REVUE D'ANALYSE NUMÉRIQUE ET DE LA THÉORIE DE L'APPROXIMATION, Tome 3, N° 1, 1974, pp. 27–32

ON RIVLIN'S CONJECTURE FOR UNISOLVENT FAMILIES*

By

GH. CIMOCA
(Cluj)

G. D. TAYLOR*
(East Lansing, Michigan)

1. The following problem posed by T. J. RIVLIN [4] is well-known: Let $\mathcal{C}$ be the normed linear space of continuous real valued functions on the interval [0, 1], endowed with the uniform norm. Characterize those n-tuples of algebraic polynomials $(p_0, p_1, \ldots, p_{n-1})$ such that the degree of p_i is i, $i = 0, 1, \ldots, n-1$, for which there exists a function $f \in \mathcal{C}$ so that the polynomial of best approximation of degree i to f in the sense of Chebyshev is p for each $i = 0, 1, \ldots, n-1$.

The anologous problem for n-tuples of elements from nonlinear unisolvent families [3] or [8], was stated in [1], where the necessary condition of Rivlin for polynomials was obtained. Moreover, it was shown for this more general problem that in the case of a pair (φ_1, φ_2) of elements $\varphi_i \in F_i$, F_i a nonlinear unisolvent family of degree i, $i = 1, 2$, Rivlin's condition is also sufficient. Particular cases of Rivlin's problem were studied in [2], [5], [6] and [7].

The purpose of this note is to prove the following:

THEOREM. *Given the families $\mathcal{G}$ and $\mathcal{H}$, where $\mathcal{G}$ is unisolvent of degree l on [0, 1] and $\mathcal{H}$ is unisolvent of degree k on [0, 1] with $1 \leq l < k$, $\mathcal{G} \subset \mathcal{H}$ and the elements φ, ψ such that $\varphi \in \mathcal{G}$, $\psi \in \mathcal{H}$ and $\psi \notin \mathcal{G}$, then there exists a function $f \in \mathcal{C}$ which satisfies:*

$$||f - \varphi|| = \inf_{g \in \mathcal{G}} ||f - g||$$

*Supported in part by AFOSR–72–2271.

and

$$||f - \psi|| = \inf_{h \in \mathcal{H}} ||f - h||,$$

if and only if the function $\psi - \varphi$ *changes sign at (at least)* l *distinct points in* $[0, 1]$.

For polynomials this theorem was proved by D. Sprecher [5] and the nonlinear case for $l = 1$, $k = 2$ is found in [1].

2. Before giving the proof of the theorem, we introduce some notation and make some observations.

Let φ and ψ be the functions given in the hypothesis of the theorem. Then $\chi = \psi - \varphi$ changes sign at least l times in $[0, 1]$, i.e. there exists the points:

$$0 < x_1 < x_2 < \ldots < x_l < 1, \tag{1}$$

such that

$$\chi(x_i) = 0 \text{ for } i = 1, 2, \ldots, l$$

and $\operatorname{sgn} \chi(x_i - \varepsilon) = -\operatorname{sgn} \chi(x_i + \varepsilon)$ for $i = 1, \ldots, l$ and all $\varepsilon > 0$ sufficiently small. Letting $x_0 = 0$ and $x_{l+1} = 1$, we set:

$$m_i = \max \{|\chi(x)|,\ x \in [x_i, x_{i+1}]\},\ i = 0, 1, \ldots, l$$

$$m = \min \{m_i,\ i = 0, 1, \ldots, l\}$$

$$\rho = \frac{m}{2}.$$

By continuity of the function χ we can choose the point sets:

$$T = \{t_1, t_2, \ldots, t_l\} \text{ and } Z = \{z_1, z_2, \ldots, z_l\}$$

such that:

$$\begin{aligned} &0 < t_i < x_i < z_i < 1, && i = 1, 2, \ldots, l \\ &z_i < t_{i+1}, && i = 1, 2, \ldots, l-1 \\ &\chi(t_i) = -\chi(z_i) = \varepsilon\rho, && i = 1, 2, \ldots, l,\ \varepsilon = \pm 1 \end{aligned}$$

and

$$\rho = \max \{|\chi(x)|,\ x \in [t_i, z_i],\quad i = 1, 2, \ldots, l]\}$$

Let $E = T \cup Z$.

D e f i n i t i o n 1. *A point* $e \in E$ *is a minus point of* χ *if* $\chi(e) = -\rho$ *and a plus point if* $\chi(e) = \rho$.

If $M = ||\chi||$, *let* μ *be a real number such that* $\mu > M$.

L e m m a 1. *If* $\alpha(x) = \max \{\psi(x) - \mu,\ \varphi(x) - \mu - \rho\},\ x \in [0, 1]$

and

$$\beta(x) = \min \{\psi(x) + \mu,\ \varphi(x) + \mu + \rho\},\ x \in [0, 1]$$

then

$$\beta(x) - \alpha(x) > M \quad \text{for} \quad x \in [0, 1].$$

Proof: First we observe that the functions $\psi(x) + \mu$ and $\varphi(x) + \mu + \rho$ coincide at the plus points of χ from the set E while the functions $\psi(x) - \mu$ and $\varphi(x) - \mu - \rho$ coincide at the minus points of χ from the set E.

Now we distinguish the following cases:

(i) for x between two minus points we have:

$$\beta(x) - \alpha(x) = \chi(x) + 2\mu + \rho \geqslant 2\mu - \mu + \rho > M$$

in the case when $\chi(x) + \rho \leqslant 0$ and

$$\beta(x) - \alpha(x) = 2\mu > M,$$

in the case when $\chi(x) + \rho \geqslant 0$;

(ii) for x between two plus points we have

$$\beta(x) - \alpha(x) = -\chi(x) + 2\mu + \rho \geqslant 2\mu - \mu + \rho > M,$$

in the case when $\chi(x) - \rho \geqslant 0$ and

$$\beta(x) - \alpha(x) = 2\mu > M$$

in the case when $\chi(x) - \rho \leqslant 0$;

(iii) for x between a plus point and a minus point or between a minus point and a plus point we get

$$\beta(x) - \alpha(x) = 2\mu > M;$$

(iv) for $x \in [0, t_1]$ or for $x \in [z_l, 1]$ where t_1 and z_l are minus points we have:

$$\beta(x) - \alpha(x) = \chi(x) + 2\mu + \rho \geqslant 2\mu - \mu + \rho > M$$

in the case when $\chi(x) + \rho \leqslant 0$ and

$$\beta(x) - \alpha(x) = 2\mu > M$$

in the case when $\chi(x) + \mu \geqslant 0$;

(v) for $x \in [0, t_1]$ or for $x \in [z_l, 1]$ where t_1 and z_l are plus points we have:

$$\beta(x) - \alpha(x) = -\chi(x) + 2\mu + \rho \geqslant 2\mu - \mu + \rho > M$$

in the case when $\chi(x) - \rho \geqslant 0$ and

$$\beta(x) - \alpha(x) = 2\mu > M,$$

in the case when $\chi(x) - \rho \leqslant 0$.
Thus $\beta(x) - \alpha(x) > M$ for each $x \in [0, 1]$.

We finish this section with:

Definition 2. *The function $g \in \mathcal{C}$ is said to alternate n times if there exists an $n+1$ point set $\{x_0, x_1, \ldots, x_n\}$, $0 \leqslant x_0 < x_1 < \ldots < x_n \leqslant 1$ such that, for $i = 0, 1, \ldots, n$:*

$$|g(x_i)| = \|g\| \text{ and } g(x_i) = (-1)^i g(x_0).$$

The set $\{x_0, x_1, \ldots, x_n\}$ is called an alternate of g of length $n+1$.

3. In this section we give the proof of the theorem from first section. The necessity of the condition in the theorem was proved in [1].

To verify the sufficiency of the condition, let us consider the elements ψ and φ as in the previous section and fix a point x_i, $1 \leqslant i \leqslant l$ in the set (1) such that t_i is a minus point of χ and z_i is a plus point of χ. In the case $l = 1$, a point with these properties may not exist, but the same arguments apply to the case that t_i is a plus point and z_i is a minus point.

Let $p = k - l \geqslant 1$. First we consider the case where p even i.e. $p = 2r$. Then in the interval (t_i, z_i) we choose the points:

(2) $$y_1 < y_2 < \ldots < y_{2r-1} < y_{2r}$$

and from the set E we choose a subset of points

(3) $$e_1, e_2, \ldots, e_{i-1}, t_i, z_i, e_{i+1}, \ldots, e_l$$

where $e_j = t_j$ or z_j, $j = 1, 2, \ldots, i-1, i+1, \ldots, l$ so that (3) becomes a sequence of points with alternating sign according to the definition 1.

Now in the plane we consider the points

$$P_j(v_j, w_j), \qquad j = 1, 2, \ldots, k+1$$

where

(i) for $v_j = e_j$, e_j from the set (3), we put $w_j = \beta(e_j) = \psi(e_j) + \mu$ if e_j is a plus point of χ and $w_j = \alpha(e_j) = \psi(e_j) - \mu$ if e_j is a minus point;

(ii) for $v_j = y_j$, $j = 1, 2, \ldots, 2r$, we put: $w_j = \beta(y_j) = \psi(y_j) + \mu$ if j is odd and $w_j = \alpha(y_j) = \psi(y_j) - \mu$ if j is even:

(iii) for $v_j = t_i$ and $v_j = z_i$ we put $w_j = \psi(t_i) - \mu$ and $w_j = \psi(z_i) + \mu$ respectively.

Finally, form $P_0(0, \delta)$ and $P_{k+2}(1, \delta)$, two additional points in the plane with δ a real number; let $\lambda(x)$ be the piecewise linear function with vertices P_j, $j = 0, 1, \ldots, k+2$.

In the case p odd. i.e. $p = 2r + 1$ we construct similarly a piecewise linear function $\lambda(x)$ by first choosing a point $y_{2r+1} \in (z_i, t_{i+1})$ and then choosing the set:

(4) $$e_1, e_2, \ldots, e_{i-1}, t_i, z_i, y_{2r+1}, e_{i+1}, \ldots, e_l$$

where $e_j = t_j$ or z_j, $j = 1, 2, \ldots, i-1, i+1, \ldots, l$ such that (4) is a sequence of points with alternating sign, and the point y_{2r+1} is considered a minus point of χ. In this case the vertex of the function $\lambda(x)$ with the abscissa y_{2r+1} will have the ordinate $\psi(y_{2r+1}) - \mu$.

Define a function f by:

$$f(x) = \begin{cases} \beta(x) & \text{if} \quad \lambda(x) > \beta(x) \\ \alpha(x) & \text{if} \quad \lambda(x) < \alpha(x) \\ \lambda(x) & \text{otherwise} \end{cases}$$

This function is defined and continuous on the interval [0, 1] and is simultaneously approximated best by ψ in the family $\mathcal{H}$ and by φ in the family of $\mathcal{Q}$. This follows from the corollary 1 [8].

Indeed, by Lemma 1 we have:

$$\|f - \varphi\| = \mu + \rho \quad \text{and} \quad \|f - \psi\| = \mu.$$

In the case p even the union of the point sets (2) and (3) is an alternant for $f - \psi$ of length $k+1$ while the set (3) is an alternant for $f - \varphi$ of length $l + 1$.

The same conclusion in the case p is odd follows in a similar manner.

REFERENCES

[1] Cimoca, Gh., *On a problem of simultaneous best approximation*. Revue d'analyse numérique et de la théorie de l'approximation, **3**, 21–24 (1974).

[2] Deutsch, F., Morris, P. D., Singer, I., *On a problem of T.J. Rivlin in approximation theory*. J. Approx. Theory, **2**, 342–354 (1969).

[3] Motzkin, T. S., *Approximation by curves of a unisolvent family*. Bull. Amer. Math. Soc., **55**, 789–793 (1969).

[4] Rivlin, T. J., *A duality theorem and upper bounds for approximations. Abstract Spaces and Approximations*. Eds., P.L. Butzer and B. Sz-Nagy, 274–280, Birkhäuser Verlag, (Basel), 1969.

[5] Sprecher, D. A., *Simultaneous Best Approximations with Two Polynomials*. J. Approx. Theory, **2**, 384–388 (1969).

[6] Sprecher, D. A., *On Simultaneous Chebyshev Approximations*. J. Approx. Theory, **4**, 137–146 (1971).

[7] Subrahmanya, M. R., *A note on a problem of Rivlin*. J. Approx. Theory, **6**, 359–361 (1972).

[8] Tornheim, L., *On n-Parameter Families of Functions and Associated Convex Functions*. Trans. Amer. Math. Soc. **69**, 457–467 (1950).

Received 11. II. 1974.

Michigan State University
East Lansing, MI 48824 U.S.A.