, u^nv^n)) = (g_1(x^1, x^2, \ldots, x^n), g_2(x^1, x^2, \ldots$$
$$\ldots, x^n), \ldots, g_n(x^1, x^2, \ldots, x^n)),$$

where g_i are the functions of $x^1, x^2, \ldots, x^n$, for $i = 1, 2, \ldots, n$.

If we want to express the mapping $P_{u,v,w}$ by using the notation in terms of f, we have a quadratical matrix,

$$P_{u,v,w} = ||a_{ij}||, \qquad i, j = 1, 2, \ldots, n,$$

where

$$a_{ij} = 0 \text{ for } i > j,\ i = 2, \ldots, n\,;\ j = 1, 2, \ldots, n - 1\,;$$
$$a_{ij} = f(w^1, \ldots, w^{i-1}, u^iv^iw^i, u^{i+1}, \ldots, u^n)$$
$$\text{for } i = j,\ i = 2, 3, \ldots, n - 1\,;$$

$$a_{11} = f(u^1v^1w^1, u^2, \ldots, u^n)\,;\ a_{nn} = f(w^1, \ldots, w^{n-1}, v^nu^nw^n)\,;$$
$$a_{ln} = f(w^1, \ldots, w^{l-1}, v^lw^l, v^{l+1}, \ldots, v^{n-1}, u^nv^n),\ l = 2, \ldots, n - 1\,;$$
$$a_{1n} = f(v^1w^1, v^2, \ldots, v^{n-1}, v^nu^n)\,;$$
$$a_{1k} = f(v^1w^1, v^2, \ldots, v^{k-1}, u^kv^k, u^{k+1}, \ldots, u^n),\ k = 2, \ldots, n - 1\,;$$
$$a_{ij} = f(w^1, \ldots, w^{i-1}, v^iw^i, v^{\ +1}, \ldots, v^{j-1}, u^jv^j, u^{j+1}, \ldots, u^n)$$

$$\text{for } i < j,\ i = 2, 3, \ldots, n - 2\,;\ j = 3, 4, \ldots, n - 1,$$

and

$$f(w^1, \ldots, w^{i-1}, u^iv^iw^i, u^{i+1}, \ldots, u^n) =$$

$$= \frac{f(w^1, \ldots, w^{i-1}, u^iv^i, u^{i+1}, \ldots, u^n) - f(w^1, \ldots, w^{i-1}, u^iv^i, u^{i+1}, \ldots, u^n)}{v^i - w^i} =$$

$$= \frac{f(w^1, \ldots, w^{i-1}, u^i, \ldots, u^n) - f(w^1, \ldots, w^{i-1}, v^i, u^{i+1}, \ldots, u^n}{(u^i - v^i)(v^i - w^i)} -$$

$$- \frac{f(w^1, \ldots, w^{i-1}, u^i, \ldots, u^n) - f(w^1, \ldots, w^{i-1}, w^i, u^{i+1}, \ldots, u^n}{(u^i - w^i)(v^i - w^i)}$$

$(i = 1, 2, \ldots, n)$, and

$$f(w^1, \ldots, w^{j-1}, v^jw^j, v^{j+1}, \ldots, v^{n-1}, u^nv^n) =$$

$$= \frac{f(w^1, \ldots, w^{j-1}, v^j, \ldots, v^{n-1}, u^nv^n) - f(w^1, \ldots, w^j, v^{j+1}, \ldots, v^{n-1}, u^nv^n)}{v^j - w^j}$$

From the other divided difference $\tilde{P}_{u,v}$ we obtain the second divided difference of the same type, which is not identical with the preceding.

E x a m p l e 2. Let $X = R^n$, $Y = R^n$ and

$$P(x) = (f_1(x), f_2(x), \ldots, f_n(x)), \; x = (x^1, x^2, \ldots, x^n).$$

Then as a divided difference we have the matricial application, whith the matrix

$$\left\| \begin{matrix} f_1(u^1v^1, u^2, \ldots, u^n) & f_2(u^1v^1, u^2, \ldots, u^n) & \ldots & f_n(u^1v^1, u^2, \ldots, u^n) \\ f_1(v^1, u^2v^2, u^3, \ldots, u^n) & f_2(v^1, u^2v^2, u^3, \ldots, u^n) & \ldots & f_n(v^1, u^2v^2, u^3, \ldots, u^n) \\ \cdot\; \cdot\; \cdot & \cdot\; \cdot\; \cdot & \cdot\; \cdot\; \cdot & \cdot\; \cdot\; \cdot \\ f_1(v^1, \ldots, v^{n-1}, u^nv^n) & f_2(v^1, \ldots, v^{n-1}, u^nv^n) & \ldots & f_n(v^1, \ldots, v^{n-1}, u^nv^n) \end{matrix} \right\| =$$

$$= (f_j(v^1, \ldots, v^{i-1}, u^iv^i, u^{i+1}, \ldots, u^n), \quad \begin{matrix} i = 1, 2, \ldots, n; \\ j = 1, 2, \ldots, n; \end{matrix}$$

We can change the $f_j(v^1, \ldots, v^{i-1}, u^iv^i, v^{i+1}, \ldots, u^n)$ with $f_j(u^1, \ldots \ldots, u^{i-1}, u^iv^i, v^{i+1}, \ldots, v^n)$, obtaining a new divided difference.

The above exposed Proposition and Examples show that the divided differences are not unique and symmetrical. We mention that the symmetry can be assured.

P r o p o s i t i o n 3. *For every mapping* $P : X \to Y$, *there exists a symmetrical divided difference.*

Proof. For every mapping having a divided difference $P_{u,v}$[2], we can put

$$Q_{u,v} = \frac{1}{2}(P_{u,v} + P_{v,u}).$$

$P_{v,u}$ is a divided difference of the mapping P in points u and v, because

$$P(u) - P(v) = -P_{v,u}(v - u) = P_{v,u}(u - v)$$

hence $Q_{u,v}$ is a symmetrical divided difference of P points u and v.

The divided difference of a differentiable mapping is not always convergent to the derivative, as is illustrated by the following.

E x e m p l e 3. Let be the identical mapping of R^2, with the usual norm, $h = (h^1, h^2)$ a point of R^2, $\emptyset = (0, 0)$ the origin of R^2, and $u = (u^1, u^2) \in R^2$. We have

$$I'(\emptyset) \cdot h = I \cdot h = h.$$

We consider the following divided difference of the mapping I in the different points u and $\emptyset$:

$$I_{u,\emptyset}h = \frac{h^1}{u^1}[I(u) - I(\emptyset)] = \frac{h^1}{u^1}u = \frac{h^1}{u^1}(u^1, u^1) = (h^1, h^1),$$

$$\lim_{u \to \emptyset} I_{u\emptyset}h = \lim_{u \to \emptyset}(h^1, h^1) = (h^1, h^1) \neq I'(\emptyset) \cdot h.$$

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R E F E R E N C E S

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