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ON THE FUNCTIONAL $[f(z_1) / f'(z_2)]$, FOR TYPICALLY-REAL FUNCTIONS

by

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ABSTRACT. *The authors obtain the domain of variability of the functional* $[f(z_1) / f'(z_2)]$ *as* $f(z)$ *ranges over the set of all analytic functions* $f(z) = z + \dots$ *that are typically-real in the unit disc for fixed* z_1 *and* z_2, $0 < z_1 < z_2 < 1$.

1. At the Conference on Analytic Functions held in Łodz, Poland, in 1966, one of us posed the following related problems.

P r o b l e m *A* [1; p. 316]. *Let* $\mathcal{F}$ *be a non-empty compact family of functions* $f(z) = z + \dots$ *that are analytic and univalent in the unit disc* Δ. *For each pair of points* z_1 *and* z_2 *in* Δ, *there exists a point* $\zeta \equiv \zeta(z_1, z_2; f)$, *on the line segment joining* z_1 *and* z_2 *and a complex number* $\lambda \equiv \lambda(z_1, z_2; f)$, $|\lambda(z_1, z_2; f)| \leqslant 1$, *such that*

$$f(z_2) - f(z_1) = \lambda f'(\zeta)(z_2 - z_1)$$

holds. Determine

$$l(z_1, z_2; \mathcal{F}) \equiv \min\, [|\lambda(z_1, z_2; f)|\,|\, f \in \mathcal{F}].$$

P r o b l e m *B* [2; p. 133]. *For* $\mathcal{F}$, z_1, z_2 *in Problem A. determine the set*

$$D[z_1, z_2; \mathcal{F}] \equiv [(f(z_1) / f'(z_2) \mid f \in \mathcal{F}].$$

In this note we offer a solution to both problems for the case that $\mathcal{F}$ is the set $\mathcal{T}$ of all typically-real $f(z) = z + \dots$ and either z_1 or z_2 is real.

Then we use that solution to find the formula for $l(z_1, z_2; f)$ when $0 < z_1 < z_2 < 1$ holds.

2. If $f(z) = z + a_2 z^2 + \ldots$ is typically-real in the unit disc Δ, then $f(z)$ has the representation [4; p. 567]

(1) $$f(z) = \int_{-1}^{1} \frac{z\,d\mu(t)}{1 + 2tz + z^2},$$

where $\mu(t)$ is a unit mass function on the interval $-1 \leqslant t \leqslant 1$. From (1) we obtain

(2) $$f'(z) = \int_{-1}^{1} \frac{(1 - z^2)\,d\mu(t)}{(1 + 2tz + z^2)^2},$$

which is valid in Δ.

Formulas (1) and (2) lead to our first result.

THEOREM 1. *The domain of variability*

(3) $$D[z_1, r_2, \mathcal{T}] \equiv \left[\frac{f(z_1)}{f'(r_2)} \middle| f \in \mathcal{T}\right],$$

where z_1 and r_2 are in Δ, r_2 real, and where $\mathcal{T}$ is the set of all functions $f(z) = z + \ldots$ that are analytic and typically-real in Δ, is the closed convex hull of the curve

(4) $$w = w(t) = \frac{z_1}{(1 + 2t z_1 + z_1^2)} \frac{(1 + 2t r_2 + r_2^2)^2}{1 - r_2^2}, \quad -1 \leqslant t \leqslant 1.$$

Proof. We shall use a clever device introduced by PILAT for a similar purpose [3; pp. 54—56].

Since $f(z)$ is typically real in Δ, it follows from (2) that $f'(z) \neq 0$ for $z \in \Delta$, z real. If we use the unit mass function defined by the formula

$$\sigma(t) = \int_{-1}^{t} \frac{1 - r_2^2}{(1 + 2tr_2 + r_2^2)^2 f'(r_2)} d\mu(t),$$

along with (1) and (2), then we obtain

(5) $$\frac{f(z_1)}{f'(r_2)} = \int_{-1}^{1} \frac{z_1}{1 + 2t z_1 + z_1^2} \frac{(1 + 2t r_2 + r_2^2)^2}{1 - r_2} d\sigma(t).$$

Now it is a fundamental result due to Piłat that the domain of variability (3) can be found by considering the right-hand member of (5) as $\sigma(t)$ ranges over all unit mass functions on the interval $-1 \leqslant t \leqslant 1$. From this it follows that the domain (3) is indeed the closed convex hull of the curve (4).

In a similar way we can show that the domain of variability of the functional $[r_2 f'(z_1)/f(r_2)]$, as f ranges over $\mathcal{T}$, where z_1 and r_2 are in Δ, r_2 real, is the closed convex hull of the curve

$$w = w_1(t) = \frac{(1 - z_1^2)(1 + 2tr_2 + r_2^2)}{(1 + 2tz_1 + z_1^2)^2}.$$

3. We now use *Theorem* 1 to obtain the solution to *Problem A* for the special case that $\mathcal{F}$ is the set $\mathcal{T}$ of normalized typically-real functions analytic in Δ and where r_1 and r_2 are real, $0 < r_1 < r_2$.

THEOREM 2. *If* $0 < r_1 < r_2 < 1$, *then*

(6) $$l(r_1, r_2, \mathcal{T}) \equiv \min\left[|\lambda(r_1, r_2; f)| \mid f \in \mathcal{T}\right] = \frac{(1 - r_1 r_2)(1 - r_2)}{(1 - r_1)^2(1 + r_2)}$$

Proof. The transform of $f(z)$,

$$g(z) \equiv \left[\left(f\left(\frac{z + r_1}{1 + r_1 z}\right) - f(r_1)\right)\Big/(1 - r_1^2) f'(r_1)\right]$$

is typically-real in Δ whenever $f(z)$ is typically-real. Elementary calculations yield the relation

(7) $$\begin{aligned}\lambda(r_1, r_2; f) &= \frac{f(r_2) - f(r_1)}{(r_2 - r_1) f'(\zeta)}, \quad 0 < r_1 \leqslant \zeta \leqslant r_2 < 1,\\ &= \frac{(1 - r_1^2)\, g(a)}{(r_2 - r_1)(1 + r_1 x)^2 g'(x)}, \quad 0 \leqslant x \leqslant a < 1\end{aligned}$$

where

(8) $$a \equiv \frac{r_2 - r_1}{1 - r_1 r_2}, \quad x = \frac{\zeta - r_1}{1 - r_1 \zeta}.$$

If we now use *Theorem* 1, it follows that the domain of variability of $[g(a)/g'(x)]$ is the closed interval on the real axis given by (4) in the form

(9) $$w(t) = \frac{a}{(1 + 2ta + a^2)} \frac{(1 + 2tx + x^2)^2}{(1 - x^2)}, \quad -1 \leqslant t \leqslant 1.$$

From (7), (8) and (9) it follows that we wish to obtain the value ot

(10) $$l = \min\left[\frac{1 - r_1^2}{(r_2 - r_1)} \frac{1}{(1 + r_1 x)^2} Q(x, t) \,\middle|\, -1 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant a\right]$$

where

(11) $$Q(x, t) \equiv \frac{(1 + 2tx + x^2)^2}{(1 - x^2)(A + 2t)}, \quad A \equiv a + \frac{1}{a}.$$

Since

$$\frac{\partial Q}{\partial t} = \frac{2(1 + 2tx + x^2)}{(1 - x^2)(A + 2t)^2} [-1 + 2x(A + t) - x^2],$$

we consider three special cases.

(i) If $0 \leqslant x \leqslant (A + 1 - \sqrt{(A + 1)^2 - 1})$, then $\frac{\partial Q}{\partial t} \leqslant 0$ so that $\min_t Q(x, t) = Q(x, 1)$.

(ii) If $(A - 1 - \sqrt{(A - 1)^2 - 1}) \leqslant x \leqslant a$, then $\frac{\partial Q}{\partial t} \geqslant 0$ so that $\min_t Q(x, t) = Q(x, -1)$.

(iii) If $(A + 1 - \sqrt{(A + 1)^2 - 1} < x < (A - 1 - \sqrt{(A - 1)^2 - 1}$, then $Q(x, t) \geqslant Q(x, t_0)$, where t_0 is the *unique* solution to the equation $\frac{\partial Q}{\partial t} = 0$. Here we note that $\frac{\partial Q}{\partial t}$ is negative for $t = -1$ and $\frac{\partial Q}{\partial t} > 0$ is positive for $t = +1$, and hence $Q(x, t)$ has its minimum at $t = t_0$.

It now follows that in order to evaluate the formula in (10) we can first find the minimum of the function

(12)
$$R(x) = \begin{cases} Q(x, 1) \equiv \dfrac{a(1 + x)^3}{(1 + a)^2(1 - x)}, & 0 \leqslant x \leqslant (A + 1 - \sqrt{(A + 1)^2 - 1}), \\ Q(x, t_0) \equiv \dfrac{4x(a - x)(1 - ax)}{a(1 - x^2)}, & (A + 1 - \sqrt{(A + 1)^2 - 1}) \leqslant x \leqslant (A - 1 - \sqrt{(A - 1)^2 - 1}) \\ Q(x, -1) \equiv \dfrac{a(1 - x)^3}{(1 - a)^2(1 + x)}, & (A - 1 - \sqrt{(A - 1)^2 - 1}) \leqslant x \leqslant a, \end{cases}$$

and then multiply by the minimum of $1/(1 + r_1 x)^2$ for $0 \leqslant x \leqslant a$.

Now for $0 \leqslant x \leqslant (A + 1 - \sqrt{(A + 1)^2 - 1})$ we find

$$\frac{dR}{dx} = \frac{a}{(1 + a)^2} \frac{2(1 + x)^2(2 - x)}{(1 - x)^2} > 0,$$

so that $R(x)$ is increasing on this interval. For $(A - 1 - \sqrt{(A - 1)^2 - 1}) \leqslant x \leqslant a$, we find

$$\frac{dR}{dx} = \frac{a}{(1 - a)^2} \frac{2(1 - x)^2}{(1 + x)^2} (-2 - x) < 0,$$

so that $R(x)$ is a decreasing function on this interval. For the remaining interval $(A + 1 - \sqrt{(A + 1)^2 - 1}) < x < (A - 1 - \sqrt{(A - 1)^2 - 1})$ we find

$$\frac{dR}{dx} = \frac{4}{(1 - x^2)^2} [1 - 2Ax + 4x^2 - x^4].$$

$$= \frac{4}{(1 - x^2)^2} [(1 - x)^2(1 - 2x - x^2) + (2 - A)x].$$

Now a study of the curves $y = (1 - x)^2(1 - 2x - x^2)$ and $y = (2 - A)x$, where $A > 2$, shows that $\frac{dR}{dx}$ vanishes exactly once in the interval $0 \leqslant x \leqslant 1$. But $\frac{dR}{dx}$ is *positive* for $x = (A + 1 - \sqrt{(A + 1)^2 - 1})$ and $\frac{dR}{dx}$ is *negative* for $x = (A - 1 - \sqrt{(A - 1)^2 - 1})$ Hence it follows that $R(x)$ is a strictly concave function in the interval $(A - 1 - \sqrt{(A + 1)^2 - 1}) \leqslant x \leqslant (A - 1 - \sqrt{(A - 1)^2 - 1})$. Hence the minimum value of $R(x)$ given by (12), that is

(13) $$\min_{0 \leqslant x \leqslant a} R(x) = \min [R(0), R(a)] = R(a) = \frac{a(1 - a)}{1 + a}.$$

If we combine (10) and (13) we obtain

$$l \equiv \min [|\lambda(r_1, r_2; f)| \mid f \in \mathfrak{F}] = \frac{1 - r_1^2}{(r_2 - r_1)} \frac{1}{(1 + ar_1)^2} \frac{a(1 - a)}{(1 + a)},$$

which, because of (8), yields (6).

The function $k(z) = z/(1 - z)^2$ is typically-real, and achieves the value given in (6), so that our result is sharp. Indeed, the minimum is achieved by a univalent function!

4. The extension of (6) to the more general case when either r_1 and/or r_2 is *complex* presents serious computational problems which we have not been able to surmount. Even the case for $-1 < r_1 < 0$ and $0 < r_2 < 1$ has not yielded a precise answer to us, although *Theorem 1* was still available as a first step.

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