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ON A PROBLEM OF SIMULTANEOUS BEST APPROXIMATION*

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1. T.J. RIVLIN [5] has posed the following question. Let $\mathcal{C}$ be the linear normed space of continuous real-valued functions on the interval [0, 1], endowed with the uniform norm. Characterize those n-tuples of algebraic polynomials $(p_0, p_1, \ldots, p_{n-1})$ such that the degree of p_i is i, $i = 0, 1, \ldots, n-1$, for which there exists a function $f \in \mathcal{C}$ so that the polynomial of best approximation of degree i to f in the sense of Chebyshev is p_i for each $i = 0, 1, \ldots, n-1$.

A necessary condition was given by T.J. RIVLIN himself [5].

F. DEUTSCH, P.D. MORRIS, I. SINGER [1], D.A. SPRECHER [6], [7] and recently M.R. SUBRAHMANYA [8] have proved the sufficiency of this condition in particular cases.

The purpose of this note is to give some results related to a more general problem and to make a remark about the case when $n = 2$ in the Rivlin's problem.

2. Basically, we will consider n-tuples of elements from nonlinear families instead the polynomials for the above problem. For an exposition of the nonlinear approximating families the reader is referred to [3].

Definition 1. ([9] or [3]). *A family $\mathcal{G} \subset \mathcal{C}$ is said to be unisolvent of degree n on* [0, 1] *or to be of type.* I_n {[0, 1]}, *if given a set of distinct n points $x_1, x_2, \ldots, x_n$ from* [0, 1] *and any n real numbers $y_1, y_2, \ldots, y_n$ there is exactly one $\varphi \in \mathcal{G}$ such that $\varphi(x_i) = y_i$, $i = 1, 2, \ldots, n$.*

Definition 2. ([3]). *The sets $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$, $\mathcal{G}_i \subset \mathcal{C}$, $i = 1, 2, \ldots, k$ form an interpolating section on* [0, 1] *if:*

1. *$\mathcal{G}_i$ is an unisolvent family of degree i on* [0, 1] for $i = 1, 2, \ldots, k$.

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2. $\mathcal{J}_i \subset \mathcal{J}_{i+1}$, $i = 1, 2, \ldots, k-1$.

We can now enunciate the following problem. Given the interpolating section $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n$ characterize those n-tuples of functions $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ such that $\varphi_i \in \mathcal{J}_i$, $i = 1, 2, \ldots, n$, $\varphi_i \notin \mathcal{J}_{i-1}$, $i = 2, 3, \ldots, n$ for which there exists a function $f \in \mathcal{C}$ so that the element of best approximation from $\mathcal{J}_i$ to f in the sense of Chebyshev is φ_i $i = 1, 2, \ldots, n$.

3. By the characterization theorem of best approximations from unisolvent family [9], we may obtain the necessary condition of Rivlin for polynomials.

THEOREM 1. *Let $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n$ be an interpolating section on* $[0, 1]$ *and let $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ be an n-tuple of elements such that $\varphi_i \in \mathcal{J}_i$, $i = 1, 2, \ldots, n$ and $\varphi_i \notin \mathcal{J}_{i-1}$, $i = 2, 3, \ldots, n$. If there exists a function $f \in \mathcal{C}$ such that:*

$$\|f - \varphi_i\| = \inf\{\|f - \psi_i\|,\ \psi_i \in \mathcal{J}_i\},\ i = 1, 2, \ldots, n.$$

then for each pair of indices l, k with $1 \leq l < k \leq n$, the function $\varphi_k - \varphi_l$ changes sign at at least l distinct points in $[0, 1]$.

Proof: We follow the idea from [5] and fix a pair of indices l, k with $1 \leq l < k \leq n$. By the characterization theorem for best approximations from an unisolvent family [9], there exist $l + 1$ distinct points $t_1, \ldots, t_{l+1}$ in $[0, 1]$ such that:

$$f(t_1) - \varphi_l(t_1) = -[f(t_2) - \varphi_l(t_2)] = \ldots = (-1)^l[f(t_{l+1}) - \varphi_l(t_{l+1})] = \pm\|f - \varphi_l\|. \tag{1}$$

Because $\varphi_i \notin \mathcal{J}_{i-1}$, $i = 2, 3, \ldots, n$ we have:

$$\|f - \varphi_k\| < \|f - \varphi_l\|. \tag{2}$$

Consequently, it is obviously by (2) and (1) that the function $\varphi_k - \varphi_l = (f - \varphi_l) - (f - \varphi_k)$ has at least l sign changes on $[0, 1]$.

We shall show that in the case when $n = 2$ the condition of the above theorem is also sufficient. For linear families see [8].

THEOREM 2. *Given the interpolating section $\mathcal{J}_1, \mathcal{J}_2$ and the elements φ_1, φ_2 such that $\varphi_i \in \mathcal{J}_i$, $i = 1, 2$, $\varphi_2 \notin \mathcal{J}_1$, then there exists a function $f \in \mathcal{C}$ which satisfies:*

$$\|f - \varphi_i\| = \inf\{\|f - \psi_i\|,\ \psi_i \in \mathcal{J}_i\},\qquad i = 1, 2.$$

if and only of the function $\varphi_2 - \varphi_1$ has one change of sign in $[0, 1]$.

Proof: By Theorem 1, $\varphi_2 - \varphi_1$ has at least one sign change in $[0, 1]$. The fact that $\varphi_2 - \varphi_1$ has at most one change of sign in the interval $[0, 1]$ follows from the definition of unisolvency.

The proof of sufficiency follows from considering many cases. We shall consider only one case here since the same arguments apply to all cases. Assume that the condition is satisfied, i.e. $\varphi = \varphi_2 - \varphi_1$ has one change of sign in $[0, 1]$. Let $t_0 \in (0, 1)$ such that $\varphi(t_0) = 0$. Then, by continuity, we can choose the points t_1 and t_2 with $0 < t_1 < t_2 < 1$ such that $\varphi(t_1) + \varphi(t_2) = 0$ and $\varphi(t)$ lies between $\varphi(t_1)$ amd $\varphi(t_2)$ when $t \in [t_1, t_2]$. Let be $t_3 \in [0, 1]$ such that $\varphi(t_3) = \|\varphi\|$.

We shall construct a function $f \in \mathcal{C}$ with the desired properties only in the case when $\delta = \varphi(t_2) > 0$ and $t_2 < t_3 < 1$. Let ρ be a real number such that $\rho \geq 2\|\varphi\|$. The desired function $f \in \mathcal{C}$ is the piecewise linear function with vertices:

$$\left(0, \frac{\varphi_1(0) + \varphi_2(0)}{2}\right),\ (t_1, \varphi_2(t_1) - \rho),\ (t_2, \varphi_2(t_2) + \rho).$$

$$(t_3, \varphi_2(t_3) - \rho),\ \left(1, \frac{\varphi_1(1) + \varphi_2(1)}{2}\right).$$

For this function (the estimations are almost the same from [8]) we have

$$\|f - \varphi_1\| = \rho + \delta, \qquad \|f - \varphi_2\| = \rho$$

and also:

$$-[f(t_1) - \varphi_1(t_1)] = f(t_2) - \varphi_1(t_2) = \rho + \delta,$$

$$-[f(t_1) - \varphi_2(t_1)] = f(t_2) - \varphi_2(t_2) = -[f(t_3) - \varphi_2(t_3)] = \rho$$

i.e. φ_1 and φ_2 are best approximations for $f \in \mathcal{C}$.

4. D.A. SPRECHER [6] has given a nice proof for the sufficiency of Rivlin's condition for the case of polynomials of degree 0 and 1, constructing a piecewise continuous function as the solution of the Rivlin's problem.

We have studied under which conditions this piecewise continuous function is also a convex function.

By the results of T. POPOVICIU [4] about the position of the alternation points in the Chebyshev's characterization theorem of best polynomial approximation for continuous convex functions we obtain:

THEOREM 3. *Let $p_0(t) = c$, $p_1(t) = at + b$, $(a \neq 0)$ be given polynomials such that $c = at_0 + b$ for some t_0, $0 < t_0 < 1$. If $a > 0$, $(a < 0)$ and $t_0 \in \left(\frac{1}{2}, 1\right)$* (respectively $t_0 \in \left(0, \frac{1}{2}\right)$), *then there exists a continuous convex function f such that p_0 and p_1 are the polynomials of best approximation of degree 0 and respectively of degree 1, for f in the sense of Chebyshev.*

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