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A NOTE ON STARSHAPEDNESS PRESERVING LINEAR OPERATORS

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1. Introduction

In a recent paper [4] LUPAŞ proved that the Hirschman-Widder operators and the generalized Bernstein power series preserve starshaped functions. In [5] ROULIER and in [1] ROULIER and BOJANIC obtain theorems about bounded linear operators in general which preserve convexity. Many other results have appeared on this latter subject; see [2] and [3]. It is the task of this note to obtain criteria to recognize starshapedness preserving operators. The method used and the results obtained are the same type as in [1]. We first show that each continuous starshaped function can be uniformly approximated by positive linear combinations of „basic starshaped functions". It then follows from the linearity and boundedness of the operators that if it sends each „basic starshaped function" into a continuous starshaped function, then it does it for all continuous starshaped functions.

2. The main theorems

Let $B[0, 1]$ be the class of bounded functions on $[0, 1]$, and let $C[0, 1]$ be the continuous functions on $[0, 1]$.

D e f i n i t i o n. *Let* $f\colon [0, 1] \to R$. *f is starshaped on* $[0, 1]$ *if for every* $a \in [0, 1]$ *and every* $x \in [0, 1]$ *we have*

$$f(ax) \leqslant af(x).$$

We define $\bar{S}[0, 1]$ to be the class of starshaped functions on $[0, 1]$ which satisfy

(1) $$f(0) = 0$$

(2) $$f(x) \geqslant 0 \text{ on } [0, 1].$$

We define $S[0, 1]$ to be all functions in $\bar{S}[0, 1]$ which satisfy also

(3) $$f \in C[0, 1].$$

For each $b \in [0, 1]$ we define the *basic starshaped functions*

$$\Phi_b(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < b \\ x & \text{for } b \leqslant x \leqslant 1. \end{cases}$$

With these definitions, we have the following

THEOREM 1. *Let $f \in S[0, 1]$. f may be approximated uniformly on $[0, 1]$ by functions g of the from*

$$g(x) = \sum_{j=1}^{n} a_j \Phi_{b_j}(x)$$

where $0 < b_1 < \ldots < b_n < 1$ and $a_j \geqslant 0$ for $j = 1, 2, \ldots, n$.

THEOREM 2. *Let T be a bounded linear operator mapping $B[0, 1]$ into $B[0, 1]$. If for each $b \in [0, 1]$ we have $T(\Phi_b, \cdot) \in S[0, 1]$, then $T: S[0, 1] \to S[0, 1]$.*

The proof of the theorem is made easier by a few preliminary lemmas.

Lemma 1. *If $f \in S[0, 1]$ and $g \in S[0, 1]$ then $f + g$, $f \cdot g$, and af (for $a \geqslant 0$) are all in $S[0, 1]$. Also, f is increasing on $[0, 1]$.*

Proof. Elementary

Lemma 2. *For each $b \in [0, 1]$ $\Phi_b \in \bar{S}[0, 1]$.*

Proof. We only have to show that Φ_b is starshaped on $[0, 1]$ since (1) and (2) are obvious. Let $a \in [0, 1]$. If $ax < b$ then $\Phi_b(ax) = 0$. Thus $\Phi_b(ax) \leqslant a\Phi_b(x)$. If $ax \geqq b$ then $\Phi_b(ax) = ax$. But $x \geqq ax \geqq b$ implies that $\Phi_b(x) = x$. Thus $a\Phi_b(x) = ax = \Phi_b(ax)$. Hence Φ_b is starshaped on $[0, 1]$.

Lemma 3. *Let $f \in S[0, 1]$ and let $b \in (0, 1]$. Choose c so that $c\Phi_b(b) = f(b)$.* $\left(\text{That is, let } c = \frac{f(b)}{b}\right)$. *Then for $b \leqslant x \leqslant 1$ we have $c\Phi_b(x) \leqslant f(x)$.*

Proof. We may write $b = ax$ for some $a \in (0, 1]$. Thus $f(ax) \leqslant af(x)$. Also, $f(ax) = f(b) = cb = cax = ca\Phi_b(x)$. Hence $c\Phi_b(x) \leqslant f(x)$.

Lemma 4. *If $\{h_n\}_{n=0}^{\infty}$ is a sequence of functions which converges uniformly to f on $[0, 1]$ and if $h_n \in S[0, 1]$ for $n = 0, 1, 2, \ldots$ then $f \in S[0, 1]$.*

Proof. (1), (2), and (3) are clear. To show that f is starshaped on $[0, 1]$ let $a \in [0, 1]$ and assume that for some $x \in [0, 1]$ we have

$$f(ax) > af(x).$$

Let $\delta = f(ax) - af(x) > 0$. But $\lim_{n\to\infty} (h_n(ax) - ah_n(x)) = f(ax) - af(x) = \delta$. This is impossible since for each n, $h_n(ax) - ah_n(x) \leqslant 0$.

PROOF OF THEOREMS. *Let $f \in S[0, 1]$. We will construct a sequence of functions $\{g_n\}_{n=2}^{\infty}$ for which*

(4) $$f(x) - \frac{f(1)}{n} \leqslant g_n(x) \leqslant f(x) \text{ for } n = 2, 3, \ldots \text{ and } 0 \leqslant x \leqslant 1.$$

Moreover, we will write

(5) $$\begin{cases} g_n(x) = \sum_{k=2}^{n} a_k \Phi_{b_k}(x) \text{ where} \\ 0 < b_2 < \ldots < b_n < 1 \text{ and} \\ a_2, \ldots, a_n \text{ are non-negative.} \end{cases}$$

Choose $b_2 > 0$ so that $f(b_2) = \frac{f(1)}{n}$ and choose a_2 so that $a_2 b_2 = f(b_2)$. Thus $a_2 = \frac{f(b_2)}{b_2} \geqslant 0$. Now choose $b_3 > b_2$ so that $f(b_3) = \frac{2f(1)}{n}$, and choose a_3 so that $a_3 b_3 + a_2 b_3 = f(b_3)$. Thus $a_3 = \frac{f(b_3) - a_2 b_3}{b_3}$. But $f(b_3) - a_2 b =_3$ $= f(b_3) - a_2 \Phi_{b_2}(b_3) \geqslant 0$ by lemma 3 since $a_2 = \frac{f(b_2)}{b_2}$. Hence, $a_3 \geqslant 0$. Proceeding inductively we choose $b_j > b_{j-1}$ so that $f(b_j) = \frac{(j-1)f(1)}{n}$ and we choose a_j so that

$$a_j b_j + a_{j-1} b_j + \ldots + a_2 b_j = f(b_j).$$

Thus

$$a_j = \frac{f(b_j) - Kb_j}{b_j},$$

where $K = a_2 + \ldots + a_{j-1}$. But $Kb_{j-1} = a_2 b_{j-1} + \ldots + a_{j-1} b_{j-1} = f(b_{j-1})$. Also $Kb_{j-1} = K\Phi_{b_{j-1}}(b_{j-1})$. So by lemma 2 we have

$$Kb_j = K\Phi_{b_{j-1}}(b_j) \leqslant f(b_j).$$

Thus $a_j \geqslant 0$. We now write $g_n(x) = \sum_{k=2}^{n} a_k \Phi_{b_k}(x)$. If $b_j \leqslant x < b_{j+1}$ we have

$$g_n(x) = \sum_{k=2}^{j} a_k \Phi_{b_k}(x) = \sum_{k=2}^{j} a_k x = x\left(\sum_{k=2}^{j} a_k\right)$$

also $g_n(b_j) = b_j\left(\sum_{k=2}^{j} a_k\right) = f(b_j)$. Thus by lemma 3, $g_n(x) \leqslant f(x)$ for $b_j \leqslant x < b_{j+1}$. Notice if $0 \leqslant x < b_2$ that $g_n(x) = 0$ and that if $b_n \leqslant x \leqslant 1$ we have $g_n(b_n) = f(b_n)$ and $g_n(x) \leqslant f(x)$. Thus $g_n(x) \leqslant f(x)$ for all x in $[0, 1]$, and $g_n(0) = f(0)$ and $g_n(b_j) = f(b_j)$ for $j = 2, 3, \ldots, n$. If $b_j \leqslant x < b_{j+1}$ notice that

$$f(x) - \frac{f(1)}{n} \leqslant f(b_{j+1}) - \frac{f(1)}{n} = f(b_j) = g_n(b_j) \leqslant g_n(x).$$

This also follows if $b_n \leqslant x \leqslant 1$ and if $0 \leqslant x < b_2$. Thus we have (4) and (5). Hence $g_n \to f$ uniformly on $[0, 1]$. Thus $T(g_n, \cdot) \to T(f, \cdot)$ uniformly on $[0, 1]$. But $T(g_n, \cdot) = \sum_{k=2}^{n} a_k T(\Phi_{b_k}, \cdot)$. By hypothesis $T(\Phi_{b_k}, \cdot) \in S[0, 1]$ and by lemma 1, $T(g_n, \cdot) \in S[0, 1]$ since $a_k \geqslant 0$ for $k = 2, \ldots, n$. Thus by lemma 4 $T(f, \cdot) \in S[0, 1]$.

3. Remarks

Obviously Theorem 2 has an analogous statement if T maps each Φ_b into any given convex set M.

The two theorems in LUPAŞ [4] may be proven using Theorem 2 although it is not much easier to do so.

Theorem 2 holds true if we replace $B[0, 1]$ by any subspace containing all the appropriate functions.

REFERENCES

[1] Bojanic, R., and Roulier, J. A., *Approximation of convex functions by convex splines and convexity preserving linear operators.* (to appear).

[2] Karlin, S., and Studden, W. J., *Tchebycheff Systems: With Applications in Analysis and Statistics.* Interscience, New York, 1966.

[3] Karlin, S., *Total Positivity.* Volume I. Stanford University Press, Stanford, 1968.

[4] Lupaş, L., *On starshapedness preserving properties of a class of linear positive operators.* Mathematica, **25**, 105–109 (1970).

[5] Roulier, J. A., *Linear operators invariant on non-negative monotone functions.* SIAM J. Numer. Anal., **8**, 30–35 (1971).

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