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SOME PROPERTIES OF THE LINEAR POSITIVE OPERATORS (III)

by

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1. Introduction

There are many approximation processes constructed by means of linear positive operators which enable us to approximate from a qualitative point of view. This means that the operators preserve the shape of the elements from the domain. Likewise, for some concrete operators the remainder-term on the class of non-concave functions has a constant sign. With such remarkable properties, we note the operators that are attributed to BERNSTEIN, SZÁSZ–MIRAKYAN, MEYER–KÖNIG and ZELLER. The case of one variable was treated by many authors [2]–[6], [8]–[9], [13]. It is natural to ask what happens in the case of many variables.

The aim of this paper is to give a partial answer. Firstly we prove that if $L : C(K) \to C(K)$, K being a compact convex set from $\mathbf{R}^m$, is a linear positive operator which reproduces the affine functions, then $f - Lf \leqslant 0$ for every non-concave function f on K. The case of one variable was treated by the author in [4]. Further, some properties of the sequence of Bernstein operators are investigated. It is shown that, in the case of two variables, this sequence is non-increasing on the class of non-concave functions. Finally, we prove that the Bernstein operators defined on a simplex preserve the S-convexity (in the sense of I.Schur). The proofs are made for the two-dimensional case.

At the end of this paper we get an example of polynomial operator which interpolates at the vertices of a convex polygon and is positive in its interior.

2. The sign of the remainder-term

We use the following notations and terminology:
K is a compact, convex set in $\mathbf{R}^m$, $m \geqslant 1$;

$$c = (c_1, c_2, \ldots, c_m), \quad x = (x_1, x_2, \ldots, x_m), \quad t = (t_1, \ldots, t_k, \ldots, t_m)$$

$$e_0(t) = 1, \quad e_{1,k}(t) = t_k, \quad \langle c, x\rangle = \sum_{k=1}^{m} c_k x_k;$$

If $f: K \to \mathbf{R}$, then *the epigraph of* f is

$$\mathrm{Epi}(f) = \{(x, y) | x \in K, \ y \in \mathbf{R}, \ y \geqslant f(x)\}.$$

A function $f: K \to \mathbf{R}$ is called *non-concave* on K iff for every $y_j \in K$, $j = 1, 2, \ldots, p$, $p \geqslant 2$.

$$f\Big(\sum_{j=1}^{p} a_j y_j\Big) \leqslant \sum_{j=1}^{p} a_j f(y_j)$$

whenever $a_j \in [0, 1]$, $j = 1, 2, \ldots, p$, $a_1 + a_2 + \ldots + a_p = 1$.

By $B(K)$ resp. $C(K)$ we denote the linear normed space of all functions $K \to \mathbf{R}$ which are bounded, respectively continuous on K. They are normed by means of uniform norm.
An affine function on K is defined as

$$e(x) = \langle c, x\rangle + r, \qquad x \in K,$$

where r is a real number. Let $\mathcal{E}$ be the collection of all such affine functions. A linear operator $L: B(K) \to B(K)$ *preserves the affine functions* iff

(1) $$Le = e \quad \text{for every} \quad e \in \mathcal{E}.$$

It is clear that (1) is equivalent with

$$Le_0 = e_0, \quad Le_{1,k} = e_{1,k}, \ k = 1, 2, \ldots, m.$$

THEOREM 1. *If* $L: C(K) \to C(K)$ *is a linear positive operator which preserves the affine functions, then*

$$f(x) \leqslant (Lf)(x), \qquad x \in K$$

for every $f \in C(K)$ *which is non-concave on* K.

Proof. The continuity and non-concavity of f imply that Epi(f) is a convex body in $\mathbf{R}^{m+1}$. Let

$$H = \{(x, y) | x \in \mathbf{R}^m, \ y \in \mathbf{R}, \ \langle c, x\rangle + c_{n+1} y + c_{n+2} = 0\}$$

be an arbitrary closed hyperplane in $\mathbf{R}^{m+1}$ which bounds Epi(f), say

$$\langle c, x\rangle + c_{n+1} y + c_{n+2} \geqslant 0 \ \text{for} \ (x, y) \in \mathrm{Epi}(f).$$

Because for each $t \in K$ the point $(t, f(t))$ belongs to Epi(f), we may write

$$\langle c, t\rangle + c_{n+1} f(t) + c_{n+2} \geqslant 0.$$

By means of the monotonicity property of L

(2) $$\langle c, x\rangle + c_{n+1}(Lf)(x) + c_{n+2} \geqslant 0, \quad x \in K.$$

In conclusion, if Epi(f) lies on one side of an arbitrary closed hyperplane, then $\{(x, (Lf)(x))\}$ lies on the same side. If we assume

$$\{(x, (Lf)(x))\} \cap \mathrm{Epi}(f) = \emptyset,$$

then, according to the second separation theorem of convex sets (see [1] p. 58 or [12] p. 65), there exists a closed hyperplane H_1 in $\mathbf{R}^{m+1}$ strictly separating $\{(x, (Lf)(x))\}$ and Epi(f). Thus for

$$H_1 = \{(x, y) | x \in \mathbf{R}^m, \ y \in \mathbf{R}, \ \langle \bar{c}, x\rangle + \bar{c}_{n+1} y + \bar{c}_{n+2} = 0\}$$

one has

$$\langle \bar{c}, x\rangle + \bar{c}_{n+1} y + \bar{c}_{n+2} > 0 \ \text{for} \ (x, y) \in \mathrm{Epi}(f)$$

and

$$\langle \bar{c}, x\rangle + \bar{c}_{n+1}(Lf)(x) + \bar{c}_{n+2} < 0.$$

But this contradicts (2) and the proof is complete.

As an application of the above theorem we may prove the following two-dimensional variant of a well known result by T. POPOVICIU [9]. Though this result was established by V. I. VOLKOV [14] we present a new, shorter proof.

THEOREM 2. *Let K be a compact, convex set in $\mathbf{R}^2$ and $L_n : C(K) \to C(K)$, $n = 1, 2, \ldots$, be a sequence of linear positive operators which preserve the affine functions. If $\Omega \in C(K)$ is defined as*

$$\Omega(x, y) = x^2 + y^2$$

and

$$\lim_{n\to\infty} \|\Omega - L_n\Omega\| = 0,$$

then

$$\lim_{n\to\infty} \|f - L_n f\| = 0 \quad \textit{for every} \quad f \in C(K).$$

Proof. Let $C^{(2)}(K)$ be the subspace of $C(K)$ formed with all functions which have continuous partial derivatives of the second order on K. For $f \in C^{(2)}(K)$ let us denote

$$\delta_f^+ = \frac{1}{2}\left[f''_{x^2} + f''_{y^2} + \sqrt{(f''_{x^2} - f''_{y^2})^2 + 4 f''^2_{xy}}\right]$$

$$\delta_f^- = \frac{1}{2}\left[f''_{x^2} + f''_{y^2} - \sqrt{(f''_{x^2} - f''_{y^2})^2 + 4 f''^2_{xy}}\right]$$

as well as

(3) $$m_f = \min_{(x,y)\in K} \delta_f^-(x, y), \qquad M_f = \max_{(x,y)\in K} \delta_f^+(x, y)$$

For (x, y) arbitrary in K put

$$\varphi[\lambda; (x, y)] = \lambda^2 - \lambda[f''_{x^2}(x, y) + f''_{y^2}(x, y)] + $$
$$+ f''_{x^2}(x, y) f''_{y^2}(x, y) - [f_{xy}(x, y)]^2.$$

It may be seen that

$$\varphi[m_f; (x, y)] \geqslant 0, \qquad \varphi[M_f; (x, y)] \geqslant 0$$

(4) $$\varphi[f''_{x^2}(x, y); (x, y)] \leqslant 0.$$

An element g from $C^{(2)}(K)$ is non-concave if its Hessian matrix

$$\begin{Vmatrix} g''_{x^2} & g''_{xy} \\ g''_{xy} & g''_{y^2} \end{Vmatrix}$$

is positive semi-definite for every $(x, y) \in K$ (see [11], p. 27). Let $f \in C^{(2)}(K)$ and

$$g = \frac{1}{2} M_f \cdot \Omega - f, \qquad g = f - \frac{1}{2} m_f \cdot \Omega.$$

According to (3)—(4) one observes that g, h are non-concave on K. For instance, this may be motivated by the equalities

$$g''_{x^2} = M_f - f''_{x^2} \geqslant 0$$

$$g''_{x^2} \cdot g''_{y^2} - g''^2_{xy} = \varphi[M_f; \cdot] \geqslant 0.$$

Therefore, theorem 1 implies

$$g \leqslant L_n g$$

$$h \leqslant L_n h \quad \text{on } K,$$

which are of course equivalent with

(5) $$\frac{1}{2} m_f[L_n\Omega - \Omega] \leqslant L_n f - f \leqslant \frac{1}{2} M_f[L_n\Omega - \Omega], \; n = 1, 2, \ldots.$$

If the hypothesis is verified, then (5) furnishes

$$\lim_{n\to\infty} \|f - L_n f\| = 0 \quad \text{for every} \quad f \in C^{(2)}(K).$$

Finally, the fact that $C^{(2)}(K)$ is dense in $C(K)$ and $\|L_n\| = 1$, $n = 1, 2, \ldots$, proves our theorem.

Another consequence of theorem 1 is the following representation of the remainder-term in the approximation by means of the operators $L_n : C[a, b] \to C[a, b]$, $n = 1, 2, \ldots$. Some similar ideas were exposed by the present author in [4].

An operator $L : C[a, b] \to C[a, b]$ is called *strictly positive* relative to $K_1 \subseteq [a, b]$, iff

$$f \in C[a, b], \quad f \geqslant 0, f \neq 0 \quad \text{on} \quad [a, b]$$

implies

$$Lf \geqslant 0 \quad \text{on} \quad [a, b], \; Lf > 0 \text{ on } K_1.$$

From the proof of the theorem 1 we see that for such an operator $L : C[a,b] \to C[a, b]$ which is moreover linear and preserves the linear functions one has

$$Lh - h > 0 \quad \text{on} \quad K_1$$

whenever h is a convex function on $[a, b]$, i.e., $[x_1, x_2, x_3; h] > 0$ x_1, x_2, x_3 being arbitrary distinct points on $[a, b]$. For $x_0 \in K_1$ let $F_{x_0} : C[a, b] \to \mathbf{R}$ be defined as

$$F_{x_0}(f) = (Lf)(x_0) - f(x_0)$$

Then

$$F_{x_0}(h) > 0$$

for any convex function $h \in C[a, b]$. An earlier result of T. POPOVICIU [10] asserts that there exist three distinct points ξ_i from $[a, b]$ so that

$$F_{x_0}(f) = F_{x_0}(e_2)[\xi_1, \xi_2, \xi_3; f], \quad e_2(t) = t^2.$$

In this way we have proved

THEOREM 3. *If $L : C[a, b] \to C[a, b]$ is a linear, strictly positive operator relative to $K_1 \subseteq [a, b]$, and*

$$Le_k = e_k, \quad k = 0, 1, \quad e_j(t) = t^j,$$

then to each function $f \in C[a, b]$ corresponds a system ξ_1, ξ_2, ξ_3 of distinct points from $[a, b]$ such that

$$(Lf)(x_0) - f(x_0) = [(Le_2)(x_0) - e_2(x_0)] \cdot [\xi_1, \xi_2, \xi_3; f], \quad x_0 \in K_1.$$

By $[\xi_1, \xi_2, \xi_3; f]$ we have denoted the divided difference of the second order at the knots ξ_1, ξ_2, ξ_3. In this way we see that the remainder-term, in the approximtaion by means of linear strictly-positive operators which preserve the linear functions, has a *simple from.*

3. The behaviour of Bernstein's operators on the class of non-concave functions of two variables

Let us denote

$$K_1 = \{(x, y) \in \mathbf{R}^2 \mid x \in [0, 1], y \in [0, 1]\}$$

(6) $$K_2 = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}.$$

$$b_{n,j}(t) = \binom{n}{j} t^j (1 - t)^{n-j}$$

(7) $$p_{n,k,i}(x, y) = \binom{n}{k}\binom{n-k}{i} x^k y^i (1 - x - y)^{n-k-i}.$$

The Bernstein operators $B_{n,m} : B(K_1) \to B(K_1)$, $n, m = 1, 2, \ldots$, $B_n : B(K_2) \to B(K_2)$, $n = 1, 2, \ldots$, are defined respectively by

(8) $$(B_{n,m}f)(x, y) = \sum_{k=0}^{n}\sum_{i=0}^{m} b_{n,k}(x)\, b_{m,i}(y) f\left(\frac{k}{n}, \frac{i}{m}\right), \qquad (x, y) \in K_1,$$

(9) $$(B_n f)(x, y) = \sum_{k=0}^{n}\sum_{i=0}^{n-k} p_{n,k,i}(x, y) f\left(\frac{k}{n}, \frac{i}{n}\right), \qquad (x, y) \in K_2.$$

THEOREM 4. *If $f \in B(K_1)$ is a non-concave function on K_1, then for every $(x, y) \in K_1$*

$$(B_{n,m}f)(x, y) \geq (B_{n+1,\, m+1}f)(x, y), \quad n, m = 1, 2, \ldots.$$

Proof. We have

$$(B_{n,m}f)(x, y) = \sum_{k=0}^{n}\sum_{i=0}^{m} b_{n,k}(x)\, b_{m,i}(y) [(1 - x)(1 - y) + y(1 - x) +$$
$$+ x(1 - y) + xy] f\left(\frac{k}{n}, \frac{i}{m}\right) =$$
$$= \sum_{k=0}^{n}\sum_{i=0}^{m} b_{n+1,k}(x)\, b_{m+1,i}(y) \frac{(n - k + 1)(m - i + 1)}{(n + 1)(m + 1)} f\left(\frac{k}{n}, \frac{i}{m}\right) +$$
$$+ \sum_{k=0}^{n}\sum_{i=1}^{m+1} b_{n+1,k}(x)\, b_{m+1,i}(y) \frac{i(n - k + 1)}{(n + 1)(m + 1)} f\left(\frac{k}{n}, \frac{i - 1}{m}\right) +$$
$$+ \sum_{k=1}^{n+1}\sum_{i=0}^{m} b_{n+1,k}(x)\, b_{m+1,i}(y) \frac{k(m - i + 1)}{(n + 1)(m + 1)} f\left(\frac{k - 1}{n}, \frac{i}{m}\right) +$$
$$+ \sum_{k=1}^{n+1}\sum_{i=1}^{m+1} b_{n+1,k}(x)\, b_{m+1,i}(y) \frac{ki}{(n + 1)(m + 1)} f\left(\frac{k - 1}{n}, \frac{i - 1}{m}\right).$$

Let us denote

$$\alpha_{ik}^{(1)} = \frac{(n - k + 1)(m - i + 1)}{(n + 1)(m + 1)}, \qquad \alpha_{ik}^{(2)} = \frac{i(n - k + 1)}{(n + 1)(m + 1)}$$

$$\alpha_{ik}^{(3)} = \frac{k(m - i + 1)}{(n + 1)(m + 1)}, \qquad \alpha_{ik}^{(4)} = \frac{ki}{(n + 1)(m + 1)}$$

$$z_{ik}^{(1)} = \left(\frac{k}{n}, \frac{i}{m}\right), \quad z_{ik}^{(2)} = \left(\frac{k}{n}, \frac{i - 1}{m}\right), \quad z_{ik}^{(3)} = \left(\frac{k - 1}{n}, \frac{i}{m}\right)$$

$$z_{ik}^{(4)} = \left(\frac{k - 1}{n}, \frac{i - 1}{m}\right), \quad z_{ik} = \left(\frac{k}{n + 1}, \frac{i}{m + 1}\right)$$

$$D_{ik}(f) = \alpha_{ik}^{(1)} f(z_{ik}^{(1)}) + \alpha_{ik}^{(2)} f(z_{ik}^{(2)}) + \alpha_{ik}^{(3)} f(z_{ik}^{(3)}) + \alpha_{ik}^{(4)} f(z_{ik}^{(4)}) - f(z_{ik})$$
$$i, k = 1, 2, \ldots, n.$$

Since

$$\alpha_{ik}^{(1)} + \alpha_{ik}^{(2)} + \alpha_{ik}^{(3)} + \alpha_{ik}^{(4)} = 1, \qquad i, k = 0, 1, \ldots, n+1,$$

we write

$$D_{00}(f) = 0, \; D_{0k}(f) = \frac{n-k+1}{n+1} f\left(\frac{k}{n}, 0\right) + \frac{k}{n+1} f\left(\frac{k-1}{n}, 0\right) - f\left(\frac{k}{n+1}, 0\right)$$

$$k = 1, 2, \ldots, n+1$$

and similarly

$$D_{i0}(f) = \frac{m-i+1}{m+1} f\left(0, \frac{i}{m}\right) + \frac{i}{m+1} f\left(0, \frac{i-1}{m}\right) - f\left(0, \frac{i}{m+1}\right)$$

$$i = 1, 2, \ldots, m+1$$

$$D_{m+1, n+1}(f) = 0$$

$$D_{i, n+1}(f) = \frac{m-i+1}{m+1} f\left(1, \frac{i}{m}\right) + \frac{i}{m+1} f\left(1, \frac{i-1}{m}\right) - f\left(1, \frac{i}{m+1}\right)$$

$$i = 1, 2, \ldots, m$$

$$D_{m+1, k}(f) = \frac{n-k+1}{n+1} f\left(\frac{k}{n}, 1\right) + \frac{k}{n+1} f\left(\frac{k-1}{n}, 1\right) - f\left(\frac{k}{n+1}, 1\right)$$

$$k = 1, 2, \ldots, n.$$

In the same time, for $i = 0, 1, \ldots, m+1$, $k = 0, 1, \ldots, n+1$

$$\alpha_{ik}^{(1)} z_{ik}^{(1)} + \alpha_{ik}^{(2)} z_{ik}^{(2)} + \alpha_{ik}^{(3)} z_{ik}^{(3)} + \alpha_{ik}^{(4)} z_{ik}^{(4)} = z_{ik}.$$

If $f: K_1 \to \mathbf{R}$ is non-concave on its domain, then

$$(10) \qquad D_{ik}(f) \geqslant 0 \qquad \begin{pmatrix} i = 0, 1, \ldots, m+1 \\ k = 0, 1, \ldots, n+1 \end{pmatrix}.$$

But from the above equalities and taking into account (10)

$$(B_{n,m} f)(x, y) - (B_{n+1, m+1} f)(x, y) =$$

$$= \sum_{k=0}^{n+1} \sum_{i=0}^{m+1} b_{n+1, k}(x)\, b_{m+1, i}(y)\, D_{ik}(f) \geqslant 0.$$

A similar result may be established for the sequence of operators whose images are defined in (9).

THEOREM 5. *For an arbitrary* $f \in B(K_2)$ *which is non-concave on* K_2

$$(B_n f)(x, y) \geqslant (B_{n+1} f)(x, y), \qquad (x, y) \in K_2, \; n = 1, 2, \ldots.$$

Proof. One introduces the numbers

$$\Delta_{ki}(f) = \frac{i}{n+1} f\left(\frac{k}{n}, \frac{i-1}{n}\right) + \frac{k}{n+1} f\left(\frac{k-1}{n}, \frac{i}{n}\right) + \frac{n-k-i+1}{n+1} f\left(\frac{k}{n}, \frac{i}{n}\right) -$$

$$- f\left(\frac{k}{n+1}, \frac{i}{n+1}\right)$$

where for instance we have tacitly assumed that

$$\text{for } k = 0, \qquad \frac{k}{n+1} f\left(\frac{k-1}{n}, \frac{i}{n}\right) = 0, \qquad i = 0, 1, \ldots, n,$$

and

$$\Delta_{00}(f) = \Delta_{0, n+1}(f) = 0, \; \ldots .$$

We have

$$(B_n f)(x, y) = \sum_{k=0}^{n} \sum_{i=}^{n-k} p_{n,k,i}(x, y)[(1 - x - y) + x + y] f\left(\frac{k}{n}, \frac{i}{n}\right) =$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{n+1-k} \frac{n-k-i+1}{n-1} f\left(\frac{k}{n}, \frac{i}{n}\right) p_{n+1, k, i}(x, y) +$$

$$+ \sum_{i=0}^{n} \frac{n-i+1}{n+1} f\left(0, \frac{i}{n}\right) p_{n+1, 0, i}(x, y) +$$

$$+ \sum_{k=1}^{n} \sum_{i=0}^{n+1-k} \frac{k}{n+1} f\left(\frac{k-1}{n}, \frac{i}{n}\right) p_{n+1, k, i}(x, y) +$$

$$+ p_{n+1, n+1, 0}(x, y) f(1, 0) +$$

$$+ \sum_{i=1}^{n+} \frac{i}{n+1} f\left(0, \frac{i-1}{n}\right) p_{n+1, 0, i}(x, y) +$$

$$+ \sum_{k=1}^{n} \sum_{i=0}^{n+1-k} \frac{i}{n+1} f\left(\frac{k}{n}, \frac{i-1}{n}\right) p_{n+1, k, i}(x, y) =$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{n+1-k} \Delta_{k, i}(f)\, p_{n+1, k, i}(x, y) + (B_{n+1} f)(x, y) +$$

$$+ \sum_{i=0}^{n+1} \Delta_{0i}(f)\, p_{n+1, 0, i}(x, y).$$

In other words

$$(B_n f)(x, y) - (B_{n+1} f)(x, y) = \sum_{k=0}^{n} \sum_{i=0}^{n+1-k} \Delta_{ki}(f)\, p_{n+1,k,i}(x, y).$$

But $f \in B(K_2)$, f non-concave on K_2, assures the validity of the inequalities

$$\Delta_{ki}(f) \geqslant 0, \qquad \begin{pmatrix} k = 0, 1, \ldots, n \\ i = 0, 1, \ldots, n+1-k \end{pmatrix}.$$

Taking into account that $p_{n+1,k,i}(x, y) \geqslant 0$, $(x, y) \in K_2$, the proof is complete.

Remark. Any non-concave function from $C(K_1)$ or from $C(K_2)$ may be uniform approximated by a non-increasing sequence of polynomials.

4. The Bernstein operators and (S)-convexity

Let $S = ||s_{ik}||$ $i, k = 1, 2, \ldots, m$, be a doubly-stochastic matrix, i.e.

$$s_{ik} \geqslant 0, \sum_{i=1}^{m} s_{ik} = \sum_{k=1}^{m} s_{ik} = 1, \qquad i, k = 1, 2, \ldots, m.$$

If $x = (x_1, x_2, \ldots, x_m) \in \mathbf{R}^m$ then the *Schur-transform* of x is the poin

$$y = Sx = (y_1, y_2, \ldots, y_m)$$

where

$$y_i = \sum_{k=1}^{m} s_{ik} x_k, \qquad i = 1, 2, \ldots, m.$$

A subset D from $\mathbf{R}^m$ is called an *admissible domain* iff it verifies:

i) $x = (x_1, x_2, \ldots, x_m) \in D$ implies $x_\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}) \in D$, π being an arbitrary permutation of $\{1, 2, \ldots, m\}$.

ii) for every matrix S and any point $x \in D$, the Schur-transform Sx belongs likewise to D. Examples of such admissible domains in $\mathbf{R}^2$ are the sets K_1, K_2 defined by (6).

According to A. OSTROWSKI [7] a function $f: D \to \mathbf{R}$, D being an admissible domain in $\mathbf{R}^m$, $m \geqslant 2$, is called *S-convex (in the sense of I. Schur)* if for every matrix S and any point $x \in D$

$$f(Sx) \leqslant f(x).$$

He notes that a S-convex function must be symmetric on its domain. Also, if $f: D \to \mathbf{R}$ has on D continuous partial derivatives of the first order, then a sufficient condition for S-convexity is

$$(11) \qquad (x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0 \text{ on } D.$$

If D is open then (11) is also a necessary condition.

L e m m a *The Bernstein operator $B_n: B(K_2) \to B(K_2)$ preserves the symmetry, that is*

$$f \in B(K_2), \quad f(x, y) = f(y, x)$$

implies

$$(B_n f)(x, y) = (B_n f)(y, x), \qquad (x, y) \in K_2,$$

Proof. From

$$\sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ik} = \sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki}, \qquad \binom{n}{k}\binom{n-k}{i} = \binom{n}{i}\binom{n-i}{k}$$

we get

$$(B_n f)(y, x) = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \binom{n}{i}\binom{n-i}{k} y^k x^i (1 - x - y)^{n-k-i} f\left(\frac{k}{n}, \frac{i}{n}\right) =$$

$$= \sum_{k=0}^{n} \sum_{i=0}^{n-k} p_{n,k,i}(x, y) f\left(\frac{i}{n}, \frac{k}{n}\right) = (B_n f)(x, y)$$

where $p_{n,k,i}$ was defined as in (7).

Further we show that the Schur-convexity remains invariant under B_n. In the case of one variable such preserving properties for positive linear operators were exposed in [2]—[6], [8]. It is worth mentioning that the convexity-preserving property for the usual BERNSTEIN operator (see [8]) was used in statistics by W. WEGMÜLLER [15].

THEOREM 6. *Let $f \in B(K_2)$ be a symmetric function which is S-convex on K_2. Then $B_n f$, $n = 1, 2, \ldots$, are likewise S-convex functions on K_2.*

Proof. We find

$$\frac{\partial B_n f}{\partial x} = n \sum_{k=0}^{n-1} \sum_{i=0}^{n-1-k} p_{n-1,k,i} \left[f\left(\frac{k+1}{n}, \frac{i}{n}\right) - f\left(\frac{k}{n}, \frac{i}{n}\right)\right]$$

$$\frac{\partial B_n f}{\partial y} = n \sum_{k=0}^{n-1} \sum_{i=0}^{n-1-k} p_{n-1,k,i} \left[f\left(\frac{k}{n}, \frac{i+1}{n}\right) - f\left(\frac{k}{n}, \frac{i}{n}\right)\right].$$

On account of the above lemma we shall use (11). Put

$$D(B_nf,\ (x,\ y)) = \frac{1}{n}(x-y)\left(\frac{\partial B_nf}{\partial x} - \frac{\partial B_nf}{\partial y}\right)$$

and

$$\Delta f(\alpha,\ x,\ y) = f(x,\ y) - f[\alpha x + (1-\alpha)y,\ (1-\alpha)x + \alpha y].$$

It is easy to see that $f\colon K_2 \to \mathbf{R}$ is S-convex on K_2 if and only if

$$\Delta f(\alpha,\ x,\ y) \geqslant 0 \text{ for every } (x,\ y) \in K_2, \quad \alpha \in [0,\ 1].$$

For n fixed let us denote

$$q_{i,k}(x,\ y) = \binom{n-1}{i}\binom{n-1-i}{k-i} x^i y^i (1-x-y)^{n-k-1}(x-y)(x^{k-2i} - y^{k-2i}),$$

$$(x,\ y) \in K_2$$

These functions have the properties

$$(12)\begin{cases} q_{i,k}(x,\ y) \geqslant 0, \quad (x,\ y) \in K_2 \\ (x-y)[p_{n-1,\,2k-i,i}(x,\ y) - p_{n-1,i,2k-i}(x,\ y)] = q_{i,2k}(x,\ y) \\ \left(i = 0,\ 1,\ \ldots,\ k-1, \quad k = 1,\ 2,\ \ldots,\ \left[\frac{n-1}{2}\right]\right) \\ (x-y)[p_{n-1,\,2k+1-i,i}(x,\ y) - p_{n-1,i,2k-i+1}(x,\ y)] = q_{i,2k+1}(x,\ y) \\ \left(i = 0,\ 1,\ \ldots,\ k, \quad k = 0,\ 1,\ \ldots,\ \left[\frac{n-2}{2}\right]\right). \end{cases}$$

By means of the summation-trick

$$\sum_{k=0}^{n-1}\sum_{i=0}^{n-1-k} A_{ik} = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} A_{ii} + \sum_{k=1}^{\left[\frac{n-1}{2}\right]}\sum_{i=0}^{k-1}(A_{i,2k-i} + A_{2k-i,i}) +$$

$$+ \sum_{k=0}^{\left[\frac{n-2}{2}\right]}\sum_{i=0}^{k}(A_{i,2k+1-i} + A_{2k+1-i,i})$$

we get succesively

$$D(B_nf,(x,\ y)) = (x-y)\sum_{k=0}^{n-1}\sum_{i=0}^{n-1-k} p_{n-1,k,i}(x,\ y)\left[f\left(\frac{k+1}{n},\ \frac{i}{n}\right) - f\left(\frac{k}{n},\ \frac{i+1}{n}\right)\right] =$$

$$= \sum_{k=1}^{\left[\frac{n-1}{2}\right]}\sum_{i=0}^{k-1} q_{i,2k}(x,\ y)\left[f\left(\frac{2k-i+1}{n},\ \frac{i}{n}\right) - f\left(\frac{2k-i}{n},\ \frac{i+1}{n}\right)\right] +$$

$$+ \sum_{k=0}^{\left[\frac{n-2}{2}\right]}\sum_{i=0}^{k} q_{i,2k+1}(x,\ y)\left[f\left(\frac{2k-i+2}{n},\ \frac{i}{n}\right) - f\left(\frac{2k-i+1}{n},\ \frac{i+1}{n}\right)\right] =$$

$$= \sum_{k=1}^{\left[\frac{n-1}{2}\right]}\sum_{i=0}^{k-1} q_{i,2k}(x,\ y)\Delta f\left(\frac{2k-2i}{2k-2i+1},\ \frac{2k-i+1}{n},\ \frac{i}{n}\right) +$$

$$+ \sum_{k=0}^{\left[\frac{n-2}{2}\right]}\sum_{i=0}^{k} q_{i,2k+1}(x,\ y)\Delta f\left(\frac{2k-2i+1}{2k-2i+2},\ \frac{2k-i+2}{n},\ \frac{i}{n}\right).$$

Therefore

$$(13) \quad D(B_nf,\ \cdot) = \sum_{k=1}^{n-1}\sum_{i=0}^{\left[\frac{k-1}{2}\right]} q_{i,k}(\cdot)\cdot\Delta f\left(\frac{k-2i}{k-2i+1},\ \frac{k-i+1}{n},\ \frac{i}{n}\right).$$

Now the S-convexity of f enables us to write

$$\Delta f\left(\frac{k-2i}{k-2i+1},\ \frac{k-i+1}{n},\ \frac{i}{n}\right) \geqslant 0, \qquad \begin{matrix} k = 1,\ 2,\ \ldots,\ n-1 \\ i = 0,\ 1,\ \ldots,\left[\frac{k-1}{2}\right]. \end{matrix}$$

Combining these inequalities with (12)—(13) we conclude with

$$D(B_nf,\ \cdot) \geqslant 0 \text{ on } K_2$$

and (11) finishes the proof.

5. A method of positive interpolation

Let $P_1, P_2, \ldots, P_n$ be the successive vertices of a convex polygon $C_n \subset \mathbf{R}^2$ with n sides. If $f: C_n \to \mathbf{R}$ then we may formulate the following interpolation problem: „to find a linear operator $L_{n-2}: B(C_n) \to B(C_n)$ with the properties

1) $(L_{n-2}f)(P_k) = f(P_k)$, $k = 1, 2, \ldots, n$,

2) $(L_{n-2}f)(x, y)$ is a polynomial of degree $n - 2$ in x and y.

3) if $f \geqslant 0$ in C_n then $L_{n-2}f \geqslant 0$ on the same set." A method for constructing such an interpolation operator is as follows: let $d_i(x, y) = a_i x + b_i y + c_i$, $i = 1, 2, \ldots, n$, such that $d_i(x, y) = 0$ is the equation of the hyperplane $(P_i P_{i+1})$, $i = 1, 2, \ldots, n$, $(P_n P_{n+1}) = (P_n P_1)$. Putting

$$l_{nk}(x, y) = \prod_{\substack{i=1 \\ i \neq k, k-1}}^{n} \frac{d_i(x, y)}{d_i(x_k, y_k)}, \quad P_k = (x_k, y_k), \; k = 1, 2, \ldots, n.$$

we have

$$l_{nk}(P) \geqslant 0 \quad \text{for} \quad P = (x, y) \in C_n$$

$$l_{nk}(P_j) = \begin{cases} 1 & \text{for} \quad j = k \\ 0 & \text{for} \quad j \neq k, \end{cases}$$

and if we define $L_{n-2}: B(C_n) \to B(C_n)$ as

$$L_{n-2}f = L_{n-2}[C_n; f, \cdot] = \sum_{k=1}^{n} f(P_k) l_{nk}(\cdot), \quad n = 3, 4, \ldots, \tag{14}$$

the problem is solved. We want to use this operator in the following approximation problem, which is yet unsolved: let $K = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 \leqslant 1\}$ and $Bd \cdot K = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$. To find, if it is possible, a „dense" system of distinct points $P_{1n}, P_{2n}, \ldots, P_{nn}$ on $Bd \cdot K$, such that

$$\lim_{n \to \infty} L_{n-2}[P_{1n}, P_{2n}, \ldots, P_{nn}; f, (x, y)] = f(x, y), \quad (x, y) \in Bd \cdot K,$$

whenever $f \in C(K)$.

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