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ON THE JACKSON MÜNTZ APPROXIMATION

by

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Introduction

The well-known Jackson theorem gives estimates on the rate of approximation of functions in $C[0, 1]$ by means of polynomials $P_n(x) = \sum_{k=0}^{n} a_k x^k$ of degree $\leqslant n$. Namely, if $\omega(f, \delta) = \sup_{\substack{|h| \leqslant \delta \\ 0 \leqslant x \leqslant 1}} |f(x+h) - f(x)|$ denotes the modulus of continuity of f, then

$$d_n(f) = \inf_{P_n} ||f - P_n||_\infty \leqslant C\omega\left(f, \frac{1}{n}\right).$$

This result is best possible in that there exists a function f and a constant D such that $d_n(f) \geqslant D\omega\left(f, \frac{1}{n}\right)$. Similar results are known for functions in $L_p[0, 1]$, $1 \leqslant p < \infty$, where $\omega(f, \delta)$ is replaced by the p-th modulus of continuity $\omega_p(f, \delta) = \sup_{|h| \leqslant \delta} ||f(x+h) - f(x)||_p$.

Let $\lambda_0 = 0 < \lambda_1 < \ldots < \lambda_n \to \infty$, $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$. Then it is known (Müntz) that the linear combinations — „polynomials" — $\sum_{k=0}^{n} a_k x^{\lambda_k}$, $n > 0$, are dense in $C[0, 1]$ and $L_p[0, 1]$, $1 \leqslant p < \infty$. The question naturally arises as to how well to do these polynomials of „degree" $\leqslant \lambda_n$ approximate functions in the above-mentioned spaces.

To be specific, let $\lambda_1 = 0 < \lambda_1 < \ldots < \lambda_n$ and let P_Λ denote the set of all polynomials $\sum_{i=0}^{n} a_i x^{\lambda_i}$. For $1 \leqslant p \leqslant \infty$ and $f \in L_p[0, 1]$ (we identify for convenience in notation $L_\infty[0, 1]$ with $C[0, 1]$) let $d_p(f, P_\Lambda) = \inf_{P \in P_\Lambda} \|f - P\|_p$. We are seeking the smallest possible number n_p such that for any $f \in L_p[0, 1]$

$$d_p(f, P_\Lambda) \leqslant A\omega_p(f, n_p)$$

where A is some absolute constant not depending on f.

Some historical background. The first to try this problem was D. J. NEWMAN in a paper in the Amer. J. Math. 1965 [6] where he assumes $p = 2$ and $\lambda_{i+1} - \lambda_i \geqslant 2$ for all i. Here the tools are Hilbert space tools; distance formula is available etc. The result obtained is that n_2 is equivalent to $\exp\left(-2\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)$. Then in a series of papers in 1968—1970 M. VON GOLITSCHEK [8, 9, 10] discussed the problem in $C[0, 1]$ under the assumption that

$$A(\lambda_n)^\delta \leqslant \exp\left\{\sum_{i=1}^{n} 1/\lambda_i\right\} \leqslant B(\lambda_n)^\Delta,\quad n = 1, 2, \ldots, \text{for some } 0 < A,\ B < \infty$$

$0 < \delta < \Delta < \infty$. The restriction on the λ's gives the impression that most cases covered are where $\lambda_i \sim ir$ (r some constant) or where sections have this property with different r's. We finally came up with the following result [5] that included von Golitschek's.

If $\sigma_n = \max_{1 \leqslant i \leqslant n} \left\{\frac{1}{\lambda_i} \exp\left[-2\sum_{k=i+1}^{n} 1/\lambda_k\right]\right\}$, then

$$d_\infty(f, P_\Lambda) \leqslant C\omega(f, \sigma_n). \tag{1}$$

The method of proof which used von Golitschek's ideas was very simple. Knowing an exact expression for the distance in $C[0, 1]$ of a monomial x^m (m an integer) from P_Λ one can approximate f by an ordinary polynomial $p_N = \sum_{i=0}^{N} a_i x^i$ as best one can by Jackson's theorem. Then approximate each monomial x^i by its best approximator Q_i in P_Λ and estimate the distance $d_\infty(f, P_\Lambda) \leqslant \|f - P_N\| + \sum_{i=0}^{N} |a_i|\,\|x^i - Q_i\|$. We have not proven (1) is best possible and in fact one should not expect it to be so because of the generous estimations that had been used. (It is not best possible in general but it is in surprisingly many cases.) At about the same time] TORD GANELIUS and S. WESTLUND [4] and following them DONALD NEWMAN and J. BAK [1] estimated $d_p(f, P_\Lambda)$, $1 \leqslant p \leqslant \infty$, under the assumptions $\lambda_m \geqslant Sm$ $(S > 2)$ $m = 1, 2, \ldots$ and $\lambda_m \geqslant 2m$, $m = 1, 2, \ldots$, respectively. They obtained estimates some of which were best possible. The methods used were quite involved and in the case $p = \infty$ gave the same esitmates we have obtained by a very simple method. In fact for $\lambda_m \geqslant 2m$, $m = 1, 2, \ldots$, our estimate is best possible, that is, σ_n is equivalent to n_∞.

Anyway there still was the question of estimating the distance in the general case. This has recently been done first when NEWMAN [7] concluded the case $p = \infty$ and later when J. BAK, D. J. NEWMAN, J. TZIMBALARIO and the author [2] have concluded the case $2 \leqslant p < \infty$, giving some partial results for $1 \leqslant p < 2$. Recently TORD GANELIUS and DONALD NEWMAN [3] have closed the problem altogether by filling up the gap for $1 \leqslant p < 2$. Also recently another paper by M. VON GOLITSCHEK [11] improved his earlier results to include many other special cases.

We shall only state the main result here leaving the details of the proofs to be found in [2] and [3].

Main results. Let $B_p(z)$ be the Blaschke product with zeroes at $\lambda_m + 1/p$, i.e.,

$$B_p(z) = \prod_{m=0}^{n} \frac{z - (\lambda_m + 1/p)}{z + (\lambda_m + 1/p)},$$

and let $\varepsilon_p = \max_{\mathrm{Re}\, z = 1} |B_p(z)/z|$. We have

THEOREM. *Let $1 \leqslant p \leqslant \infty$ be fixed. Then there exists a constant A such that for any $f \in L_p[0, 1]$ there exists a $P \in P_\Lambda$ wiih*

$$\|f - P\|_p \leqslant A\omega(f, \varepsilon_p).$$

Furthermore this result is best possible in that there exists an $f \in L_p[0, 1]$ and a constant B such that for any $P \in P_\Lambda$, $\|f - P\|_p \geqslant B\omega_p(f, \varepsilon_p)$. Interesting special cases are that of separated λ's (S.) where $\lambda_m \geqslant 2m$, $m = 1, 2, \ldots$ (this includes the case $\lambda_{m+1} - \lambda_m \geqslant 2$, $m = 0, 1, 2, \ldots$) and that of non-separated λ's (N.S.) where $\lambda_{m+1} - \lambda_m \leqslant 2$, $m = 0, 1, 2, \ldots$. In case S. $\varepsilon_p \sim \exp\left(-2\right) \sum_{m=1}^{n} 1/\lambda_m)$ and in case N.S. $\varepsilon_p \sim \left(\sum_{m=1}^{n} \lambda_m\right)^{-1/2}$.

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