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ON THE GENERALIZED SUM OF INTERVALS

by

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1. Complex sum and generalized sum of intervals

In the set **I** of real closed intervals

$$A = [a_1, a_2], \qquad B = [b_1, b_2], \ \ldots$$

the complex sum (or simply: sum) of intervals [2] is defined as

$$A + B = \{u + v \mid u \in A \ \& \ v \in B\} = [a_1 + b_1, a_2 + b_2].$$

If $a_1 \in \mathbf{R}$ (a_1 is a real number) then it is customary to denote $a_1 = [a_1, a_1]$, hence $\mathbf{R} \subset \mathbf{I}$.

The pair $(\mathbf{H}, \oplus)$ is called a *generalized sum* of intervals, if the following conditions are satisfied:

S_1) **H** is a set of some ordered pairs of intervals.

S_2) $\oplus$ is a rule which associates with each ordered pair $(A, B) \in \mathbf{H}$ an unique interval, denoted by $A \oplus B$.

S_3) If $(A, B) \in \mathbf{H} \cap \mathbf{R}^2$ (i.e. A and B are numbers and $(A, B) \in \mathbf{H}$), then

$$A \oplus B = A + B$$

We can give the following examples.

1) If $\mathbf{H} = \mathbf{I}^2$ and the rule is $(A, B) = \{u + v \mid u \in A \ \& v \in B\}$, then we have the complex sum of intervals.

2) Let $\bar{A}$ be the width of A (i.e. $\bar{A} = a_2 - a_1$). If $k > 1/2$ and

$$\mathbf{H} = \{(A,\ B) \mid \bar{A} \leqslant \bar{B}\}$$

then by the rule

$$(A,\ B) \to \left[\frac{(a_1 + a_2 + b_1 + b_2)k - a_1 - b_2}{2k - 1},\ \frac{(a_1 + a_2 + b_1 + b_2)k - a_2 - b_1}{2k - 1}\right]$$

we have the *k-quasisum* of intervals [1]. The k-quasisum is denoted by $A \oplus_k B$.

3) If $f = (f_1,\ f_2,\ f_3,\ f_4)$ is a system of 4 real numbers, then for

$$\mathbf{H} = \{(A,\ B) \mid \bar{A}(f_1 - f_3) + \bar{B}(f_2 - f_4) \leqslant 0\}$$

the rule

$$(A,\ B) \mapsto [a_1 + b_1 + \bar{A}f_1 + \bar{B}f_2,\ a_1 + b_1 + \bar{A}f_3 + \bar{B}f_4]$$

defines a generalized sum, denoted by $A \oplus_f B$.

4) If u and v are functions of 4 real arguments, then for

$$\mathbf{H} = \{(A,\ B) \mid u(a_1,\ a_2,\ b_1,\ b_2) - u(a_1,\ a_1,\ b_1,\ b_1) \leqslant v(a_1,\ a_2,\ b_1,\ b_2) - v(a_1,\ a_1,\ b_1,\ b_1)\}$$

the rule

$$(A,\ B) \mapsto [a_1 + b_1 + u(a_1,\ a_2,\ b_1,\ b_2) - u(a_1,\ a_1,\ b_1,\ b_1),\ a_1 + b_1 + v(a_1,\ a_2,\ b_1,\ b_2) - v(a_1,\ a_1,\ b_1,\ b_1)]$$

defines a generalized sum denoted by $A \oplus_{u,v} B$.

5) For the generalized sum we can give the following representation: let α and β be functions of 4 real arguments such that for $a_1,\ b_1 \in \mathbf{R}$ it follows:

$$\alpha(a_1,\ a_1,\ b_1,\ b_1) = \beta(a_1,\ a_1,\ b_1,\ b_1) = a_1 + b_1;$$

then for

$$\mathbf{H} = \{(A,\ B) \mid \alpha(a_1,\ a_2,\ b_1,\ b_2) \leqslant \beta(a_1,\ a_2,\ b_1,\ b_2)\}$$

we have the generalized sum given by the mapping

$$(A,\ B) \mapsto [\alpha(a_1,\ a_2,\ b_1,\ b_2),\ \beta(a_1,\ a_2,\ b_1,\ b_2)].$$

In the present paper we shall give some results about the sums 2), 3) and 4).

2. On the quasisums of intervals

In connection with some interval-equations we have the so-called *quasioperations:* if $\mathbf{H} \subseteq \mathbf{I}^3$, $f: \mathbf{H} \to \mathbf{I}$, $g: \mathbf{H} \to \mathbf{I}$ and for $(A,\ B,\ X) \in \mathbf{H} \cap \mathbf{R}^3$ the real equation $f(A,\ B,\ X) = g(A,\ B,\ X)$ has the solution $X = A + B$, then the interval solution of the interval equation

$$f(A,\ B,\ X) = g(A,\ B,\ X) \qquad (A,\ B,\ X,\ \in \mathbf{I})$$

is called quasisum. It is immediate the generalization for a quasioperation generated by a real binary or n-ary operation.

In the paper [1] is given the proof of the following theorem:

THEOREM 1. *Let* $s_k : \mathbf{I}^2 \to \mathbf{I}$ $(k = 1,\ 2,\ 3,\ \ldots)$ *the maps defined by*

$$s_1(A,\ B) = A - B,\ s_k(A,\ B) = A - (A - s_{k-1}(A,\ B))(k = 2, 3, 4, \ldots)$$

where $A - B = \{u - v \mid u \in A\ \&\ v \in B\}$. *The interval equation* $s_k(X,\ A) = B$ *has an interval-solution if and only if* $\bar{A} \leqslant \bar{B}$. *The solution is denoted by* $A \oplus_k B$ *and is equal with the interval*

$$\left[\frac{(a_1 + a_2 + b_1 + b_2)k - a_1 - b_2}{2k - 1},\ \frac{(a_1 + a_2 + b_1 + b_2)k - a_2 - b_1}{2k - 1}\right].$$

This is an interval also for real $k > 1/2$, *hence we have the generalized quasisum* $A \oplus_k B$ *(k-quasisum) for real* $k > 1/2$. *If* $k = \bar{B}/(\bar{A} + \bar{B})$ *then the k-quasisum is the complex sum* $A + B$ *(defined for* $\bar{A} \leqslant \bar{B}$*).*

We define for $k = 1/2$ and $k = \infty$ the asymptotic quasisums

$$A \oplus_{1/2} B = (-\infty,\ +\infty) \text{ and } A \oplus_\infty B = \frac{a_1 + a_2 + b_1 + b_2}{2}$$

where $A \oplus_{1/2} B$ is the improper interval (the set of real numbers) and $A \oplus_\infty B$ is a real number.

For the width of quasisum we have the formula

$$\overline{A \oplus_k B} = \frac{\bar{B} - \bar{A}}{2k - 1}.$$

In the case of complex operations the inclusions

$$A \subset C \text{ and } B \subset D \quad (\text{denoted } (A,\ B) \subset (C,\ D))$$

imply that

$$A \circ B \subset C \circ D,$$

i.e. the interval arithmetic is inclusion monotonic [2].

For the quasioperations the interval arithmetic is not inclusion monotonic. First we give the following theorem.

THEOREM 2. *If* $(A,\ B) \subset (C,\ D)$ *and* $\bar{A} < \bar{B},\ \bar{C} < \bar{D}$, *then we can give the following parametrical forms for these intervals:*

$$A = [a_1,\ a_1 + a],\ \ B = [b_1, [b_1 + a + b],\ \ C = [a_1 - (b + c + d)s,\ a_1 + a + (b + c + d)(1 - s)t],$$

$$D = [b_1 - c,\ b_1 + a + b + d],$$

where

$a_1,\ b_1 \in \mathbf{R},\ a,\ b,\ c,\ d \in \mathbf{R}^+$ (i.e. are positive numbers) *and* $0 < s,\ t < 1$.

Proof. The above parametrical forms follow for A, B and C, hence from

$$\bar{A} < \bar{B} \text{ and } B \subset D.$$

From $A \subset C$ it follows: $C = [a_1 - x,\ a_1 + a + y]$ where $x,\ y \in \mathbf{R}^+$. From $\bar{C} < \bar{D}$ we have the inequations

$$x < b + c + d, \qquad y < b + c + d - x,$$

which has the solution

$$x = (b + c + d)s, \qquad y = (b + c + d)(1 - s)t$$

and the theorem is proved.

For the quasisum in the above hypothesis we have

THEOREM 3. *If* $(A,\ B) \subset (C,\ D)$ *and the intervals* $A \oplus_k B,\ C \oplus_k D$ *exist, then for the parametrical form of intervals* $A,\ B,\ C,\ D$ *given in theorem* 2, *we have*

$$A \oplus_k B = \left[a + a_1 + b_1 + \frac{(k-1)b}{2k-1},\ a + a_1 + b_1 + \frac{kb}{2k-1}\right],$$

$$C \oplus_k D = \left[a + a_1 + b_1 + \frac{k(b - c + d) - b - d + (b + c + d)((1 - s)tk - sk + s)}{2k-1},\right.$$

$$\left. a + a_1 + b_1 + \frac{k(b - c + d) + c + (b + c + d)((1 - s)\,tk - sk - (1 - s)t)}{2k-1}\right].$$

This form of quasisums follows immediately from the Theorem 1.

THEOREM 4. *For the quasisums given in theorem* 3 *we have:*

$$A \oplus_k B = A + B \Leftrightarrow k = k_1 = \frac{a + b}{2a + b}$$

and

$$C \oplus_k D = C + D \Leftrightarrow k = k_2 = \frac{a + b + c + d}{2(a + b + c + d) - \bar{s}\bar{t}(b + c + d)}.$$

We denote $\bar{s} = 1 - s$ *and* $\bar{t} = 1 - t$.

Proof. We have (for the parametrical forms given in theorem 2)

$$\bar{A} = a,\ \ \bar{B} = a + b,\ \ \bar{C} = a + b + c + d - st(b + c + d),\ \ \bar{D} = a + b + c + d,$$

therefore (theorem 1)

$$k_1 = \frac{\bar{B}}{\bar{A} + \bar{B}} = \frac{a + b}{2a + b} \text{ and } k_2 = \frac{\bar{D}}{\bar{C} + \bar{D}} = \frac{a + b + c + d}{2(a + b + c + d) - \bar{s}\bar{t}(b + c + d)}.$$

The theorem is proved.

THEOREM 5. *For the quasisums given in theorem* 3 *we have*

$$(A \oplus_k B = A + B\ \&\ C \oplus_k D = C + D) \Leftrightarrow \bar{s}\bar{t} = \frac{b(a + b + c + d)}{b(a + b + c + d) + a(c + d)}$$

$$\&\ k = \frac{a + b}{2a + b}.$$

The theorem is an immediately consequence of theorem 4.

THEOREM 6. *The quasisums* $A \oplus_k B$ *and* $C \oplus_k D$ *are complex sums for a common value of* k, *if and only if*

$$\bar{s}\bar{t} = \frac{b(a + b + c + d)}{b(a + b + c + d) + a(c + d)},$$

i.e. for

$$s = \frac{b(a + b + c + d)}{b(a + b + c + d) + a(c + d)s'},\quad t = \frac{b(a + b + c + d) + a(c + d)s'}{b(a + b + c + d) + a(c + d)}$$

where $0 < s' < 1$ *is arbitrary.*

The proof is immediately.

For the following theorem we recall the definitions of some binary relations in $\mathbf{I}$ (i.e. subsets of $\mathbf{I}^2$; see [1]) : for the intervals A and B we have

$$A < B \Leftrightarrow a_2 < b_1,\ A \dashv B \Leftrightarrow a_1 < b_1 < a_2 < b_2,\ A \subset B \Leftrightarrow b_1 < a_1 < a_2 < b_2,$$
$$A > B \Leftrightarrow B < A,\ A \vdash B \Leftrightarrow B \dashv A,\ A \supset B \Leftrightarrow B \subset A.$$

THEOREM 6. *If* $\rho \in \{<, >, \dashv, \vdash, \subset, \supset\}$ *then in the hypothesis of theorems* 2 *and* 3 *it follows*

$$(A \oplus_k B)\rho(C \oplus_k D) \Leftrightarrow [-b,\ 0]\ \rho[kp - b - d + qs,\ \ kp + c - q(1 - s)t],$$

where

$$p = d - c + (b + c + d)((1 - s)t - s)\ \text{ and }\ q = b + c + d.$$

Proof. By addition to the endpoints of the intervals

$$[-b,\ 0)\ \text{and}\ [kp - b - d + qs,\ kp + c - q(1 - s)t]$$

of the constant number kb, next by division with the positive number $2k - 1$ and next by addition of the constant number $a + a_1 + b_1$ we obtain the endpoints of the quasisums $A \oplus_k B$ and $C \oplus_k D$. The theorem is proved.

THEOREM 7. *With the notations of theorem* 6 *we have the following equivalences :*

$$A \oplus_k B > C \oplus_k D \Leftrightarrow kp < q(1 - s)t - b - c,$$

$$A \oplus_k B \vdash C \oplus_k D \Leftrightarrow$$
$$\Leftrightarrow q(1 - s)t - b - c < kp < \begin{cases} q(1 - s)t - c & \text{if } (1 - s)t + s < \dfrac{c + d}{b + c + d} \\ d - qs & \text{if } (1 - s)t + s > \dfrac{c + d}{b + c + d} \end{cases}$$

$$A \oplus_k B \subset C \oplus_k D \Leftrightarrow (1 - s)t + s < \frac{c + d}{b + c + d}\ \&\ q(1 - s)t - c < kp < d - qs,$$

$$A \oplus_k B \supset C \oplus_k D \Leftrightarrow (1 - s)t + s > \frac{c + d}{b + c + d}\ \&\ d - qs < kp < q(1 - s)t - c,$$

$$A \oplus_k B \dashv C \oplus_k D \Leftrightarrow$$
$$\Leftrightarrow \left.\begin{cases} \text{if } (1 - s)t + s < \dfrac{c + d}{q} & \text{then } d - qs \\ \text{if } (1 - s)t + s > \dfrac{c + d}{q} & \text{then } g(1 - s)t - c \end{cases}\right\} < kp < b + d - qs$$

$$A \oplus_k B < C \oplus_k D \Leftrightarrow kp > b + d - qs.$$

We give only the *proof* of the equivalences related to $A \oplus_k B \vdash C \oplus_k D$. The proofs of the other equivalences can be made similarly. We have (from theorem 6) : $A \oplus_k B \vdash C \oplus_k D \Leftrightarrow [-b, 0] \vdash [kp - b - d + qs, kp + c - c(1 - s)t] \Leftrightarrow kp - b - d + qs < -b < kp + c - c(1 - s)t < 0 \Leftrightarrow q(1 - s)t - b - c < kp < min\,(g(1 - s)t - c, d - qs) \Leftrightarrow (q(1 - s)t - b - c < kp < q(1 - s)t - c\ \&\ q(1 - s)t - c < d - qs) \vee (g(1 - s)t - b - c < kp < d - qs\ \&\ d - qs < q(1 - s)t - c)$.

Since

$$q(1 - s)t - c < d - qs \Leftrightarrow (1 - s)t + s < \frac{c + d}{q} < \frac{c + d}{b + c + d}$$

and

$$q(1 - s)t - c > d - qs \Leftrightarrow (1 - s)t + s > \frac{c + d}{b + c + d}$$

the equivalence follows.

In the end of this paragraph we give an example. We consider the intervals

$$A = [a_1,\ a_1 + 3],\quad B = [a_1 + 6,\ a_1 + 10],\quad c = [a_1 - 2,\ a_1 + 4],$$
$$D = [a_1 + 5,\ a_1 + 11 + u]$$

where $a_1 \in \mathbf{R}$ and $u \in \mathbf{R}^+$. These intervals satisfy the conditions of theorems 2 and 3 with the parameters

a_1	b_1	a	b	c	d	s	t	$\bar{s}$	$\bar{t}$
a_1	$a_1 + 6$	3	1	1	$1 + u$	$\frac{2}{3 + u}$	$\frac{1}{1 + u}$	$\frac{1 + u}{3 + u}$	$\frac{u}{1 + u}$

In this way we have

$$A \oplus_k B = \left[\frac{(4a_1 + 19)k - 2a_1 - 10}{2k - 1},\ \frac{(4a_1 + 19)k - 2a_1 - 9}{2k - 1}\right]$$

and

$$C \oplus_k B = \left[\frac{(4a_1 + 18 + u)k - 2a_1 - 9 - u}{2k - 1},\ \frac{(4a_1 + 18 + u)k - 2a_1 - 9}{2k - 1}\right].$$

For the comparison of the quasisums $A \oplus_k B$ and $C \oplus_k D$ we have the following equivalences:

$$A \oplus_k B < C \oplus_k D \Leftrightarrow u > 1 \ \& \ k > \frac{u}{u-1} \Leftrightarrow (u,k) \in D_1$$

$$A \oplus_k B \dashv C \oplus_k D \Leftrightarrow u > 1 \ \& \ 1 < k < \frac{u}{u-1} \Leftrightarrow (u,k) \in D_2$$

$$A \oplus_k B \subset C \oplus_k D \Leftrightarrow u > 1 \ \& \ \frac{1}{2} < k < 1 \Leftrightarrow (u,k) \in D_3$$

$$A \oplus_k B \supset C \oplus_k D \Leftrightarrow u < 1 \ \& \ k < 1 \Leftrightarrow (u,k) \in D_4$$

$$A \oplus_k B \vdash C \oplus_k D \Leftrightarrow u < 1 \ \& \ 1 < k < \frac{1}{1-u} \Leftrightarrow (u,k) \in D_5$$

$$A \oplus_k B > C \oplus_k D \Leftrightarrow u < 1 \ \& \ k > \frac{1}{1-u} \Leftrightarrow (u,k) \in D_6.$$

In the figure 1 are represented the domains $D_1 - D_6$.

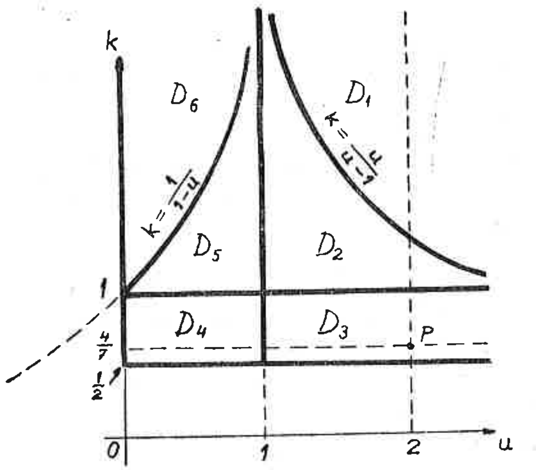

Fig. 1

Observations. Only the points of the domain $u > 0$ & $k > \frac{1}{2}$ define the quasisums $A \oplus_k B$ and $C \oplus_k D$.

From theorem 6 we have for $A \oplus_k B$ and $C \oplus_k D$ the complex sum for a common value of k, if:

$$\frac{1+u}{3+u} \cdot \frac{u}{1+u} = \frac{6+u}{4(3+u)}$$

hence for $u = 2$. We have the complex sum for the point P $\left(\text{with } u = 2 \text{ and } k = \frac{4}{7}\right)$ (Fig. 1.). The monotonic law valid for P (interval arithmetic i.e. complex sum) is valid also in a neighbourhood of P, namely in the domain D_3.

3. Some considerations on the generalized sum $A \oplus_k B$(§ 1; 3)

If $f = (f_1, f_2, f_3, f_4)$ is a system of 4 real numbers, then for

$$\mathbf{H} = \{(A, B) \mid A(f_1 - f_3) + B(f_2 - f_4) < 0\}$$

we define the generalized sum $A \oplus_f B$ as the interval

$$[a_1 + b_1 + \bar{A}f_1 + \bar{B}f_2, \ a_1 + b_1 + \bar{A}f_3 + \bar{B}f_4]$$

THEOREM 8. *For the generalized sum $A \oplus_f B$ we have one of the following 3 formes:*

1) $f = (f_1, f_2 + g, f_1 + f, f_2)$, $\bar{B} < \frac{\bar{A}f}{g}$;

2) $f = (f_1 + f, f_2, f_1, f_2 + g)$, $\bar{B} > \frac{\bar{A}f}{g}$;

3) $f = (f_1, f_2, f_1 + f, f_2 + g)$,

where $f_1, f_2 \in \mathbf{R}$ and $f, g \in \mathbf{R}^+$ are arbitrar real numbers. The quasisum $A \oplus_k B$ has the form 2), namely with

$$f_1 = f_2 = \frac{k-1}{2k-1} \ \text{and} \ f = g = \frac{1}{2k-1}, \quad (\bar{B} < \bar{A})$$

therefore:

$$A \oplus_k B = A \oplus_{\left(\frac{k}{2k-1}, \frac{k-1}{2k-1}, \frac{k-1}{2k-1}, \frac{k}{2k-1}\right)} B$$

Proof. From the condition $\bar{A}(f_1 - f_3) + \bar{B}(f_2 - f_4) < 0$ holds one of the following possibilities:

$$f_3 - f_1 > 0 \ \& \ f_4 - f_2 < 0 \ \& \ \bar{B} < \frac{\bar{A}(f_3 - f_1)}{f_2 - f_4},$$

$$f_3 - f_1 < 0 \ \& \ f_4 - f_2 > 0 \ \& \ \bar{B} > \frac{\bar{A}(f_1 - f_3)}{f_4 - f_2},$$

or

$$f_3 - f_1 > 0 \ \& \ f_4 - f_2 > 0$$

and the parametrical forms given in theorem hold. An immediately proof gives that the quasisum has the form 2.

THEOREM 9. *The generalized sum* $A \oplus_f B$ *gives a quasisum if and only if*

$$f_1 + f_3 = f_2 + f_4 = 1 \text{ and } \bar{A}(f_3 - f_1) + \bar{B}(f_4 - f_2) > 0.$$

The *proof* is very simple.

4. An example related to the generalized sum $A \oplus_{f,g} B$ (§1, 4)

The above sum is not in the present paper in general. We give only the following example.

If

$$u(a_1, a_2, b_1, b_2) = a_1a_2 + b_1b_2 \text{ and } v(a_1, a_2, b_1, b_2) = (a_1 + b_1)(a_2 + b_2),$$

then

$$A \oplus_{u,v} B$$

is the interval

$$[a_1 + b_1 + a_1a_2 + b_1b_2 - a_1^2 - b_1^2, \ a_1 + b_1 + (a_1 + b_1)(a_2 + b_2) - (a_1 + b_1)^2],$$

defined for

$$\bar{A}b_1 + \bar{B}a_1 < 0.$$

5. A historical remark

The complex sums have been considered (in 1895) by F.G. FROEBENIUS (1849—1917) remarking that for complexes (subsets of a group) we can define the group operation; namely if $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of

a multiplicative group, then $\mathfrak{A}\mathfrak{B}$ is the set of elements AB where A is in $\mathfrak{A}$ and B in $\mathfrak{B}$. The product is uniquely determined, but is not uniquely inversable. See: Sitzungsberichte Preuss. Akad. Wiss. Berlin 1895, pp. 163—164.

The study of some aspects connected with inversability of complex operations (in a particular case of intervals) is the scope of the investigation about the quasisums, generalized sums and generally of the quasioperations (see [1]).

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