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APPROXIMATION OF MONOTONE FUNCTIONS BY MONOTONE POLYNOMIALS IN HAUSDORFF METRIC

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In [1] G. G. LORENTZ and K. L. ZELLER have studied the order of approximation of monotone functions by means of monotone polynomials in the uniform metric. The purpose of this paper is to obtain the exact order of monotone approximation in Hausdorff's metric. From our results and the relation between uniform and Hausdorff's metric, the estimates, obtained in [1], follow.

1.

Let us recall the main results from [1].

The 2π-periodic function f will be called bell-shaped if f is even and if f decreases in $[0, \pi]$. The analogue to the approximation of monotone functions by monotone polynomials in the 2π-periodic case is the approximation of bell-shaped functions by means of bell-shaped trigonometric polynomials.

Let $\omega(f, \delta)$ be the modulus of continuity of the continuous 2π-periodic function f:

$$\omega(f;\ \delta) = \max |f(x) - f(y)|\ ;\ |x - y| \leq \delta.$$

In [1] the following theorem is proved:

There exists a constant C with the following property: for each bell-shaped function f one can find a bell-shaped trigonometric polynomial T_n of n-th order such that

$$\max_x |f(x) - T_n(x)| \leq c\omega(f;\ 1/n). \tag{1}$$

This theorem shows that the order of approximation of bell-shaped functions by means of bell-shaped trigonometric polynomials of n-th order in general is the same as the order of approximation by means of arbitrary trigonometric polynomials of n-th order (compare with the Jackson's theorem, for example [2]).

For the algebraical case in [1] the following result is obtained:

There exists a constant C_0 with the following property: if f is an increasing function on $[-1, 1]$, then there exists a sequence of polynomials p_n of n-th degree, increasing in $[-1, 1]$, such that

$$|f(x) - p_n(x)| \leq C_0\omega(f;\ \Delta_n(x)) \quad \text{for } x \in [-1, 1] \tag{2}$$

where $\Delta_n(x) = \max\ [\sqrt{1 - x^2}/n,\ 1/n^2]$, $n = 1, 2, 3, \ldots$

In this paper we shall consider the Hausdorff distance between functions (see [3], [4] and **2.** for the definition). We shall prove the following.

THEOREM 1. *There exists a constant c such that for every $\alpha > 0$ and for every bell-shaped function f there exists a sequence of bell-shaped trigonometric polynomials $\{T_n\}_{n=1}^{\infty}$ (T_n is of n-th order), which satisfy*

$$r(\alpha;\ f,\ T_n) \leq c\omega(f;\ 1/n)\,\frac{\ln e(1 + \alpha n\omega(f;\ 1/n))}{1 + \alpha n\omega(f;\ 1/n)}, \tag{3}$$

where $r(\alpha;\ f,\ T_n)$ is the Hausdorff distance (with a parameter α) between f and T_n.

Since for f continuous

$$\lim_{\alpha\to 0} r(\alpha;\ f,\ T_n) = r(0;\ f,\ T_n) = \max_x |f(x) - T_n(x)|,$$

from (3), letting $\alpha \to 0$, we obtain Lorentz and Zeller's result (1).

On the other hand, setting $\alpha = 1$ we obtain the following estimate:

$$r(f,\ T_n) = r(1;\ f,\ T_n) = O(\ln n/n),$$

which shows that the order of monotone Hausdorff approximation is the same as the order of approximation by means of arbitrary trigonometric polynomials in the Hausdorff metric (see [3]).

In the algebraic case we have

THEOREM 2. *Let f be an increasing function on $[-1, 1]$. There exists a sequence of algebric polynomials $\{P_n\}_{n=1}^{\infty}$(P_n is of n-th degree and is increasing in $[-1, 1]$), such that*

$$|f(x) \dot{-} P_n(x)|_\alpha \leq c_1\omega(f;\ \Delta_m(x))/(1 + \alpha\Delta_m^{-}\ (x)\omega(f;\ \Delta_m(x))), \tag{4}$$

where $\Delta_m(x) = \sqrt{1 - x^2}/m + m^{-2}$, $m = [n/\ln e(1 + \alpha M m^2)]$, $M = \max\limits_{|x|\leq 1} |f(x)|$, c_1 is a constant and $|f(x) \dot{-} P_n(x)|_\alpha$ is the Hausdorff difference with a parameter α between f and P_n in the point x. (See **2.** for the definition).

Since

$$\lim_{\alpha\to 0} |f(x) \dot{-} P_n(x)|_\alpha = |f(x) \dot{-} P_n(x)|_0 = |f(x) - P_n(x)|$$

for f continuous at the point x, from (4), setting $\alpha \to 0$, we obtain (2).

2.

Let us recall the definition of the Hausdorff distance with a parameter $\alpha > 0$ between two bounded functions f and g in the interval Δ.

Let $\bar{f}$ be the completed graph of the function f:

$$\bar{f} = \bigcap G, \quad f \subset G, \quad G \in F_\Delta,$$

where F_Δ denotes the set of all point sets in the plane, convex with respect to the y axis and the projection of which on the axis x coincides with Δ, and f denotes the graph of the function f.

The Hausdorff distance with parameter α between f and g is defined by

$$r(\alpha;\ f,\ g) = \max\{\max_{(x,y)\in\bar{f}}\ \min_{(\xi,\eta)\in\bar{g}}\ \max\ [\alpha^{-1}|x - \xi|,\ |y - \eta|], \tag{5}$$

$$\max_{(x,y)\in\bar{g}}\ \min_{(\xi,\eta)\in\bar{f}}\ \max\ [\alpha^{-1}|x - \xi|,\ |y - \eta|]\}.$$

If f and g are continuous functions, then (5) can be expressed as follows

$$r(\alpha;\ f,\ g) = \max_{x\in\Delta} |f(x) \dot{-} g(x)|_\alpha, \tag{6}$$

where

$$|f(x) \dot{-} g(x)|_\alpha = \max\{\min_{\xi\in\Delta}\ \max\ [\alpha^{-1}|x - \xi|,\ |f(x) - g(\xi)|],$$

$$\min_{\xi\in\Delta}\ \max\ [\alpha^{-1}|x - \xi|,\ |f(\xi) - g(x)|]\}. \tag{7}$$

is the Hausdorff difference with a parameter α between f and g in the point x.

From (6) and (7) when f and g are continuous we obtain

(8) $$\lim_{\alpha\to 0} |f(x) \dot{-} g(x)|_\alpha = |f(x) - g(x)|,$$

(9) $$\lim_{\alpha\to 0} \tau(\alpha\,;\, f,\, g) = \tau(0\,;\, f,\, g) = \max_{x\in\Delta} |f(x) - g(x)|.$$

Let f be a 2π-periodic integrable function and let m and τ be positive integers. Let us consider the trigonometric polynomial of order $\leqq m\tau$:

$$T_{m,\tau}(f;\, x) = \mu_{m,\tau} \int_{-\pi}^{\pi} f(x+t)\psi_{m,\tau}(t)dt,$$

$$\psi_{m,\tau}(t) = (\sin(mt/2)/\sin(t/2))^{2\tau},$$

where $\mu_{m,\tau}$ is defined by

(10) $$\mu_{m,\tau} \int_{-\pi}^{\pi} \psi_{m,\tau}(t)dt = 1.$$

If f is a bell-shaped function, the polynomial $T_{m,\tau}(f;\, x)$ may not be a bell-shaped function in general. But if f is a bell-shaped step function with jumps in the points $k\pi/m$, then, similarly as in [1], it is easy to see that $T_{m,\tau}(f;\, x)$ is also a bell-shaped function.

Let f be a bell-shaped step function with jumps c_k in the poitns $k\pi/m$, $k = 1, 2, 3, \ldots, m-1$. Then

$$T_{m,\tau}(f;\, x) = \mu_{m,\tau} \sum_{k=1}^{m-1} c_k \int_{-k\pi/m}^{k\pi/m} \psi_{m,\tau}(x-t)\, dt.$$

The functions $\varphi_k(x) = \int_{-k\pi/m}^{k\pi/m} \psi_{m,\tau}(x-t)dt$, $k = 1, 2, 3, \ldots, m-1$, are bell-shaped, because $\varphi_k(x)$ are even and for $x \in [0, \pi]$ $\varphi_k'(x) \leqq 0$ (compare with [1]):

$$\varphi_k'(x) = \psi_{m,\tau}(x + k\pi/m) - \psi_{m,\tau}(x - k\pi/m) =$$

$$= \sin^{2\tau}(mx/2 + k\pi/2)\,[1/\sin^{2\tau}((x + k\pi/m)/2) - 1/\sin^{2\tau}((x - k\pi/m)/2] \leqq 0,$$

since

$$|\sin((x + k\pi/m)/2)| \geqq |\sin((x - k\pi/m)/2)| \text{ for } x \in [0, \pi]$$

$$(\sin(\alpha+\beta) \geqq |\sin(\alpha-\beta)| \text{ for } 0 \leqq \alpha,\ \beta \leqq \pi/2).$$

Therefore we have the following

Lemma 1. *Let f be a bell-shaped step function with jumps at the points $k\pi/m$. Then, if m and τ are positive integers, the trigonometric polynomial of order $\leqq m\tau$*

$$T_{m,\tau}(f;\, x) = \mu_{m,\tau} \int_{-\pi}^{\pi} f(x+t)\,[\sin(mt/2)/\sin(t/2)]^{2\tau}\, dt$$

is a bell-shaped function.

3. Let f be a bell-shaped function and let us consider the function

$$f_m(x) = \begin{cases} f(10k\pi/m) & \text{for } 10k\pi/m \leqq x < 10\pi(k+1)/m, \\ & k = 0, 1, 2, \ldots, k_0,\ k_0 = \max \nu,\ 10(\nu+1) \leqq m, \\ f(\pi) & \text{for } 10\pi k_0/m \leqq x \leqq \pi, \\ f(-x) & \text{for } -\pi \leqq x \leqq 0, \end{cases}$$

Obviously

(11) $$\tau(\alpha\,;\, f,\, f_m) \leqq \min\,[10\pi/\alpha m,\ \omega(f;\, 10\pi/m)].$$

We denote

$$x_i = (2i+1)5\pi/m,\ i = 1, 2, 3, \ldots, k_0 - 1\,;\ x_0 = 0,\ x_{k_0} = \pi.$$

If $m = 10l$, l — positive integer, then $T_{m,\tau}(f_m\,;\, x)$ is *a* bell-shaped function (lemma 1). From now on we suppose that $m = 10l$.

From the definition of the Hausdorff distance and from the fact that f_m and $T_{m,\tau}(f_m\,;\, x)$ are bell-shaped functions, it follows that in order to estimate $\tau(\alpha\,;\, f_m,\, T_{m,\tau}(t_m))$ it is sufficient to estimate

(12) $$|f_m(x_i) - T_{m,\tau}(f_m\,;\, x_i)|,\ i = 0, 1, 2, \ldots, k_0.$$

Let us now estimate (11). From (10) we have

$$1 = \mu_{m,\tau} \int_{-\pi}^{\pi} \psi_{m,\tau}(t)\, dt \geqq 2\mu_{m,\tau} \int_{0}^{\pi/m} \psi_{m,\tau}(t)\, dt \geqq$$

$$\geqq 2\mu_{m,\tau} \int_{0}^{\pi/m} ((mt/\pi)/(t/2))^{2\tau}\, dt = 4(2m/\pi)^{2\tau-1}\mu_{m,\tau}$$

or

$$0 < \mu_{m,\tau} \leqq (\pi/2m)^{2\tau-1}/4.$$

Then we have

$$|T_{m,\tau}(f_m;\ x_i) - f_m(x_i)| \leqq \mu_{m,\tau} \int_{-\pi}^{\pi} |f_m(x_i + t) - f(x_i)|\, \psi_{m,\tau}(t)\, dt \leqq$$

$$\leqq 2\mu_{m,\tau} \sum_{k=1}^{k_0} \omega(f;\ 10k\pi/m) \int_{5k\pi/m}^{\pi} \psi_{m,\tau}(t)\, dt \leqq$$

$$\leqq [(\pi/2m)^{2\tau-1}/(4\tau - 2)] \sum_{k=1}^{\infty} \omega(f;\ 10k\pi/m)\, \pi^{2\tau}(5k\pi/m)^{-2\tau+1} \leqq$$

$$\leqq \pi(\pi/10)^{2\tau-1}\omega(f;\ 10\pi/m) \left(\sum_{k=1}^{\infty} k^{-2\tau+2}\right)/(2\tau - 1).$$

Let $\tau \geqq 2$. Since

$$\sum_{k=1}^{\infty} k^{-s} < 2^{s-1}/(2^{s-1} - 1), \qquad s \geqq 2,$$

we obtain

$$|f_m(x_i) - T_{m,\tau}(f_m;\ x_i)| \leqq 2\pi(\pi/10)^{2\tau-1}\omega(f;\ 10\pi/m)/(2\tau - 1) \leqq \tag{13}$$
$$\leqq 2\pi e^{-\tau}\, \omega(f;\ 10\pi/m)/\tau; \qquad i = 0,\ 1,\ 2,\ \ldots,\ k_0.$$

Let $\beta = \ln e^2(1 + \alpha n\, \omega(f; 1/n))$, setting in (13) $\tau = [\beta]$, $m = 10[n/10\tau]$ we see that there exists an absolute constant C, such that

$$|T_{m,\tau}(f_m;\ x_i) - f_m(x_i)| \leqq C\omega(f;\ 1/n)/(1 + \alpha n\omega(f;\ 1/n)), \tag{14}$$
$$i = 0,\ 1,\ 2,\ \ldots,\ k_0$$

and $T_{m,\tau}(f;x)$ is a bell-shaped trigonometric polynomial of order at most n.
From (14) and the definition of Hausdorff distance it follows that

$$\tau(\alpha;\ T_{m,\tau}(f_m),\ f_m) \leqq$$
$$\leqq \max\left\{C\frac{\omega(f;1/n)}{1 + \alpha n\omega(f;1/n)},\ \min\left[\frac{10\pi}{\alpha m},\ \omega\left(f;\ \frac{10\pi}{m}\right) + C\frac{\omega(f;\ 1/n)}{1 + \alpha n\omega(f;\ 1/n)}\right]\right\} \leqq \tag{15}$$
$$\leqq C_1\omega(f;\ 1/n)\frac{\ln e^2(1 + \alpha n\omega(f;\ 1/n))}{1 + \alpha n\, \omega(f;\ 1/n)},$$

where C_1 is a constant.

From (11) we obtain ($m = 10[n/10[\beta]]$, $\beta = \ln e^2(1 + \alpha n\omega(f;\ 1/n))$) :

$$\tau(\alpha; f; f_m) \leqq \min\{10\pi/\alpha m,\ \omega(f;\ 5\pi/m)\} \leqq C_2\omega(f;\ 1/n)\frac{\ln e^2(1 + \alpha n\omega(f;\ 1/n))}{1 + \alpha n\omega(f;\ 1/n)}, \tag{16}$$

From (15) and (16) we obtain

$$\tau(\alpha;\ f,\ T_{m,\tau}(f_m)) \leqq C\omega(f;\ 1/n)\frac{\ln e(1 + \alpha n\omega(f;\ 1/n))}{1 + \alpha n\omega(f;\ 1/n)},$$

where $T_{m,\tau}(f_m)$ is a bell-shaped trigonometric polynomial of n-th order and C is an absolute constant. This proves Theorem 1.
As a corollary in the case $\alpha = 1$ we obtain

THEOREM 3. *There exists an absolute constant $\bar{c}$ with the properties: for every bell-shaped function f there exists a bell-shaped trigonometric polynomial T_n of n-th order such that*

$$\tau(f,\ T_n) = \tau(1;\ f,\ T\ ) \leqq \bar{c}(\ln nM)/n,$$

where $M = \sup_x |f(x)|$.

4.

Now we shall consider the algebraic case. Let f be an increasing function in $[-1,\ 1]$. We write $g(x) = f(\cos x)$. Then $g(x)$ is a bell-shaped function.
Using the notation from 3., let us estimate

$$|g_m(x_i) - T_{m,\tau}(g_m;\ x_i)|,\ i = 0,\ 1,\ 2,\ \ldots,\ k_0.$$

We have

$$|T_{m,\tau}(g_m;\ x_i) - g_m(x_i)| \leqq \mu_{m,\tau} \int_{-\pi}^{\pi} |g_m(x_i + t) - g_m(x_i)|\, \psi_{m,\tau}(t)\, dt = \tag{17}$$

$$= \mu_{m,\tau}\left\{\int_{5\pi/m}^{\pi} |g(\xi' + t) - g(\xi'')|\psi_{m,\tau}(t)\, dt + \int_{-\pi}^{-5\pi/m} |g(\eta' + t) - g(\eta'')|\,\psi_{m,\tau}(t)\, dt\right\},$$

where

$$|\xi' - x_i| \leqq 10\pi/m; \qquad |\xi'' - x_i| \leqq 10\pi/m;$$
$$|\eta' - x_i| \leqq 10\pi/m; \qquad |\eta'' - x_i| \leqq 10\pi/m.$$

Let us estimate $|g(\xi' + t) - g(\xi'')|$. It is easy to see that for $t \geqq 5\pi/m$

$$|g(\xi' + t) - g(\xi'')| = |f(\cos(\xi' + t)) - f(\cos \xi'')| \leqq \tag{18}$$

$$\leqq \omega(f;\ |\cos(\xi' + t) - \cos \eta''| \leqq c_2\omega(f;\ |t|\ |\sin x_i| + t^2)$$

where c_2 is an absolute constant.
We have also

$$|g(\eta' + t) - g(\eta'')| \leqq c_2\omega(f;\ |t|\ |\sin x_i| + t^2). \tag{19}$$

From (17), (18) and (19) we obtain for $\tau \geqq 2$:

$$|T_{m,\tau}(g_m;\ x_i) - g_m(x_i)| \leqq 2\mu_{m,\tau}c_2 \int\limits_{5\pi/m}^{\pi} \omega(f;\ |t|\,|\sin x_i| + t^2)\psi_{m,\tau}(t)dt \leqq$$

$$\leqq 4\mu_{m,\tau}c_2\omega(f;\ 5\pi\ |\sin x_i|/m + (5\pi/m)^2)\ (5\pi|\sin x_i|/m + (5\pi/m)^2)^{-1}. \times$$

$$\times \int\limits_{5\pi/m}^{\pi} (t|\sin x_i| + t^2)\psi_{m,\tau}(t)dt \leqq c_2(\pi/2m)^{2\tau-1}(5\pi|\sin x_i|/m + (5\pi/m^2)^{-1} \times$$

$$\times\ \omega(f;\ 5\pi|\sin x_i|/m + (5\pi/m)^2)\pi^{2\tau}(|\sin x_i)(m/5\pi)^{2\tau-2}/(2\tau - 2) +$$

$$+\ (m/5\pi)^{2\tau-3}/(2\tau - 3)) \leqq c_2 5\pi(\pi/10)^{2\tau-1}\omega(f;\ 5\pi|\sin x_i|/m +$$

$$+\ (5\pi/m)^2)/(2\tau - 3) \leqq 5\pi c_2 e^{-2\tau+1}\omega(f;\ 5\pi|\sin x_i|/m + (5\pi/m^2))/(2\tau - 3).$$

We have found that if $\tau \geqq 2$ then

$$|T_{m,\tau}(g_m;\ x_i) - g_m(x_i)| \leqq c_3 e^{-2\tau+1}\omega(f;\ 5\pi|\sin x_i|/m + (5\pi/m)^2))/(2\tau - 3), \tag{20}$$

where c_3 is an absolute constant.
Let $\beta = \ln e^2(1 + \alpha m^2 M)$; $M = \max\limits_x |f(x)|$. Setting $\tau = [\beta]$ in (20) we see that there exists an absolute constant c_4 such that

$$|T_{m,\tau}(g_m;\ x_i) - g_m(x_i)| \leqq c_4\omega(f;\ \Delta_m(u_i)|(1 + \alpha m^2 M) \quad {}^{*)} \tag{21}$$

$$i = 0,\ 1,\ 2, \ldots,\ k_0,$$

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* $\Delta_m(u_i) = \sqrt{1 - u_i^2}/m + m^{-2} = \sqrt{1 - \cos^2 x_i}/m + m^{-2}$

where $T_{m,\tau}(g_m)$ is a trigonometric polynomial of order $\leqq m \ln e^2 (1 + \alpha m^2 M)$ and $T_{m,\tau}(g_m;\ x)$ is a bell-shaped function if $m = 10l$, l — positive integer.
Let us denote

$$f_m(u) = g_m(\arccos u),\ u_i = \cos x_i,\ i = 0, 1, 2, \ldots, k_0;\ \ P(u) = T_{m,\tau}(g_m,\ \arccos u),$$

$P(u)$ is an increasing algebraic polynomial of degree

$$\leqq m \ \ln\ e^2(1 + \alpha m^2 M).$$

We have

$$|\cos(10k\pi/m) - \cos(10\pi(k + 1)/m)| \leqq c_5(\sqrt{1 - u_k^2}/m + m^{-2}). \tag{22}$$

Using the construction of g_m, (22) and (7) we obtain

$$|f_m(u_i) \dot{-} f(u_i)|_\alpha \leqq \min\ [c_5\Delta_m(u_i)/\alpha,\ c_6\omega(f;\ \Delta_m(u_i)]. \tag{23}$$

(7), (21) and (23) give us

$$\begin{aligned} |P(u_i) \dot{-} f(u_i)|_\alpha &\leqq |P(u_i) \dot{-} f_m(u_i)|_\alpha + |f_m(u_i) \dot{-} f(u_i)|_\alpha \\ &\leqq c_4\omega(f;\ \Delta_m(u_i))/(1 + \alpha m^2 M) + \min[c_5\Delta_m(u_i)/\alpha,\ c_6\omega(f;\ \Delta_m(u_i))]. \end{aligned} \tag{24}$$

From (24) it is easy to obtain

$$|P(u_i) \dot{-} f(u_i)|_\alpha \leqq c_7\omega(f;\ \Delta_m(u_i))/(1 + \alpha\Delta_m^{-1}(u_i)\omega(f;\ \Delta_m(u_i))), \tag{25}$$

where $\Delta_m(u_i) = \sqrt{1 - u_i^2}/m + m^{-2}$ and c_7 is an absolute constant.
Let $u_i \leqq x \leqq u_{i-1}$. From (7), (25) and the monotony of the functions f and P it follows

$$\begin{aligned} |P(x) \dot{-} f(x)|_\alpha &\leqq \min\ [|u_i - u_{i-1}|/\alpha,\ \omega(f;\ |u_i - u_{i-1}|)] + \\ &+ \max[|g(u_i) \dot{-} P(u_i)|_\alpha,\ |f(u_{i-1}) \dot{-} P(u_{i-1})|_\alpha] \leqq \\ &\leqq \min[c_5\Delta_m(u_i)/\alpha,\ \omega(f;\ \Delta_m(u_i))] + c_7 \frac{\omega(f;\ \Delta_m(u_i))}{1 + \alpha\Delta_m^{-1}(u_i)\omega(f;\ \Delta_m(u_i)))} \leqq \\ &\leqq c_0\omega(f;\ \Delta_m(x)/(1 + \alpha\Delta_m^{-1}(x)\omega(f;\ \Delta_m(x))), \end{aligned} \tag{26}$$

where c_0 is an absolute constant. Since $P(x)$ is a monotone function of degree $\leqq m \ \ln\ e^2(1 + \alpha m^2 M)$, (26) proves the Theorem 2.

As a corollary in the case $\alpha = 1$ we obtain

THEOREM 4. *There exists a constant* $\bar{c}_0$ *such that for every function* f *increasing in the interval* $[-1,1]$ *there exists a sequence of algebraic polynomials* $\{P_n\}_1^\infty$(P_n is of n-th degree and increasing in $[-1,1]$) *such that*

$$|f(x) \dot{-} P(x)|_\alpha \leqq \bar{c}_0\left(\frac{\ln(nM)}{n}\sqrt{1-x^2}+\left(\frac{\ln(nM)}{n}\right)^2\right),$$

where $M = \max_{|x|\leqq 1} |f(x)|$

If we compare the last result with the one in [5], it is seen that in both cases the order of approximation is the same.

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