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ON THE DIVIDED DIFFERENCES AND FRÉCHET DERIVATIVES

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In this paper a connection between the divided differences and the Fréchet derivatives is given. Such connection appeared axiomatically in S. ULM's paper [5], but we can give a sufficient condition for the existence of Fréchet derivatives as the limit of the divided differences.

We recall the definition of the divided difference of a mapping P of the linear normed space X into linear normed space Y.

D e f i n i t i o n: *The linear and continuous mapping* $P_{u,v}$ (*i, e.* $P_{u,v} \in \mathcal{L}(X, Y)$) *defined on* X *with values in* Y *is a divided difference of* P *in the distinct points* u, v *of* X, *iff*

$$P_{u,v}(u - v) = P(u) - P(v). \tag{1}$$

We note that the study of the divided differences is given in [3].

The fact that the divided differences are not in the strong relation with the Fréchet derivatives, was illustrated by the following example given by M. BALÁZS [1] at the suggestion of the author:

E x a m p l e . Let be the identical mapping of the R^2, with the usual norm, $h = (h^1, h^2)$ a point of the R^2, $\varnothing = (0, 0)$ the origin of the R^2, and $u = (u^1, u^1) \in R^2$

We have

$$I'(\varnothing) \cdot h = I \cdot h = h.$$

We consider the following divided difference of the mapping I in the different points u and $\varnothing$:

$$I_{u,\emptyset}h = \frac{h^1}{u^1}[I(u) - I(\emptyset)] = \frac{h^1}{u^1}\cdot u = \frac{h^1}{n^1}(u^1, u^1) = (h^1, h^1)$$

$$\lim_{u\to\emptyset} I_{u,\emptyset}h = \lim_{u\to\emptyset}(h^1, h^1) = (h^1, h^1) \neq I'(\emptyset)\cdot h.$$

We give now a sufficient condition for the existence of the Fréchet derivatives, as the limit of divided differences.

Proposition 1. *Let X be a real normed vector space, Y a real Banach space, and P a mapping of X into Y. If P has a divided difference $P_{u,v}$ with the property*

$$(2)\qquad ||P_{u,x} - P_{v,x}|| \leqslant L||u - v||,$$

for every points u, v, x belonging to X, then:

(i) *for every x_0 of X there exists $P'(x_0)$.*

(ii) $\lim_{u\to x_0} P_{u,x_0} = P'(x_0)$.

Proof. Let $(u_n)_{n\in N}$ be a sequence of points of X, with $\lim u_n = x_0$, and we consider the sequence of the divided differences $(P_{u_n,x_0})_{n\in N}$. By using the condition (2), we have

$$(3)\qquad ||P_{u_n,x_0}x - P_{u_m,x_0}x|| \leqslant L\,||u_n - u_m||\cdot||x||$$

for every x of X. Then for every fixed x of X, the sequence $(P_{u_n,x_0}x)_{n\in N}$ is a Cauchy sequence, and Y being a Banach space there exists $y = \lim P_{u_n,x_0}x$. We define the mapping A of X into Y, puting for every x of X, $Ax = y$. We observe that the mapping A is well defined (i, e. for the other sequence v_n converging to x_0, we obtain the same y). Indeed

$$||P_{u_n,x_0}x - P_{v_n,x_0}x|| \leqslant L(||u_n - x_0|| + ||v_n - x_0||)||x||.$$

The linearity of A is evidently.

From the condition (2) we obtain

$$\Big|\,||P_{u_n,x_0} - P_{u_m,x_0}||\,\Big| \leqslant \Big|\,||P_{u_m,x_0} - P_{u_n,x_0}||\,\Big| \leqslant L\,||u_m - u_n||,$$

hence the sequence P_{u_n,x_0} is a Cauchy sequence, which is bounded.

Then we have

$$||P_{u_n,x_0}|| \leqslant M \quad \text{hence} \quad ||P_{u_n,x_0}x|| \leqslant M\,||x||$$

and for $n \to \infty$, we obtain the boundeness of the mapping A.

We prove the equality

$$(4)\qquad A = P'(x_0).$$

For $m \to +\infty$, by the inequality (3) we obtain

$$||P_{u_n,x_0}x - Ax|| \leqslant L||u_n - x_0||\cdot||x||.$$

Now we have

$$||(P_{u,x_0} - A)x|| \leqslant ||(P_{u,x_0} - P_{u_n,x_0})x|| + ||(P_{u_n,x_0} - A)x|| \leqslant$$
$$\leqslant L(||u - x_0|| + 2||u_n - x_0||)\cdot||x||,$$

i.e.

$$||(P_{u,x_0} - A)x|| \leqslant L||u - x_0||\cdot||x||.$$

Hence we can write

$$\frac{||P(u) - P(x_0) - A(u - x_0)||}{||u - x_0||} = \frac{||(P_{u,x_0} - A)(u - x_0)||}{||u - x_0||} \leqslant L||u - x_0||.$$

By the unicity of the Fréchet derivative, we have (4).

Then

$$||P_{u,x_0} - P'(x_0)|| = \sup_{||x||\leqslant 1}||(P_{u,x_0} - P'(x_0))x|| \leqslant$$
$$\leqslant L\sup_{||x||\leqslant 1}\{||u - x_0||\cdot||x||\} = L||u - x_0||.$$

i.e. $\lim_{u\to x_0} P_{u,x_0} = P'(x_0)$.

On the other hand, it is possibly to put the problem of the existence of the divided difference which converges to the Fréchet derivative, in the hypothesis that P is derivable. We can give only a partial answer to this question.

Remark 1. For the real case, we obtain a consequence of the theorem of T. POPOVICIU [4].

Remark 2. If the condition (2) is satisfied in a closed ball of X, the Proposition is true in that ball.

Proposition 2. *If the mapping P of the real normed space X in the space Y of the same type has the continuous Fréchet derivative in every point x of the X, then there exists a divided difference $P_{u,v}$ of the P, with*

$$\lim_{u \to x} P_{u,x} = P'(x).$$

Proof. We consider the divided difference [2]:

$$P_{u,x} = \int_0^1 P'(x + t(u - x))\mathrm{d}t.$$

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