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ON BEST ONE-SIDED APPROXIMATION WITH INTERPOLATORY FAMILIES*

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0. Introduction

Let X be a set formed with $n + 1$ points of the real axis and $f : X \to R$ be the restriction, at X, of a polynomial of degree n. Evidently, there are polynomials P and Q of degree n, such that

$$P(x) \geqq 0, \quad Q(x) \geqq 0 \tag{0.1}$$

for $x \in X$, and

$$f = P - Q \text{ on } X. \tag{0.2}$$

Professor T. POPOVICIU has proposed the following problem: to study the existence and the uniqueness of a pair (P^*, Q^*) of polynomials of degree $\leqq n$ which is minimal, i.e.,

$$P \geqq P^* \geqq 0, \; Q \geqq Q^* \geqq 0 \quad \text{on } X \tag{0.3}$$

for every pair (P, Q) which verifies (0.2); if the problem has a solution, let this minimal pair be determined. Further, let a similar problem be solved in the case when X contains $n + 2$ points.

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From (0.1), (0,2) and (0.3) we have

$$(P^* \geqq 0 \wedge P^* - f = Q^* \geqq 0) \Rightarrow (P^* \geqq 0 \wedge P^* \geqq f) \Rightarrow P^* \geqq \geqq \max\{0, f\} = f_+$$

and

$$(Q^* \geqq 0 \wedge f + Q^* = P^* \geqq 0) \Rightarrow (Q^* \geqq 0 \wedge Q^* \geqq -f) \Rightarrow Q^* \geqq \geqq \max\{0, -f\} = (-f)_+.$$

If

(0.4) $$\rho(\Phi) = \max_{x \in X} |\Phi(x)|$$

for every $\Phi \in C[X]$, then we want to find the polynomials P^*, Q^* with properties

(0.5) $$\rho(P^* - f_+) = \min_{P \in \mathfrak{P}_n^{f_+}} \rho(P - f_+)$$

and

(0.6) $$\rho(Q^* - (-f)_+) = \min_{Q \in \mathfrak{P}_n^{(-f)_+}} \rho(Q - (-f)_+),$$

where

$$\mathfrak{P}_n^{f_+} = \{p \in \mathfrak{P}_n \mid \forall x \in X,\ p(x) \geqq f_+(x)\},$$

$$\mathfrak{P}_n^{(-f)_+} = \{p \in \mathfrak{P}_n \mid \forall x \in X,\ p(x) \geqq (-f)_+(x)\}.$$

Therefore, one observe that the above problem is a specimen of the best one-sided approximation in uniform norm.

Inspired by POPOVICIU's problem our aim in this paper is a study of several problems of best one-sided approximation. Some of our theoremes generalize known results about the same topic, and several new results are incorporated into a presentation of the previously known theory. Regarding to the interpolatory families, we note that one uses the notations of E. POPOVICIU and T. POPOVICIU (see for instance [8—10]), which seems to be practicaly.

1. Statement of the problem; existence of a best approximation

Let X be a compact set of the real axis and $C[X]$ be the real normed linear space of all functions $f: X \to R$ which are continuous on X, normed by means of

(1.1) $$p(f) = \max_{x \in X} |f(x)|,\ f \in C[X].$$

The norm p is the measure of the closeness of the approximation. For a fixed element f from $C[X]$ we denote

(1.2) $$C_B = \{g \in C[X] \mid \forall x \in X,\ g(x) \leqq f(x)\}.$$

Let $\mathcal{K} \subset C[X]$ be a subspace (closed), and

$$\mathcal{K}_B = C_B \cap \mathcal{K} = \{g \in \mathcal{K} \mid \forall x \in X,\ g(x) \leqq f(x)\}.$$

The purpose is to study the elements g_* from $\mathcal{K}_B$ such that

(1.3) $$p(f - g_*) = \min_{g \in \mathcal{K}_B} p(f - g),$$

in which case we say that $g_* = g_*(f; \mathcal{K}; X)$ is the element of best one-sided approximation from below on X of the function f by elements from $\mathcal{K}$. In the same way may be defined the elements of the best one-sided approximation from above, $g^* = g^*(f; \mathcal{K}; X)$. The following theorems deals with approximation from below; an analogous result holds for approximation from above. In our case $p(f - g) = \max_{x \in X} (f(x) - g(x))$, but in order to put in evidence the similarity with the unconstrained uniform approximation, we shall frequently use the notation introduced in (1.1). It will be supposed that

$$d = E(f; \mathcal{K}_B; X) = \inf_{g \in \mathcal{K}_B} p(f - g) > 0.$$

In the following we need statement (see for instance [4], Th. 1.4.1.):

THEOREM 1.1. *Let W be a real normed linear space which is provided with the norm $\|\cdot\|$, $\mathcal{K} \subset W$ a subspace and $\varphi: W \to R$ a continuous functional. For a fixed element $g_0 \in \mathcal{K}$ let us put*

$$S = \{g \in W \mid \varphi(g) < \varphi(g_0)\} \subset W.$$

If Q is a given set with non-void interior and by $K(\cdot)$ respectively $K[\cdot]$ one denotes the cone of admissible deplacements respectively the cone of adherent deplacements, then the functional φ attains its local minimum relativ of set $Q \cap \mathcal{K}$ at the point g_0, if

$$K(Q; g_0) \cap K(S; g_0) \cap K[\mathcal{K}; g_0] = \emptyset.$$

If these cones are convex then the above condition is likewise necessary for minimum on $Q \cap \mathcal{K}$.

It is worth mentioning that the characterization theorems for one-sided approximation may be proved directly by means of the cones of admissible deplacements as well as of the adherent cones. Other proofs may be performed by using the fact that the one-sided approximation is a particular case of some problems of uniform approximation with constraints (see for instance G. D. TAYLOR [12]). In the second of this paper we generalize the one-sided approximation in the following manner: instead of an interval $[a, b]$ one considers an arbitrary compact set and the Chebyshev space on $[a, b]$ is substituted by an interpolatory set of functions which are defined on this set.

Firstly we get two results regarding the existence problem in the case when X contains only a finite number of points as well as $\mathcal{K}$ is a finite dimensional subspace. We note that in this case any $f: X \to R$ is continuous on X. It is well known that the proof of the existence of a best approximation essentialy depend on the fact that a continuous function on a compact set attains its extreme values.

THEOREM 1.2. *Let $X = \{x_1, \ldots, x_m\}$, $x_i \in R$ and $\mathcal{K} \subset C[X]$. Then for every $f: X \to R$ there exists at least one element $g_* = g_*(f; \mathcal{K}; X)$.*

Proof. We consider the sets

$$M_i = \{g \in C[X] \mid 0 \leq f(x_i) - g(x_i) \leq c\},\ i = 1, \ldots, m$$

where c is arbitrary sufficiently large positive constant. The sets $Q = \bigcap_{i=1,\ldots,m} M_i$ and $Q \cap \mathcal{K}$ are compact $C[X]$ respectively in $\mathcal{K}$, and $d = \inf_{g \in Q \cap \mathcal{K}} p(f - g)$. The functional φ defined by

$$\varphi(g) = p(f - g) = \max_{i=1,\ldots,m} |f(x_i) - g(x_i)|,\ g \in C[X],$$

being continuous on $C[X]$, attains its minimum at least on a point $g_* \in Q \cap \mathcal{K}$, and we have

$$\varphi(g_*) = p(f - g_*) = \min_{g \in Q \cap \mathcal{K}} p(f - g) = \min_{g \in \mathcal{K}_B} p(f - g). \ \square$$

THEOREM 1.3. *If $X \subset R$ is a compact and $\mathcal{K}$ is a finite dimensional subspace of $C[X]$, then for every $f \in C[X]$ there exists at least one element $g_*(f; \mathcal{K}; X)$ of best one-sided approximation.*

Proof. Let $\mathcal{K} = \operatorname{span}\{g_1, \ldots, g_n\}$, where $g_1, \ldots, g_n$ is any basis, $g_i \in C[X]$ for $i = 1, \ldots, n$. Let C_B be the set

$$C_B = \{g \in C[X] \mid \forall x \in X,\ g(x) \leq f(x)\}.$$

Further the set $V = C_B \cap \mathcal{K}$ is convex and closed in $\mathcal{K}$. The functional p defined by (1.1), is continuous on $\mathcal{K}$. Now, let us consider

$$M = \{g \in C[X] \mid p(f - g) \leq d + \delta\},$$

where δ is an arbitrary positive number. Taking into account that

$$M \cap \mathcal{K}_B \neq \emptyset, \quad \inf_{g \in M \cap \mathcal{K}_B} p(f - g) = \inf_{g \in \mathcal{K}_B} p(f - g) = d,$$

we see that it is enough to seek the minimum of $\varphi(g) = p(f - g)$ on $M \cap \mathcal{K}_B$. The set $M \cap \mathcal{K}_B$ is compact in $\mathcal{K}$, because it is closed and bounded in $\mathcal{K}$. The continuity of $p(f - g)$ with respect to g, implies that there is at least one element $g_* \in M \cap \mathcal{K}_B$ such that

$$p(f - g_*) = \min_{g \in M \cap \mathcal{K}_B} p(f - g) = \min_{g \in \mathcal{K}_B} p(f - g).$$

One observes that the above theorems follow from a more general situation. Namely, if Q is a closed subset from $C[X]$ such that its intersections with every subset $\{g \in C[X] \leq \mid p(f - g) \leq \rho\}$, $\rho \in R$, is compact, then $f \in C[X]$ admits at least one element of best approximation by means of elements from Q. In particular, this is true if Q is a closed subset from a finite dimensional space $\mathcal{K}$.

2. Characterization

(i) General case

The following important theorem is a modification of the characterization theorem for approximation with restricted range due to G. D. TAYLOR [13, Theorem 3.2.]. In this section X is a compact set which contains at least n distinct points, and $\mathcal{K} \subset C[X]$ is arbitrary subspace (closed with respect to norm p).

For a fixed $g \in \mathcal{K}_B$ we denote

$$E_+(g) = \{x \in X \mid f(x) - g(x) = p(f - g)\},$$
$$C_-(g) = \{x \in X \mid f(x) - g(x) = 0\},$$

and

$$A(g) = E_+(g) \cup C_-(g),$$

$$\sigma_g(x) = \begin{cases} 1 \text{ if } & x \in E_+(g), \\ -1 \text{ if } & x \in C_-(g). \end{cases}$$

Points in $E^+(g)$ are called (plus) extremal points and points in $C_-(g)$ are called points of the (lower) contact. The points in $A(g)$ are called critical points, relativ of function $e = f - g$. Analogous symbols we have for approximation from above. If $g_* = g_*(f; \mathcal{K}; X)$, then $p(f - g_*) = d_*$ is called the deviation and $f - g_* = e_*$ is called the deviation function. It is known that even in order to develop a theory of best approximation it is necessary to assume that the approximating set has at least one element which verifies $g_0(x) < f(x)$ for all $x \in X$. This fact we shall frequently symbollically denote with $g_0 < f$.

THEOREM 2.1. *If there exists an element $g_0 \in \mathcal{H}$ which verifies $g_0 < f$, then a necessary and sufficient condition which must be verified by $g_* \in \mathcal{H}$ such that g_* be an element of best one-sided approximation from f, namely $g_* = g_*(f; \mathcal{H}; X)$, is that there does not exists $g \in \mathcal{H}$ such that*

$$\sigma(x)\, g(x) > 0 \text{ for all } x \in A(g_*), \tag{2.1}$$

where $\sigma = \sigma_{g_*}$.

Proof. The functional $\varphi: C[X] \to R$ defined by $\varphi(g) = p(f-g)$ is continuous and convex. Therefore the sets

$$Q = \{g \in C[X] \mid p(f-g) < p(f-g^*)\}$$

and

$$C_B = \{g \in C[X] \mid \forall x \in X,\ f(x) - g(x) \geqq 0\},$$

are convex. Since $g_0 \in \overset{\circ}{C}_B$ one has $\overset{\circ}{C}_B \cap \mathcal{H} \neq \emptyset$. In this way the conditions of the Theorem 1.1 are fulfilled. Thus $g_*(f; \mathcal{H}; X)$ verifies

$$K(Q; g_*) \cap K(C_B; g_*) \cap K[\mathcal{H}; g_*] = \emptyset. \tag{2.2}$$

But it is known (see for instacne [4] or [2]),

$$K(Q; g_*) = \{g \in C[X] \mid \forall x \in E_+(g_*),\ g(x)\cdot \text{sign}\,(f(x) - g_*(x)) > 0\}, \tag{2.3}$$

$$K(C_B; g_*) = \{g \in C[X] \mid \forall x \in C_-(g_*),\ g(x) < 0\}, \tag{2.4}$$

$$K[\mathcal{H}; g_*] = \mathcal{H}. \tag{2.5}$$

In our case sign $(f(x) - g_*(x)) = +1$ for all $x \in E_+(g_*)$ and (2.2) reduces to

$$\{g \in C[X] \mid \forall x \in E_+(g_*),\ g(x) > 0\} \cap \{g \in C[X] \mid \forall x \in C_-(g_*),$$
$$g(x) < 0\} \cap \mathcal{H} = \emptyset,$$

which proves our assertion. $\square$

(ii) The case with finite dimensional subspace

Firstly we note that in this case the set $A(g_*)$ which is defined in (i), contained only finited much of points. Let X be a compact of real line which contain at least n distinct points as well as $\mathcal{H} \subset C[X]$ be a subspace of dimension n. We may write $\mathcal{H} = \text{span}\,\{g_1, \ldots, g_n\}$ where $g_1, \ldots, g_n$

is a basis of $\mathcal{H}$, $g_i \in C[X]$, $i = 1, \ldots, n$. Then we may transpose our problem to the same topic in R^n. This is motivated by taking into account the linear biunivoque correspondence between $\mathcal{H}$ and R^n:

$$g = \sum_{k=1}^{n} \alpha_k g_k \in \mathcal{H} \longleftrightarrow \alpha = (\alpha_1, \ldots, \alpha_n) \in R^n.$$

Therefore to the set C_B corresponde the set

$$V_B = \left\{\alpha \in R^n \mid \forall x \in X,\ \sum_{k=1}^{n} \alpha_k g_k(x) \leqq f(x)\right\}. \tag{2.6}$$

Further we define the continuous application $a: X \leqq R^n$ by

$$a(x) = (g_1(x), \ldots, g_n(x)),\ x \in X$$

and put

$$a(A(g)) = \{a(x) \mid \forall x \in A(g)\}.$$

The following characterization theorem is valid.

THEOREM 2.2. *Let us assume that exists an element $g_0 \in \mathcal{H}$ such that $g_0 < f$, and f be a given function in $C[X]$, $f \notin \mathcal{H}$. Then each of the follwing is a necessary and sufficient condition for $g_* \in \mathcal{H}$ to be an element of best one-sided approximation, $g_* = g_*(f; \mathcal{H}; X)$:*

(a) *There does not exist $g \in \text{span}\,\{g_1, \ldots, g_n\}$ such that $\sigma(x)g(x) > 0$ for all $x \in A(g_*)$.*

(b) *The zero n-vector θ_{R^n} is in the convex hull of the set of n-tuples $\{\sigma(x)(g_1(x), \ldots, g_n(x)) \mid \forall x \in A(g_*)\}$.*

(c) *There exists the points $x_1, \ldots, x_n$ in $A(g_*)$, $m \leqq n+1$, and positive numbers γ_i such that a linear (and continuous) functional $l \in C[X]^*$ defined on $C[X]$ of the „point evlatuation" type*

$$l(f) = l\begin{pmatrix}\gamma_1, & \ldots, & \gamma_m \\ x_1, & \ldots, & x_m\end{pmatrix}; f \Bigg) = \sum_{i=1}^{m} \sigma(x_i)\, \gamma_i f(x_i),\ f \in C[X]$$

which satisfies

$$l(g) = 0 \text{ for all } g \in \mathcal{H}.$$

Proof. We shall show that: $(g_* \in g_*(f; \mathcal{H}; X)) \Leftrightarrow (a) \Leftrightarrow (b) \Leftrightarrow (c)$. In view of Theorem 2.1, firstly part is established. To show $(a) \Leftrightarrow (b)$, let $\langle \cdot, \cdot \rangle$ denote the scalar product in R^n. Then the set defined in (2.6) may be written in the form

$$V_B = \{\alpha \in R^n \mid \forall\ x \in X,\ \langle \alpha, a(x) \rangle \leqq f(x)\}$$

and our problem reduces to the finding of a point $\alpha_* \in R^n$ for which

$$\varphi(\alpha_*) = \min_{\alpha \in V_B} \varphi(\alpha)$$

where

$$\varphi(\alpha) = p\left(f - \sum_{k=1}^{n} \alpha_i g_i\right).$$

By using the formulae (2.3)—(2.5), we find

$$\begin{aligned} K(Q\,;g_*) &= \{g \in C[X] \mid \forall x \in E_+(g_*),\ g(x) > 0\} = \\ &= \{\alpha \in R^n \mid \forall x \in E_+(g_*),\ \langle \alpha, a(x) \rangle > 0\} = \\ &= \{\alpha \in R^n \mid \forall x \in a(E_+(g_*)),\ \langle \alpha, \beta \rangle > 0\}. \\ K(C_B\,;g_*) &= \{g \in C[X] \mid \forall x \in C_-(g_*),\ g(x) < 0\} = \\ &= \{\alpha \in R^n \mid \forall x \in C_-(g_*),\ \langle \alpha, -a(x) \rangle > 0\} = \\ &= \{\alpha \in R^n \mid \forall \beta \in -a(C_-(g_*)),\ \langle \alpha, \beta \rangle > 0\}. \\ K[\mathcal{H}\,;g_*] &= R^n. \end{aligned}$$

By means of the Theorem 2.1, we conclude with the fact that the system of linear inequalities in R^n

$$\begin{aligned} &\langle \alpha, \beta \rangle > 0 \quad \forall \beta \in a(E_+(g_*)) \\ &\langle \alpha, \beta \rangle > 0 \quad \forall \beta \in -a(C_-(g_*)) \end{aligned}$$

in unknown α, is inconsistent, i.e., there isn't any solution. Since

$$-a(C_-(g_*)) = \{\sigma(x_i)\, a(x_i) \mid \forall x \in C_-(g_*)\},$$

this system may be written in the form

$$\langle \alpha, \beta \rangle > 0, \text{ with } \beta \in \{\beta\} = \{\sigma(x)\, a(x) \mid \forall x \in A(g_*)\}. \tag{2.7}$$

The set $\{\beta\}$ is compact in R^n. It is known (see for instance [1, p. 19]) that the inconsistent property of system (2.7) is equivalent with

$$\theta_{R^n} \in \mathrm{co}\ (\{\beta\}) \tag{2.8}$$

which is equivalent with (b). The chaim will be completed by showing (b) $\Leftrightarrow$ (c). In order to show this it is sufficiently to show that (2.8) is equi-

valent with (c). On account of Caratheodory Theorem, there exist at most $n+1$ points $\beta_1, \ldots, \beta_m$ and positive coefficients ρ_i such that

$$\theta_{R^n} = \sum_{i=1}^{m} \rho_i \beta_i \tag{2.9}$$

with

$$\beta_i \in \{\sigma(x_i)\, a(x) \mid x_i \in A(g_*)\}, \quad \sum_{i=1}^{m} \rho_i = 1, \ \rho_i > 0, \ 1 \leqq m \leqq n+1.$$

If $x_i \in a(E_+(g_*))$ we have

$$\rho_i \beta_i = \rho_i a(x_i) = \sigma(x_i)\, \rho_i a(x_i),$$

as well as if $x_i \in -a(C_-(g_*))$, we have

$$\rho_i \beta_i = -\rho_i a(x_i) = \sigma(x_i)\, \rho_i a(x_i).$$

Thus, from (2.9), we have

$$\theta_{R^n} = \sum_{i=1}^{m} \sigma(x_i)\, \rho_i a(x_i) = \sum_{i=1}^{m} \rho_i \sigma(x_i)(g_1(x_i), \ldots, g_n(x_i)),$$

which is equivalent with

$$\sum_{i=1}^{m} \sigma(x_i)\, \rho_i g_k(x_i) = 0, \quad k = 1, \ldots, n. \tag{2.10}$$

By multiplying every equation in (2.10) with arbitrary numbers α_k, $k = 1, \ldots, n$, and by adding these equalities, we obtain

$$\sum_{i=1}^{m} \sigma(x_i)\, \rho_i \left[\sum_{k=1}^{n} \alpha_k g_k(x_i)\right] = 0, \quad x_i \in A(g_*),$$

that is

$$l(g) = \sum_{i=1}^{m} \sigma(x_i)\, \rho_i g(x_i) = 0, \quad \forall g = \sum_{k=1}^{n} \alpha_k g_k. \ \Box$$

We note that $E_+(g_*) \neq \emptyset$. Indeed, let us assume contrary. The inequality $1 \leqq m$ implies $C_-(g_*) \neq \emptyset$, i.e., we must have $A(g_*) = C_-(g_*)$ and

$$0 = \sum_{i=1}^{m} \sigma(x_i)\, \rho_i g(x_i) = -\sum_{i=1}^{m} \rho_i g(x_i)$$

for every $x_i \in A(g_*)$. In other words, for the element $\tilde{g} = g - g_0 \in \mathcal{H}$ holds

$$\sum_{i=1}^{m} \rho_i(g - g_0)(x_i) = \sum_{i=1}^{m} \rho_i(f(x_i) - g_0(x_i)) > 0,$$

(because $g_*(x_i) = f(x_i)$ for all $x_i \in A(g_*)$), which asserte that there is an element in $\mathcal{H}$ such that $l(\tilde{g}) = 0$.

In conection with the unicity we note that the conditions which we have imposed, not asures the unicity of the best approximation element for each $f \in C[X]$. In the following paragraph we see that in the case when $\mathcal{H}$ is an interpolatory subspace, then the best approximation has an unique solution as well as that the interpolatory property is also a necessary condition for the uniqueness.

3. Approximation with interpolatory sets

Let $\mathcal{H} \subset C[X]$ be an interpolatory subspace on X of dimension n i.e., $\mathcal{H}$ is linear interpolatory set of the order n on X (see [8]), and $l : C[X] \to R$ is a non-zero linear continuous functional defined by

$$l(f) = l\begin{pmatrix}\gamma_1, \ldots, \gamma_{n+1} \\ x_1, \ldots, x_{n+1}\end{pmatrix}; f\Big) = \sum_{i=1}^{n+1} \gamma_i f(x_i). \tag{3.1}$$

If $l(g) = 0$ for every $g \in \mathcal{H}$, then we have

$$\gamma_i = (-1)^{n+1-i}\gamma_{n+1} V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\end{pmatrix} : V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n\end{pmatrix} \tag{3.2}$$

and „prewronskian"

$$V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix} g_1(x_1) & g_1(x_2) & \ldots & g_1(x_n) \\ g_2(x_1) & g_2(x_2) & \ldots & g_2(x_n) \\ \vdots & & & \\ g_n(x_1) & g_n(x_2) & \ldots & g_n(x_n)\end{vmatrix}$$

differs from zero. Moreover, we have

$$l(f) = \gamma_{n+1} V\begin{pmatrix} g_1, \ldots, g_n, f \\ x_1, \ldots, x_{n+1}\end{pmatrix} : V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n\end{pmatrix}. \tag{3.3}$$

In his work T. POPOVICIU [10] establishes that the unique functional of the form (3.1) which vanishes on $\mathcal{H}$ is defined by (3.3). Likewise

he notes that in case when $X = [a, b]$, the prewronskian $V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n\end{pmatrix}$ may be assumed positive. Therefore, in this case (3.2) implies the well-known alternatory property of the coefficients $\gamma_1, \ldots, \gamma_{n+1}$. This property remains valid even if X is an arbitrary compact set on R and $\mathcal{H}$ is interpolatory of type $I_n\{[a, b]\}$ (see [8]), where $[a, b] \subset R$ is a closed interval which contain the set X.

The following theorem deals with unicity solution of best one-sided approximation. The proof of this theorem may be performed by using similar arguments with those from the unconstrained case (see for instance [4, p. 96]).

THEOREM 3.1. *Let us suppose that there exist one element $g_0 \in \mathcal{H}$ such that $g_0 < f$. In order that an unique element $g_* = g_*(f; \mathcal{H}; X)$ of best one-sided approximation may exist for every $f \in C[X]$, it is necessary and sufficient that the subspace $\mathcal{H}$ to be a interpolatory on X.*

We note that in interpolatory case, if the functional which is considered in Theorem 2.2. (c), is ortogonal on $\mathcal{H}$, i.e., $l(g) = 0$ for all $g \in \mathcal{H}$, then we must have $m = n + 1$. Indeed, let us assume $m \leq n$. Then there exist an element $\tilde{g} \in \mathcal{H}$ defined by $\tilde{g}(x_i) = \gamma_i$, $i = 1, \ldots, m$ and we have

$$l(\tilde{g}) = \sum_{i=1}^{m} \gamma_i g(x_i) = \sum_{i=1}^{m} \gamma_i^2 > 0,$$

which contradicts our hypothesis. But this means that in the interpolatory case there exist exactly $n + 1$ points which form the set $A(g_*)$ of critical points from X.

4. The alternatory case

If $\mathcal{H}$ is of the type $I_n\{[a, b]\}$, $[a, b] \supseteq X$, then the quotient of prewronskians in (3.2) has a constant sign and all coefficients γ_i, $i = 1, \ldots, n+1$ are not zero. Therefore from (3.2) we conclude that γ_i have alternating sign. Finally the fact, that the set $A(g_*) \subset X$ of critical points contain exactly $n + 1$ distinct points, enables us to assert that the deviation-function $e_* = f - g_*$ has exactly $n + 1$ points in X at which e_* takes alternatively the values d_* and 0. This situation is called t h e c a s e w i t h a l t e r n a n c e. Further we extend the Theorem 2.2 as

THEOREM 4.1. *Let $\mathcal{H} = \mathrm{span}\,\{g_1, \ldots, g_n\} \subset C[a, b]$ be an interpolatory subspace of the type $I_n\{[a, b]\}$, $X \subseteq [a, b]$ is arbitrary compact, and let us suppose that there exists an element $g_0 \in \mathcal{H}$ with $g_0(x) < f(x)$ for all $x \in X$. Then each of the following is a necessary and sufficient condition for $g_* \in \mathcal{H}$ to be a best one-sided approximation to f, i.e., $g_* = g_*(f; \mathcal{H}; X)$:*

(a) *There does not exist* $g \in \operatorname{span}\{g_1, \ldots, g_n\}$ *such that* $\sigma(x)g(x) > 0$ *for all* $x \in A(g_*)$.

(b) *The zero n-vector* θ_{R^n} *is in the convex hull of the set of n-tuples* $\{\sigma(x)(g_1(x), \ldots, g_n(x)) \mid x \in A(g_*)\}$.

(c) *There exists a linear functional* $l \in C[X]^*$ *of the form* (3.3) *which satisfies* $l(g) = 0$ *for all* $g \in \mathcal{H}$, *where* $x_1 < x_2 < \ldots < x_{n+1}$, $x_i \in E_+(g_*) \cup C_-(g_*)$, $\sigma(x_i)\gamma_i > 0$ *for* $i = 1, \ldots, n+1$, *and* $\gamma_i\gamma_{i+1} < 0$ *for* $i = 1, .., n$.

(d) (*Alternation*) *There exist the points* $x_1 < x_2 < \ldots < x_{n+1}$ *in* $A(g_*) = E_+(g_*) \cup C_-(g_*)$ *such that* $\sigma(x_{i+1}) = -\sigma(x_i)$ *for* $i = 1, \ldots, n$.

In this case we have the following consequences.

Corollary 4.2. *Let* $x_1 < \ldots < x_{n+1}$ *be critical points from* X *relative to deviation-function* $e_* = f - g_*$. *Then the coefficients* α_k *of* $g_* = g_*(f; \mathcal{H}; X)$ *verify*

$$\sum_{k=1}^{n} \alpha_k g_k(x_i) = f(x_i) - \vartheta_i d_*, \quad i = 1, \ldots, n+1 \tag{4.1}$$

where ϑ_i *are the coordinates of the alternating-vector* $\vartheta = (1, 0, 1, 0, \ldots) \in R^{n+1}$ *or of the similarly vector* $\bar{\vartheta} = (0, 1, 0, 1, \ldots) \in R^{n+1}$.

To the system of points $x_1 < \ldots < x_{n+1}$ corresponds for the functional l the system of coefficients

$$\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_r, \mu_s$$

or

$$\mu_1, \lambda_1, \mu_2, \lambda_2, \ldots, \mu_s, \lambda_r$$

with $\lambda_i > 0$ and $\mu_j < 0$ and $r + s = n + 1$. (We select one of the above system such that $d_* > 0$. This we shall show later). Similarly, if $x'_1 < \ldots < x'_{n+1}$ are critical points from X relative to the deviation-function $e^* = f - g^*$ with $g^* = g^*(f; \mathcal{H}; X)$, then the coefficients of g^* verify

$$\sum_{k=1}^{n} \alpha_k g_k(x'_i) = f(x'_i) - \eta_i d^*, \quad i = 1, \ldots, n+1 \tag{4.2}$$

where η_i, $i = 1, \ldots, n+1$ are the coordinates of alternating-vectors $\eta = (0, -1, 0, -1, \ldots) \in R^{n+1}$ respectively $\bar{\eta} = (-1, 0, -1, 0, \ldots) \in R^{n+1}$. If g_* is the element of the best one-sided approximation from below of f on $X = [a, b]$, then in [3, p. 12] it is shown that this is equivalent with the fact that $-g_*$ is the element of the best one-sided approximation

from above of $-f$, i.e., $g^*(f; \mathcal{H}; [a, b]) = -g_*(-f; \mathcal{H}; [a, b])$. This may be extended to the interpolatory case with an arbitrary compact set $X \subset [a, b]$. Therefore in the case when the knots are ordered as $x_1 < \ldots < x_{n+1}$ we can assert the following: if for the vector ϑ the system (4.1) furnishes us the element g_*, then from the same system one finds, with the vector η and for a certain system $x_1 < \ldots < x_{n+1}$, the element g^*. We remark that generally these two systems of knots are not the same.

Corollary 4.3. *If* n *is even, then the points* x_1 *and* x_{n+1} *are critical points of the same kind, i.e., both belong to* $E_+(g_*)$ *or to* $C_-(g_*)$. *For* n *odd the same points are critical of the different nature.*

Indeed, let us suppose $n = 2k \in N$ and $r + s = 2k + 1$. According to the alternatory property, one results $r = k + 1$, $s = k$ or $r = k$, $s = k + 1$. In the first case x_1 and x_{n+1} belong to $E_+(g_*)$, while in the second case these points are in $C_-(g_*)$. If $n = 2k - 1 \in N$ and $r + s = 2k$, we give $r = s = k$. But this means that x_1 and x_{n+1} are not of the same kind. Moreover, in both cases the number of contact points (in $C_-(g_*)$) is $s = \left[\frac{n}{2}\right] + 1$. □

Corollary 4.4. *Let* $X = [a, b]$, $\mathcal{H} = \operatorname{span}\{g_1, \ldots, g_n\}$ *be an interpolatory of the type* $I\{[a, b]\}$ (i.e., $\mathcal{H}$ *is a Chebyshev subspace of* $C[a, b]$ *on* $[a, b]$, *and suppose that* $g_1 = 1$. *Then, if* f *is non-polynomial with respect to* $\mathcal{H}$, *the deviation-function* $e_* = f - g_*(f; \mathcal{H}; [a, b])$ *has the end-points* a *and* b *in its set of alternance* $A(g_*)$.

Indeed, suppose that x_{i_0} is extremal, i.e., $e_*(x_{i_0}) = p(f - g_*)$, and x_{i_0+1} is a contact point, $e^*(x_{i_0+1}) = 0$. Let us suppose that there is an extremal point x_0 between x_{i_0} and x_{i_0+1} (the same study when x_0 is a contact point). The function e_* differs on $[x_{i_0}, x_0]$ of constant fnnction $p(f - g_*)$. This is motivated by the fact that the number of extremal points is finite. Therefore we find a positive number c so that the function $e_* - c = f - (g_* + c)$ vanishes at three points from $[x_{i_0}, x_{i_0+1}]$ and one results that the above function has at least $n + 2$ roots on $[a, b]$. But this contradicts the fact that $g_* + c \in \mathcal{H}$ and $\mathcal{H} + f$ is interpolatory of dimension $n + 1$. Further, let us assume that the end-point a is not critical point, i.e., $0 < e_*(a) < p(f - g_*)$. Then we can find a positive number c for which the function $e_* - c = f - (g_* + c)$ has a root on $[a, x_1]$ as well as a root on $[x_1, x_2]$. This means that $e_* - c$ has at least $n + 1$ roots on $[a, b]$, i.e., $f = g_* + c$. But this is a contradiction and the proof is complete. □

The above Corollary extend a well-known result by T. POPOVICIU [9] in the case $X = [a, b]$ and $\mathcal{H}$ is the subspace of polynomials of the degree $n - 1$.

5. On computation

Let be $X = \{x_1, \ldots, x_{n+1}\}$ fixed and l be the functional considered in Theorem 2.2.(c). Put

$$\gamma_i = \begin{cases} \lambda_i & \text{if} \quad \gamma_i > 0 \\ \mu_j & \text{if} \quad \gamma_i < 0, \end{cases}$$

and $I = \{i \in \{1, \ldots, n+1\} \mid \gamma_i > 0\}$, $J = \{i \in \{1, \ldots, n+1\} \mid \gamma_i < 0\}$. Then we have

$$\begin{aligned} l(f) = l(f - g_*) &= \sum_{i \in I} \lambda_i (f(y_i) - g(y_i)) + \sum_{j \in J} \mu_j (f(z_j) - g(z_j)) \\ &= \sum_{i \in I} \lambda_i (f(y_i) - g(y_i)) \\ &= d_* \sum_{i \in I} \lambda_i, \end{aligned}$$

$$y_i \in E_+(g_*), \quad z_j \in C_-(g_*).$$

Therefore, if $I \neq \emptyset$ and $d_* > 0$, then we see that $l(f) > 0$. In this manner we conclude that by selecting the sign of γ_{n+1} in (3.3) so that $l(f) > 0$, then by means of (3.2) one has the posibility to divide the index set $\{1, \ldots, n+1\}$ in the subsets I and J. Instead of l we shall consider the normalized functional $l_* = l / \sum_{i \in I} \lambda_i$ which satisfies $l_*(f) = d_*$. If l_* is welldefined then is known the deviation $d_* = l_*(f)$. The knowledge of the sets I, J is equivalent with the fact that are known the subsets $E_+(g_*)$ and $C_-(g_*)$. Taking into account that the coefficients α_k of $g_* = \sum_{k=1}^{n} \alpha_k g_k$ satisfy the system of equations

$$\sum_{k=1}^{n} \alpha_k g_k(x_i) = \varphi(x_i), \quad i = 1, \ldots, n+1 \tag{5.1}$$

where

$$\varphi(x_i) = f(x_i) - \frac{1 - \sigma(x_i)}{2} d_*. \tag{5.2}$$

Thus g_* is the element which interpolates the function φ on the set X. On the other hand, from the construction of φ as well as by taking into account that every element from $\mathcal{H}$ is determined by n distinct points, we may write

$$g = L(\mathcal{H}; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}; \varphi). \tag{5.3}$$

We note that the signs $\sigma(x_i)$ may be determined as follows: if (5.1) is considered as a system in the unknowns $\alpha_1, \ldots, \alpha_n, d_*$, then

$$d_* = \frac{V\begin{pmatrix} g_1, \ldots, g_n, f \\ x_1, \ldots, x_{n+1} \end{pmatrix}}{V\begin{pmatrix} g_1, \ldots, g_n, \frac{1+\sigma(\cdot)}{2} \\ x_1, \ldots, x_{n+1} \end{pmatrix}} = \frac{V\begin{pmatrix} g_1, \ldots, g_n, f \\ x_1, \ldots, x_{n+1} \end{pmatrix}}{\sum_{i=1}^{n+1} \frac{1+\sigma(x_i)}{2} D_i}$$

where D_i are the co-factors in the developing of the denominator by the elements of the last row. Because the numerator has a constant sign and the denominator is a linear combination of $(1 + \sigma(x_i))/2$, we may determined the sign of coefficients D_i such that d has a positive value and moreover a minimal one. Indeed, put

$$\sigma(x_i) = \operatorname{sign}\left(V\begin{pmatrix} g_1, \ldots, g_n, f \\ x_1, \ldots, x_{n+1} \end{pmatrix} : D_i^{-1}\right),$$

$$i = 1, \ldots, n+1.$$

The above method depends of a functional l which enables us to find d_* and $\sigma(x_i)$, $i = 1, \ldots, n+1$. Professor T. POPOVICIU in his work considers simultaneously two interpolatory-systems $\mathcal{H} = \operatorname{span}\{g_1, \ldots, g_n\}$ and $W = \operatorname{span}\{g_1, \ldots, g_n, g_{n+1}\}$ (where g_{n+1} is selected in a convenable manner). By means of this idea we shall give a more elegant solution.

Let

$$u = L(\mathcal{W}; x_1, \ldots, x_{n+1}; f) = \sum_{k=1}^{n+1} a_k g_k$$

and

$$v = L(\mathcal{W}; x_1, \ldots, x_{n+1}; f) = \sum_{k=1}^{n+1} b_k g_k,$$

and let us consider

$$h = u - d \cdot v$$

where d is a real number. If $d = d_0$ is selected such that the coefficient of g_{n+1} is zero, then the element

$$h_* = u - d_* v = \sum_{k=1}^{n} (a_k - d b_k) g_k$$

belongs to $\mathcal{H}$ and satisfies the system (5.1). Therefore we have $h^* = g^*(f; \mathcal{H}; X)$. From the above remarks and by taking into account (see for instance [8, p. 34])

$$a_{n+1} = [\mathcal{W}; x_1, \dots, x_{n+1}; u], \quad b_{n+1} = [\mathcal{W}; x_1, \dots, x_{n+1}; v],$$

if we denote $d_* = a_{n+1}/b_{n+1}$, then one finds h_*. Because the generalized Lagrange operator is linear (see for instance [8, p, 27]), we may write

$$h_* = g_*(f; \mathcal{H}; X) = L(\mathcal{W}; x_1, \dots, x_{n+1}; \varphi) \tag{5.4}$$

where φ is given by (5.2). We note that in (5.4) the coefficient of g_{n+1} is zero. It is of interest to remark that from $f(x_i) = u(x_i)$, $i = 1, \dots, n+1$, as well as $l(g) = 0$ for all $g \in \mathcal{H}$, we conclude that $l(f) = d_* = \frac{1}{b_{n+1}} [\mathcal{W}; x_1, \dots, x_{n+1}; f]$ is a divided difference (of order n relative to the interpolatory segment $\mathcal{H} \subset \mathcal{W}$).

Because (5.3) is symmetric relative to the knots and on the other hand, by taking into account the relationship between φ and f, in the following we intend to represent the element g of best one-sided approximation in a more convenient form. Namely, let us denote

$$\delta_k = f(x_k) - L(\mathcal{H}; x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}; f)(x_k), \quad i = 1, \dots, n+1.$$

It is known that (see for instance [8, p. 45])

$$\delta_k = \frac{V\begin{pmatrix} g_1, \dots, g_{n+1} \\ x_1, \dots, x_{n+1} \end{pmatrix}}{V\begin{pmatrix} g_1, \dots, g_n \\ x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1} \end{pmatrix}} [\mathcal{W}; x_1, \dots, x_{n+1}; f].$$

Let $\theta_1, \dots, \theta_{n+1}$ be non-negative numbers so that

$$\sum_{k=1}^{n+1} \theta_k = 1. \tag{5.5}$$

Put

$$h = \sum_{k=1}^{n+1} \theta_k L(\mathcal{H}; x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}; f) \tag{5.6}$$

where θ_k' must be determined in order that $h = g_*(f; \mathcal{H}; X)$. On account of the equalities

$$L(\mathcal{H}; x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}; f)(x_i) = \begin{cases} f(x_i) & \text{if } i \neq k \\ f(x_k) - \delta_k & \text{if } i = k \end{cases}$$
$$i, k = 1, \dots, n+1,$$

we may write

$$\begin{aligned} h(x_i) &= \sum_{k=1}^{n+1} \theta_k f(x_i) - \theta_i \delta_i \\ &= f(x_i) - \theta_i \delta_i, \quad i = 1, \dots, n+1. \end{aligned}$$

From this it follows

$$f(x_i) - h(x_i) = \theta_i \delta_i, \quad i = 1, \dots, n+1. \tag{5.7}$$

Let us suppose that $|\delta_i| \neq 0$, $i = 1, \dots, n+1$. (For example, this is fulfilled when $\mathcal{H}$ is interpolatory on X and f is not polynomial relative to $\mathcal{H}$ on X). Because $h = g_*$, we have

$$f(x_i) - h(x_i) = \begin{cases} d_* & \text{if } x_i \in E_+(g_*) \\ 0 & \text{if } x_i \in C_-(g_*) \end{cases}$$

and from (5.6) we conclude that

$$\theta_i = \begin{cases} \dfrac{d_*}{|\delta_i|} = \dfrac{d^*}{\delta_i} & \text{if } i \in I \\ 0 & \text{if } i \in J. \end{cases}$$

Thus (5.5) implies

$$1 = \sum_{k=1}^{n+1} \theta_k' = \sum_{i \in I} \frac{d_*}{\delta_i} = d_* \sum_{i \in I} \frac{1}{\delta_i}$$

that is

$$d^* = \frac{1}{\sum_{i \in I} \frac{1}{\delta}},$$

and finally

$$\theta_k = \begin{cases} \dfrac{\frac{1}{\delta_k}}{\sum_{k \in I} \frac{1}{\delta_k}} & \text{for } k \in I \\ 0 & \text{for } k \in J. \end{cases} \tag{5.8}$$

Therefore g_* is given by (5.6) and (5.8). For the non-restricted case, a similar result was obtained by MOTZKIN and SHARMA [7, Theorem 2]. In the case with alternance we have $\delta_i \delta_{i+1} < 0$, $i = 1, \dots, n$. This implies the remarks from the preceeding paragraph. Likewise, a similar representation may be given when X contains a finite number $m > n+1$ of distinct points. □

Further we present an algorithm for the solution of the best one-sided approximation problem. In what follows $\mathcal{H}$ is an interpolatory subspace. The algorithm is the analogue of the one-point REMES exchange algorithm for uniform unconstrained approximation (see for instance [11, p. 173]). Our algorithm starts with a set $X^{(0)}$ of $n+1$ point distinct $x_1^{(0)}, \ldots, x_{n+1}^{(0)}$ of X. If we denote $g^{(0)} = g^*(f; \mathcal{H}; X^{(0)})$, then the deviation function $f - g^{(0)}$ is now examined over X and $g^{(0)} = \sum_{k=1}^{n} \alpha_k g_k$ is compared with f. From the inclusion

$$\mathcal{H}_B(0) = \{g \in \mathcal{H} \mid \forall x \in X^{(0)}, g(x) \leq f(x)\} \supset$$

$$\supset \{g \in \mathcal{H} \mid \forall x \in X, g(x) \leq f(x)\} = \mathcal{H}_B$$

it follows

$$p^{(0)}(\Phi) = \max_{i=1,\ldots,n+1} |\Phi(x_i^{(0)})| \leq \max_{x \in X} |\Phi(x)| = p(\Phi)$$

for every $\Phi \in C[X]$. In particular

$$d^{(0)} = p^{(0)}(e^{(0)}) = \min_{g \in \mathcal{H}_B(0)} p^{(0)}(f - g^{(0)}) \leq \min_{g \in \mathcal{H}_B} p(f - g^{(0)}),$$

i.e.,

$$d^{(0)} \leq p(e^{(0)}). \tag{5.9}$$

Always there is at least a point $x_q \in X - X^{(0)}$ such that

$$e^{(0)}(x_q) = \max_{x \in X - X^{(0)}} e^{(0)}(x) = M^{(0)} \tag{5.10}$$

as well as a point $x_{q'} \in X - X^{(0)}$ such that

$$e^{(0)}(x_{q'}) = \min_{x \in X - X^{(0)}} e^{(0)}(x) = m^{(0)}. \tag{5.11}$$

By means of (5.9)—(5.10) we see that

$$M^{(0)} = p(e^{(0)}). \tag{4.12}$$

If we have simultaneously

$$m^{(0)} \geq 0 \text{ and } d^{(0)} = p(e^{(0)}), \tag{5.13}$$

then the fact $g^{(0)} \in \mathcal{H}_B$ and since $e^{(0)}$ has $n+1$ critical points (i.e., the points of $X^{(0)} \subset X$) enables us to assert that $g^{(0)} = g_*(f; \mathcal{H}; X)$. If it is not as above, then at least one relation from (5.13) is violated. A point is replaced where the violation is greatest and this point is exchanged for one of the points of $X^{(0)}$ in a certain way. On account of (5.9) and (5.10) one concludes that it is possible to have simultaneously

$$m^{(0)} < 0 \text{ and } d^{(0)} < p(e^{(0)}). \tag{5.14}$$

Let x_q respectively $x_{q'}$ be one of the points which satisfies (5.10) respectively (5.11). Setting

$$x_0^{(1)} = \begin{cases} x_q & \text{if } M^{(0)} - d^{(0)} > |m^{(0)}|, \\ x_{q'} & \text{if } M^{(0)} - d^{(0)} \leq |m^{(0)}|, \end{cases} \tag{5.15}$$

and

$$X^{(1)} = \{x_1^{(1)}, \ldots, x_{n+1}^{(1)}\} \tag{5.16}$$

where

$$x_i^{(1)} = \begin{cases} x_i^{(0)} & \text{for } i = 1, \ldots, n+1, \ i \neq i_0 \\ x_0^{(1)} & \text{for } i = i_0, \end{cases} \tag{5.17}$$

we have prepared the next step in algorithm, i.e., from $X^{(0)}$ to $X^{(1)}$. If $x_0^{(1)} = x_q$, then $x_0^{(1)} \in E_+(g^{(1)})$ and when $x_0^{(1)} = x_{q'}$ then $x_0^{(1)} \in C_-(g^{(1)})$. Similarly with the uniform non-restricted approximation (see for instance [4, p. 151]), it may be proved that the sequences $(d^{(\tau)})$, $(m^{(\tau)})$, $(M^{(\tau)})$ $\tau = 1, 2, \ldots$, and $(g^{(\tau)})$ $\tau = 1, 2, \ldots$, $(g^{(\tau)} \in \mathcal{H}_B)$, are convergent when $\tau \to +\infty$.

In the case with alternance it is easy to exchange a point $x_i^{(0)} \in X^{(0)}$ with $x_0^{(1)}$. Indeed, let $x_{i_0-1}^{(0)}$ and $x_{i_0}^{(0)}$ be two points from $X^{(0)}$ so that $x_0^{(1)} \in (x_{i_0-1}^{(0)}, x_{i_0}^{(0)})$. Since we know the sign of coefficient relative to $x_0^{(1)}$ which appears in the functional l, further we exchange with preservation of the alternance, one of $x_{i_0-1}^{(0)}, x_{i_0}^{(0)}$ with $x_0^{(1)}$.

6. The connection between one-sided and unconstrained approximation

Let $\mathcal{H} = \text{span}\{g_1, \ldots, g_n\} \subset C[X]$ be of the type $I_n\{[a, b]\}$, $[a, b] \supseteq X$ and $g_* = g_*(f; \mathcal{H}; X)$, $g^* = g^*(f; \mathcal{H}; X)$, $\bar{g} = \bar{g}(f; \mathcal{H}; X)$ be the elements of the one-sided best approximation from below, from above respectively of the unconstrained best approximation. Then we have

THEOREM 6.1. *Let g_*, g^*, $\bar{g}$ be the elements described as above. In order to exists a positive constant c such that*

$$g_* + c = \bar{g} = g^* - c \tag{6.1}$$

it is necessary and sufficient that $g_1 = 1$.

Proof. Let $g^* = g_* + 2c$ where $c > 0$, and

$$A(g_*) = E_+(g_*) \cup C_-(g_*),$$

$$A(g^*) = E_-(g^*) \cup C_+(g^*),$$

be the sets of critical points relative to g_* and g^*. Then $C_+(g^*) \cap C_-(g_*) = \emptyset$. Indeed, if we assume that the intersection is non-void, then $C_+(g_*) = C_-(g_*)$. Therefore the elements g^* and g_* have $\left[\frac{n}{2}\right] + 1$ contact-points, which implies $g^* = g_*$ (see for instance [5]). It follows that we have

$$\begin{aligned} E_-(g^*) &= C_-(g_*) \\ E_+(g_*) &= C_+(g^*). \end{aligned} \tag{6.2}$$

Taking into account (6.2) and Corollary 4.2 we conclude that g_* and g^* may be obtained from (4.1) (or (4.2)) by means of one of the pairs (ϑ, η) $(\bar{\vartheta}, \bar{\eta})$ of alternating-vectors. Evidently that $d^* = d_* = c$. From (5.3) we may write, with the pair (ϑ, η)

$$g_* = g_*(f;\, \mathcal{K};\, X) = \frac{\begin{vmatrix} V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n \end{pmatrix} & \begin{matrix} g_1 \\ \vdots \\ g_n \end{matrix} \\ f(x_1) - c \quad f(x_2) \quad f(x_3) - c \;\ldots & 0 \end{vmatrix}}{V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n \end{pmatrix}}$$

and

$$g^* = -g_*(-f;\, \mathcal{K};\, X) = \frac{\begin{vmatrix} V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n \end{pmatrix} & \begin{matrix} g_1 \\ \vdots \\ g_n \end{matrix} \\ -f(x_1) \quad -f(x_2) - c \quad -f(x_3) \ldots & 0 \end{vmatrix}}{V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n \end{pmatrix}}$$

(see for instance [8, p. 34]). Substituting these values in the equality $g^* = g_* + 2c$, we obtain

$$\begin{vmatrix} V\begin{pmatrix} g_1, \ldots, g_n \\ x_1, \ldots, x_n \end{pmatrix} & \begin{matrix} g_1 \\ \vdots \\ g_n \end{matrix} \\ 2c \quad 2c \;\ldots\; 2c & 2c \end{vmatrix} = 0. \tag{6.3}$$

Because $V\begin{pmatrix} g_2, \ldots, g_n \\ x_1, \ldots, x_n \end{pmatrix} \neq 0$, in the determinant in (6.3) the rows are linear dependent: there are the constants λ_k such that

$$\sum_{k=1}^{n} \lambda_k g_k(x_i) = 2c, \quad i = 1, \ldots, n+1. \tag{6.4}$$

From the fact that $\mathcal{K}$ is interpolatory the equality (6.4) remain valid and for every $x \in X$. This implies that $\mathcal{K}$ contains the constant functions, i.e., $g_1 = 1$.

Further, if $g_1 = 1$ and $g_* = g_*(f;\, \mathcal{K};\, X)$, then $g_* + d_* \in \mathcal{K}$ and the function $f - (g_* + d_*)$ takes the values $-d_*$ and 0 at $n+1$ distinct points. Therefore $g_* + d_* = g^* = g^*(f;\, \mathcal{K};\, X)$. It is clearly that $\bar{g} = \frac{1}{2}(g_* + g^*)$. □ The case when $X = [a, b]$ was investigated in [6] where the reader finds many references. We note that in general case (even if $g_1 = 1$) the following inequality

$$\frac{1}{d_*} + \frac{1}{d^*} \leqq \frac{1}{d}$$

is valid, [3, p. 28].

7. The approximation with positive elements

Let X be a set on the real axis which contains at least $n+1$ distinct points and $f = \sum_{k=1}^{n} a_k g_k$ be a given element from $\mathcal{K}$, $\mathcal{K}$ is assumed to be interpolatory of the type $I\{[a, b]\}$, $[a, b] \supseteq X$. By $\mathcal{K}^+$ we denote the subset of $\mathcal{K}$ which contains only non-negative elements on X. We want to find a pair (P^*, Q^*) of elements from $\mathcal{K}^+$ which verifies the following minimum property

$$P \geqq P^* \geqq 0, \quad Q \geqq Q^* \geqq 0 \quad \text{on } X \tag{7.1}$$

and

$$f = P^* - Q^* \text{ on } X. \tag{7.2}$$

If we use the same reason as in the paragraph of introduction, we have

$$p(f_+ - P^*) = \min_{P \in \mathcal{H}^{f+}} p(f_+ - P) \tag{7.3}$$

and

$$p((-f)_+ - Q^*) = \min_{Q \in \mathcal{H}^{(-f)_+}} p((-f)_+ - Q) \tag{7.4}$$

where

$$\mathcal{H}^{f_+} = \{g \in \mathcal{H}^+ \mid \forall x \in X,\ g(x) \geqq f_+(x)\}$$

$$\mathcal{H}^{(-f)_+} = \{g \in \mathcal{H}^+ \mid \forall x \in X,\ g(x) \geqq (-f)_+(x)\}.$$

Thus we have

$$P^* = g^*(f_+ ; \mathcal{H} ; X) \text{ and } Q^* = g^*((-f)_+ ; \mathcal{H} ; X). \tag{7.5}$$

Evidently if f does not change its sign on X, then the solution is trivial. If $f_+ = f$ on X, then $P^* = f$, $Q^* = 0$, and with $f_- = f$ on X we have $P^* = 0$, $Q^* = -f$.

We remark that the solution of the best approximation of this kind (which is described as above), has a mean even if X contains only n distinct points.

In the case when $X = \{x_1, \ldots, x_n\}$ from (4.3) and (7.5) one has

$$P^* = L(\mathcal{H} ; x_1, \ldots, x_n ; f_+),$$

$$Q^* = L(\mathcal{H} ; x_1, \ldots, x_n ; (-f)_+).$$

If $X = \{x_1, \ldots, x_{n+1}\}$ the solution is given by (5.6). If $\mathcal{H} = \mathfrak{L}$ we have the problem proposed by T. POPOVICIU.

Finally we note that if one of P^*, Q^* is determined then the other may be find according to (7.2).

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