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CONJUGATE POINT CLASSIFICATION WITH APPLICATION TO CHEBYSHEV SYSTEMS

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0. In the present note we shall concern about the application of the results from disconjugacy theory of the linear differential equations to derive Chebyshev spaces with special structural properties. The investigations on the structural properties of the Chebyshev spaces had a recent development by the results of V.I. VOLKOV [12], J. KIEFER and J. WOLFOWITZ [7], P. HADELER [6], R. ZIELKE [14, 15] and of the author [9, 10]. We remark also some results of V.I. VOLKOV [13] and YU. G. ABAKUMOV [1, 2] concerning the Chebyshev spaces of derivable functions. There are given examples of Chebyshev spaces of derivable functions. There are given examples of Chebyshev spases of dimension 4 of derivable functions on a closed interval, without elements with two simple zeros: one at the endpoint of the interval, the other in its interior, and without any other zero. Some examples in [13] and [1] are in particular spaces of solutions of linear differential equations and the endpoints of the interval of definition are so called „conjugate points" for the respective differential equations. This gives the idea to made an attempt of classification of conjugate points for the linear differential equations and to derive some structural properties about the space of solutions. The present note constitutes the first step in this direction.

We shall use theorems of G. PÓLYA [11], O. ARAMĂ [3], PH. HARTMAN [5], A. YU. LEVIN [8] from the disconjugacy theory. For the sake of simplicity as a reference monography we use the lecture notes of W. A. COPPEL [4].

1. Definition 1. *The n-dimensional linear subspace L_n of $C(R)$ is called a Chebyshev space on $J \subset R$, if any nonzero element of its has* *at most n-1 distinct zeros in J. A basis of a Chebyshev space is called a Chebyshev system.*

Definition 2. *The n-dimensional subspace L_n of $C^n(R)$ is called an unrestricted Chebyshev space on $J \subset R$ if any nonzero element of its has at most n-1 zeros in J, counting multiplicities. A basis of an unrestricted Chebyshev space is called an unrestricted Chebyshev system.*

Consider the n-th order differential equation

$$x^{(n)} + p_1(t)\, x^{(n-1)} + \ldots + p_n(t)\, x = 0, \tag{1}$$

where p_i, $i = 1, \ldots, n$ are continuous functions on R.

Definition 3. *The points 0 and 1 are called conjugate points for the differential equation (1) if the space L_n of solutions of (1) is an unrestricted Chebyshev space on [0,1), but this property fails on [0,1].*

If 0 and 1 are conjugate points for (1), then L_n is an unrestricted Chebyshev space also on (0,1] (see [4], Theorem 8 p. 102).

Definition 4. *The differential equation (1) is called disconjugate on an (open, closed, half closed) interval $J \subset R$, if J contains no pair of conjugate points for (1).*

2. If 0 and 1 are conjugate points for (1), then its space L_n of solutions can be a Chebyshev space in sense of the Definition 1. To give a characterization of the conjugate points for which this holds, let be $x_1, \ldots, x_n$ a fundamental system of solutions for (1). Introduce the notations

$$\Phi^{(j)}(t) = (x_1^{(j)}(t), \ldots, x_n^{(j)}(t)),\ t \in [0, 1],\quad j = 0, 1, \ldots, n-1. \tag{2}$$

A slight reformulation of the Theorem 1 in [10] gives exactly what we need:

Proposition 1. *If 0 and 1 are conjugate points for the differential equation (1), then the space L_n of solutions is a Chebyshev space on [0,1] if and only if the vectors $\Phi(0)$ and $\Phi(1)$ are linearly independent.*

As a consequence of this proposition we have

Proposition 2. *Suppose that 0 and 1 are conjugate points for (1) and that $\Phi(0)$ and $\Phi(1)$ are linearly independent vectors. Then the space L_n of solutions of (1) forms a Chebyshev space on [0,1] with the property that the elements of its cannot be extended as elements of class C^n to $[0,1+\varepsilon)$, $\varepsilon > 0$, such that the property of L_n to be a Chebyshev space be preserved.*

Proof. From the theorem of G. PÓLYA (see [4] Theorem 3 p. 93) the linear differential equation (1) is disconjugate on $[a, b] \subset R$ if and only if the space L_n of solutions has a basis $x_1, \ldots, x_n$ such that

$$x_1(t) = W(x_1; t) > 0,\ W(x_1, x_2; t) > 0,\ \ldots,\ W(x_1, \ldots, x_{n-1}; t) > 0$$

for $t \in [a, b]$, where $W(x_1, \ldots, x_i; t)$ denotes the Wronskian of the functions $x_1, \ldots, x_i$ in the point t.

Suppose that the proposition is false, i.e., the elements of L_n are extended as functions of class C^n to $[0, 1+\varepsilon)$ such that L_n, considered the space spanned by the extended functions, is a Chebyshev space on this interval. If $x_1, \ldots, x_n$ is a basis of L_n extended in this form, then obviously $W(x_1, \ldots, x_n; t) \neq 0$ for $t \in [0,1]$ and therefore exists a δ, $0 < \delta < \varepsilon$ such that this holds also for $t \in [0, 1+\delta]$. This means that $x_1 \ldots, x_n$ form a fundamental system of solutions of a differential equation of form (1) with p_i continuous functions on $[0, 1+\delta]$. Because L_n is a Chebyshev space on $[0, 1+\delta]$, according a result of O. ARAMĂ and PH. HARTMAN (see [4], Proposition 3 p. 82), the space L_n restricted to $(0, 1+\delta)$ is an unrestricted Chebyshev space, i.e., (1) will be disconjugate on this interval. Extend the coefficients of this equation continuously to **R**. Then, by the monotonity property of conjugate points proved by A. YU. LEVIN (see [4], Theorem 6, p. 100), it follows that the obtained equation has as left conjugate point for $1 + \delta/2$ a point $\eta < 0$. But this means that the equation is disconjugate on $(\eta, 1+\delta/2)$ and therefore also on [0,1]. But this is absurd because on [0, 1] it coincides with the original diffential equation.

Proposition 3. *Let 0 and 1 be conjugate points for the differential equation (1) and suppose that $\Phi(0)$ and $\Phi(1)$ are linearly independent vectors. Then the space L_n of solutions of (1) is a Chebyshev space on [0, 1] which has no unrestricted Chebyshev subspace of dimension n-1, but can have unrestricted Chebyshev subspaces of any dimension m, $m \leq n-2$.*

Proof. If L_n has an unrestricted Chebyshev subspace L_{n-1} of dimension n-1, then according to the theorem of Pólya, it has a basis $x_1, \ldots, x_{n-1}$ such that

$$x_1(t) = W(x_1; t) > 0,\ \ldots,\ W(x_1, \ldots, x_{n-2}; t) > 0,\ t \in [0, 1]. \tag{3}$$

L_{n-1} cannot contain nontrivial elements with zeros of multiplicity n-1, i.e., $W(x_1, \ldots, x_{n-1}; t) \neq 0$, $t \in [0, 1]$. By the theorem of Pólya, this, together with (3) contradicts the hypothesis that 0 and 1 are conjugate points for (1).

The fundamental system $1, t, \ldots, t^{n-3}, \sin t, \cos t$, of solutions of the differential equation

$$x^{(n)} + x^{(n-2)} = 0 \tag{4}$$

has the property that $L_m = L(1, t, \ldots, t^{m-1})$, $1 \leq m \leq n-2$ is an unrestricted Chebyshev space on all the real line. Let be δ the right conjugate point of 0 for (4). Then $\Phi(0)$ and $\Phi(\delta)$ are linearly independent vectors. Such the second part of the proposition is verified for (4) on the interval $[0, \delta]$.

Proposition 4. *Consider the differential equation* (1) *for* $n = 4$ *and suppose that* 0 *and* 1 *are conjugate points for this differential equation. If the vectors*

$$\Phi(0),\ \Phi'(0),\ \Phi(1),\ \Phi'(1) \tag{5}$$

are linearly dependent, while the systems of vectors

$$\Phi(0),\ \Phi'(0),\ \Phi(1) \quad \textit{and respectively} \quad \Phi(0),\ \Phi(1),\ \Phi'(1) \tag{6}$$

are linearly independent, then the space of solutions forms a Chebyshev space of dimension 4 *on* $[0, 1]$, *whose domain of definition can be extended at most with a single point.*

Proof. By the Proposition 1 and the linear independence of the systems of vectors (6) it follows that the space L_4 of solutions is a Chebyshev space on $[0,1]$. Consider the curve

$$x^i = x_i(t),\ i = 1, 2, 3, 4,\ t \in [0, 1] \tag{7}$$

in R^4, where x_1, x_2, x_3, x_4 is the fundamental system of solutions which occur in the definition of the vector $\Phi(t)$. The curve (7) will be denoted also by $\Phi([0,1])$. The hyperplane R^3 spanned by the vectors (5) has with this curve an intersection point of the multiplicity two at $t = 0$ and $t = 1$. Let be R^2 the subspace in R^4 spanned by $\Phi(0)$ and $\Phi(1)$ and denote by R^{4-2} its orthogonal complement in R^4. The projection $p\Phi[0, 1)]$ of $\Phi([0,1])$ into R^{4-2} will be a curve $\psi([0,1])$ such that $\psi(0) = \psi(1) = 0$ and the tangent to this curve for $t = 0$ and $t = 1$ will be contained in the line $R^1 = p\ R^3$ (p denotes the projection onto R^{4-2}). Any stright line passing through the origin intersects the curve $\psi((0,1)) = p\ \Phi((0,1))$ in at most a single point. Suppose the contrary: the line Δ passing through the origin of R^{4-2} intersects $\psi((0, 1))$ at $\psi(t_1)$ and $\psi(t_2)$. Then the hyperplane $p^{-1}\Delta$ would contain the vectors $\Phi(0)$, $\Phi(1)$, $\Phi(t_1)$, $\Phi(t_2)$, which contradicts the fact that L_4 is a Chebyshev space on $[0,1]$. From this property of $\psi((0,1))$ and from the fact that $R^1 = p\ R^3$ contains the tangents to $\psi([0,1])$ for $t = 0$ and $t = 1$, it follows also by a simple geometrical reasoning that any stright line through the origin, except R^1, intersects $\psi((0,1))$ in a point.

Suppose now that the domain of definition of L_4 can be extended with two distinct points: α_1 and α_2. Then $\Phi(0)$, $\Phi(1)$, $\Phi(\alpha_1)$, $\Phi(\alpha_2)$ will be linearly independent vectors and therefore one of $\Phi(\alpha_1)$, $\Phi(\alpha_2)$, say $\Phi(\alpha_1)$ cannot be in R^3. Consider the line Δ spanned by $p\Phi(\alpha_1)$. This line is different from R^1 and therefore intersects $\psi((0,1))$ in the point $\psi(t_0)$. But then the hyperplane $p^{-1}\Delta$ will contain the vectors $\Phi(0)$, $\Phi(1)$, $\Phi(t_0)$, and $\Phi(\alpha_1)$, which is in contradiction with the hypothesis L_4 extended to $[0, 1] \cup \{\alpha_1\}$ is a Chebyshev space.

Example. Consider the differential equation

$$x^4 + x'' = 0. \tag{8}$$

It is easy to verify that 0 and 2π are conjugate points for (8) and that the conditions in Proposition 4 are verified. Then we have that the space $L_4 = L(1,\ t,\ \sin t,\ \cos t)$ of solutions of (8) is a Chebyshev space on $[0,2\pi]$ whose domain of definition can be extended at most with a single point.

R. ZIELKE pointed out that L_4 has a Markov basis. In particular the subspace $L_3 = L(1,\ t\text{-}\sin t,\ 1\text{-}\cos t)$ of L_4 is a Chebyshev space on $[0,2\pi]$. Suppose the contrary: the nontrivial element

$$x(t) = c_1 + c_2(t\text{-}\sin t) + c_3(1\text{-}\cos t)$$

has three distinct zeros in $[0,2\pi]$. Then $x'(t) = c_2(1\text{-}\cos t) + c_3 \sin t$ will be a nontrivial element in $L_2 = L(1\text{-}\cos t,\ \sin t)$ with two distinct zeros în $(0,2\pi)$. But this is absurd, because $1\text{-}\cos t$, $\sin t$ is a Chebyshev system on $(0,2\pi)$. By theorem 5 in [9] it follows then that the domain of definition of L_4 can be extended with at least a point. We have then

Proposition 5. *The space* $L_4 = L(1,\ t,\ \sin t,\ \cos t)$ *is a Chebyshev space on* $[0, 2\pi]$ *whose domain of definition can be extended with exactly a point.*

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