MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 4, Nº 1, 1975, pp. 5 — 11

ON AN UNITARY METHOD FOR INTEGRATION, PARTIAL DERIVATION AND INTERPOLATION FORMULAE

by

ADOLF HAIMOVICI

(Iassy)

Some years ago, I have given a formula [1] for numerial integration, numerial derivation and interpolation for functions of one variable using an unitary method. Starting with an analogous method, I will generalize this formula for functions of several variables.

§1. Notations

Let $x = (x_1, x_2, \ldots, x_n)$ be a point in the n-dimensional interval $\Omega = [a, b]$, i.e:

$$a_i \leqslant x_i \leqslant b_i ;$$

with α we will denote the multindex

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \qquad \alpha_i \in \mathbf{N} ;$$

and

$$\alpha ! = \alpha_1 ! \alpha_2 ! \ldots \alpha_n !,$$

$$(x - a)^\alpha = (x_1 - a_1)^{\alpha_1} (x_2 - a_2)^{\alpha_2} \ldots (x_n - a_n)^{\alpha_n}.$$

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ then $\alpha > \beta$ and $\alpha \leqslant \beta$ mean:

$$\forall_i, \ \alpha_i > \beta_i \text{ and respectively, } \alpha_i \leqslant \beta_i,$$

and

$$\alpha + \beta = (\alpha_1 + \beta_1, \ \alpha_2 + \beta_2, \ \ldots, \ \alpha_n + \beta_n).$$

We will denote also

$$1 = (1_1, 1_2, \ldots, 1_n).$$

As usual, we will denote

$$D^\beta \varphi = \frac{\partial^{|\beta|} \varphi(x)}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_n^{\beta_n}}, \quad |\beta| = \sum_{i=1}^{n} \beta_i.$$

§2. Preliminaries and fundamental formulae

Consider in Ω the set of points $\{x^{(k)}\}$ given by

$$x^{(k)} = (x_1^{(k)},\ x_2^{(k)},\ \ldots,\ x_n^{(k)})\ k = (k_1,\ k_2,\ \ldots,\ k_n),\ k_i \in \mathbf{N},$$

where

$$x_i^{(k)} = a_i + k_i h_i, \qquad h_i = (b_i - a_i)/K_i,$$
$$K = (K_1,\ K_2,\ \ldots,\ K_n),$$

and K_i is the number of subintervals into which we have divided $[a_i,\ b_i]$; and a point $x^* = (x_1^*,\ x_2^*,\ \ldots,\ x_n^*)$. With the aid of a point $x^{(k)}$ we can decompose the interval $[a,\ b]$ in 2^n subintervals $S_i^k (i = 1, 2, \ldots, 2^n)$, each of which being characterized by the set of numbers

$$\operatorname{sgn}(x_1^{(k)} - x_1),\ \operatorname{sgn}(x_2^{(k)} - x_2),\ \ldots,\ \operatorname{sgn}(x_n^{(k)} - x_n).$$

Denote by I_{i1}^k the set of indices j such that

$$x_j^{(k)} - x_j \geqslant 0, \qquad (x_1,\ x_2,\ \ldots,\ x_n) \in S_i^k,$$

by I_{i2}^k the remaining set of indices for which

$$x_j^{(k)} - x_j < 0,$$

and by τ_i the number of indices in I_{i1}^k.

Let also be:

a_0) $v_i^{(k)}$ the vertex of the interval $S_i^{(k)}$, opposite to $x^{(k)}$,

a_1) $v_{iq_1}^{(k)}$ the projection of $x^{(k)}$ on the q_1-th 1-face of Ω, issuing from $v_i^{(k)}$,

a_2) $v_{iq_1q_2}^{(k)}$ the projection of $x^{(k)}$ on the q_1, q_2-th 2-face of Ω issuing from $v_i^{(k)}$, ...

Denote finally with S_i^*, I_{i1}^*, I_{i2}^*, v_i^*, $v_{iq_1}^*$, ..., the sets and points corresponding to x^*, analogous to those corresponding to $x^{(k)}$, and let the index $i = 1$ correspond to the subinterval of Ω with $x_j^* - x_j > 0$, $\forall_j$.

We consider now the function:

$$\Phi(x) = \sum_{i=1}^{2^n} \sum_{x^{(k)} \in S_i^*} \left\{ P_k(x^{(k)} - x) H_i^{(k)}(x) + \frac{(x^* - x)^\gamma}{\gamma!} H_i^*(x) \right\}, \tag{1}$$

where:

a) $P_k(\xi)$ is the polynomial

$$P_k(\xi) = \sum_{\alpha=0}^{m} A_\alpha^{(k)} \frac{\xi^\alpha}{\alpha!}, \qquad m = (m_1,\ m_2,\ \ldots,\ m_n),$$

α, m and k being multindices, and $A_\alpha^{(k)}$ constants;

b) $H_i^{(k)}(x)$ is the product of n Heaviside functions

$$H_i^{(k)}(x) = \prod_{j \in I_{i1}^*} H(x_j^{(k)} - x_j) \prod_{l \in I_{i2}^*} H(x_l - x_l^{(k)}), \quad x \in \Omega,$$

and

$$H(x' - x) = \begin{cases} 1 & \text{for } x' - x \geqslant 0, \\ 0 & \text{for } x' - x < 0; \end{cases}$$

c) In an analogous manner

$$H_i^*(x) = \prod_{j \in I_{i1}^*} H(x_j^* - x_j) \prod_{l \in I_{i2}^*} H(x_l - x_l^*) \quad x \in \Omega.$$

Remark now that, if $\varphi \in C^\beta(\Omega)$, then:

A. If $\beta > \alpha$,

$$\begin{aligned} \int_\Omega \frac{(x^{(k)} - x)^\alpha}{\alpha!} D^\beta \varphi(x) H_i^{(k)}(x) dx &= (-1)^{\tau_i} \sum_{p=0}^{\alpha} \frac{(x^{(k)} - v_i^{(k)})^{\alpha-p}}{(\alpha-p)!} D^{\beta-p-1} \varphi(v_i^{(k)}) \\ &+ (-1)^{\tau_i+1} \sum_{q=1}^{n} \sum_{\substack{p=0 \\ p_q=\alpha_q}}^{\alpha} \frac{(x^{(k)} - v_i^{(k)})^{\alpha-p}}{(\alpha-p)!} D^{\beta-p-1} \varphi(v_{iq}^{(k)}) + \\ &+ (-1)^{\tau_i+2} \sum_{q,r=1}^{n} \sum_{\substack{p=0 \\ p_q=\alpha_q,\, p_r=\alpha_r}}^{\alpha} \frac{(x^{(k)} - v_i^{(k)})^{\alpha-p}}{(\alpha-p)!} D^{\beta-\alpha-1} \varphi(v_{iqr}^{(k)}) + \\ &+ \ldots + (-1)^{\tau_i+n} D^{\beta-\alpha-1} \varphi(x^{(k)}). \end{aligned} \tag{2}$$

B. If $\beta = \alpha$ and $x^{(k)} \leqslant x^*$ $(x^{(k)} \in S_1^*,\ v_1^{(k)} = a)$,

$$\begin{aligned} \int_\Omega \frac{x^{(k)} - x}{\alpha!} D^\alpha \varphi\, dx\, H_1^{(k)}(x) dx &= (-1)^n \sum_{p=0}^{\alpha-1} \frac{(x^{(k)} - v_1^{(k)})^{\alpha-p}}{(\alpha-p)!} D^{\beta-p-1} \varphi(v_1^{(k)}) + \\ &+ (-1)^{n-1} \sum_{q=1}^{n} \sum_{\substack{p=0 \\ p_q=\alpha_q}}^{\alpha-1} \frac{(x^{(k)} - v_1^{(k)})^{\alpha-p}}{(\alpha-p)!} \int_{a_q}^{x_q^{(k)}} \varphi(w_{1q}^{(k)}) dx_q + \\ &+ (-1)^{n-2} \sum_{q,r=1}^{n} \sum_{\substack{p=0 \\ p_q=\alpha_q,\, p_r=\alpha_r}}^{\alpha-1} \frac{(x^{(k)} - v_1^{(k)})^{\alpha-p}}{(\alpha-p)!} \int_{a_q}^{x_q^{(k)}} \int_{a_r}^{x_r^{(k)}} \varphi(w_{1qr}^{(k)}) dx_q dx_r + \\ &+ \ldots + \int_{S_1^{(k)}} \varphi(x) dx. \end{aligned} \tag{3}$$

where $w_{1q}^{(k)}$ is the projection of the point x on the q-th 1-face, $w_{1qr}^{(k)}$ a point on the q, r-th 2-face of the interval $[a,\ x^{(k)}]$, issuing from the vertex $v_1^{(k)}$.

From this formulae, supposing $\beta > m$, we get:

$$\int_{\Omega} P_k(x^k - x) H_i^{(k)}(x) D^{\beta}\varphi(x)dx = (-1)^{\tau_i} \sum_{p=0}^{m} D^p P_k(x^{(k)} - v_i^{(k)}) D^{\beta-p-1} \varphi(v_i^{(k)}) +$$

$$(4) \qquad + (-1)^{\tau_i+1} \sum_{q=1}^{n} \sum_{\substack{p=0 \\ p_q=\alpha_q}}^{m} D^p P_k(x^{(k)} - v_{iq}^{(k)}) D^{\beta-p-1} \varphi(v_{iq}^{(k)}) +$$

$$+ (-1)^{\tau_i+2} \sum_{q,r=1}^{n} \sum_{\substack{p=0 \\ p_q=\alpha_q,\, p_r=\alpha_r}}^{\alpha} D^p P_{(k)}(x^{(k)} - v_{iqr}^{(k)}) D^{\beta-p-1} \varphi(v_{iqr}^{(k)}) +$$

$$+ \ldots + (-1)^{\tau_i+n} A_{(\alpha)}^{(k)} D^{\beta-\alpha-1} \varphi(x^{(k)}),$$

and further

$$(5) \qquad \int_{\Omega} \Phi(x) D^{\beta}(\varphi)dx = \int_{\Omega} \sum_{i=1}^{2^n} \left\{ \sum_{x^{(k)} \in S_i^*} P_k(x^* - x)\, H_i^{(k)}(x) + \right.$$

$$\left. + \frac{(x^* - x)^{\gamma}}{\gamma\,!} H_i^*(x) \right\} D^{\beta}\varphi(x)dx.$$

From this equality, we will deduce cubature, numerical derivation and interpolation formulae.

§3. Cubature formula

Taking în (5), $x^* = b$, $\beta = \gamma$ and $m < \beta$, it follows that $S_i^* = [a, b] = \Omega$, and

$$(6) \qquad \int_{\Omega} \varphi(x)\, dx = \mathcal{F}(\varphi) - \sum_k \int_{\Omega} P_k(x^{(k)} - x) D^{\gamma}\varphi(x) H_i^{(k)}(x)dx + \int_{\Omega} \Phi(x) D^{\gamma}\varphi(x)dx,$$

where $\mathcal{F}(\varphi)$ means

$$F(\varphi) = (-1)^n \sum_{p=0}^{\alpha-1} \frac{(x^* - a)^{\gamma-p}}{(\gamma - p)\,!} D^{\gamma-p-1}\varphi(a) +$$

$$+ (-1)^{n-1} \sum_{i=1}^{n} \sum_{\substack{p=0 \\ p_i=\gamma_i}}^{\gamma-1} \frac{(x^* - a)^{\gamma-p}}{(\gamma - p)\,!} \int_{a_i}^{b_i} \varphi(w_{1i}^{(k)})dx_i +$$

$$(7) \qquad + (-1)^{n-2} \sum_{i,j=1}^{n} \sum_{\substack{p=0 \\ p_i=\gamma_i,\, p_j=\gamma_j}}^{\gamma-1} \frac{(x^* - a)^{\gamma-p}}{(\gamma - p)\,!} \int_{a_i}^{b_i}\int_{a_j}^{b_j} \varphi(w_{1ij}^{(k)})dx_i dx_j + \ldots +$$

$$+ \sum_{i=1}^{n} \sum_{p_i=0}^{\gamma_i-1} \frac{(x_i - a_i)^{\gamma_i-p_i}}{(\gamma_i - p_i)\,!} \int_{a_1}^{b_1} \ldots \int_{a_{i-1}}^{b_{i-1}} \int_{a_{i+1}}^{b_{i+1}} \ldots \int_{a_n}^{b_n} \varphi(w_{112\ldots i-1,\, i+1,\, \ldots n})$$

$$dx_1 dx_2 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_n.$$

We will prove now that it is possible to choose the polynomials P_k, i.e. the coefficients $A_{\alpha}^{(k)}$ such that the absolute value of

$$(8) \qquad R = \int_{\Omega} \Phi(x) D^{\gamma}\varphi(x)dx$$

be sufficiently small, and

$$(9) \qquad \int_{\Omega} \varphi(x)dx = \mathcal{F}(\varphi) - \sum_k \int_{\Omega} P_k(x^{(k)} - x) H_i^{(k)}(x) D^{\gamma}\varphi(x)dx,$$

be considered as an approximate cubature formula for the function $\varphi(x)$ in the n-dimensional interval Ω, the remainder of which be given by (8).

Or we have

$$|R| \leqslant M \int_{\Omega} |\Phi(x)|dx, \quad M = \sup_{\Omega} |D^{\gamma}\varphi|.$$

Consider the n-dimensional subintervals

$$(10) \qquad I_k = [a + kh,\ a + (k+1)h],$$

a point $\xi^{(k)} \in I_k$, and for each k the polynomial

$$Q_k(x) = \sum_{l \geqslant k} P_l(x^{(l)} - x),$$

satisfying the following conditions:

i) the set $\{Q_k\}$ is ordered such that, if $k^{(1)} = (k_1^{(1)}, k_2^{(1)}, \ldots, k_n^{(1)})$ and $k^{(2)} = (k_1^{(2)}, k_2^{(2)}, \ldots, k_n^{(2)})$, then $Q_{k^{(1)}} < Q_{k^{(2)}}$, iff, p-being the rank of the first different components of $k^{(1)}$ and $k^{(2)}$, we have

$$k_p^{(1)} > k_p^{(2)},$$

ii) all the derivatives of $Q_k(x)$, of order less or equal to $|m|$ be equal to zero, in $\xi^{(k)}$.

It follows then

$$\int_{I_k} |\Phi(x)|dx = o\ (\|h\|^{n+m+1}),$$

where

$$\|h\| = \sqrt{\sum_{i=1}^{n} |h_i|^2},$$

and, as an immediate consequence

$$(11) \qquad \int_{\Omega} |\Phi(x)|dx = o\ (\|h\|^{m+1}).$$

Obviously, the polynomials $Q_k(x)$ are

$$Q_k(x) = \sum_{\substack{\alpha+\beta>\gamma \\ \beta\leq m}} \frac{(b-\xi)^\alpha}{\alpha!} \frac{(\xi-x)^\beta}{\beta!}.$$

After having chosen the polynomials Q_k, it follows that

$$P_s(x^{(s)}-x) = Q_s(x) - \sum_{i=1}^{n} Q_{s-1_i}(x) + \sum_{i,j=1}^{n} Q_{s-1_i-1_j}(x) \ldots + (-1)^n Q_{s-1}(x).$$

On the other hand, the function $\mathcal{F}(\varphi)$ involves integrals on the frontier of Ω. If this integrals are not known, other numerical integrations must be performed in order to know the numerical value of our integral (9). If the expression $\mathcal{F}(\varphi)$ can be calculated with error

$$\varepsilon = o(\|h\|^t) \quad t \leq |m|,$$

then it follows immediately that the error in the evaluation of (9) if $o(\|h\|^t)$.

§4. Interpolation and numerical derivation formulae

We come back to (5) for $i = 1, 2, \ldots 2^n$, and taking into account (5), (4) and (2) we deduce, for $\beta > \gamma \geq m$

$$\int_{S_i^*} \Phi(x) D^\beta \varphi(x) dx = \int_{S_i^*} \left\{ \sum_{x^{(k)} \in S_i^*} P_k(x^{(k)} - x) H_i^{(k)}(x) + \frac{(x^*-x)^\gamma}{\gamma!} H_i^*(x) \right\} D^\beta \varphi dx =$$

$$= \int_{S_i^*} \sum_{x^k \in S_i^*} P_k(x^{(k)} - x) H_i^{(k)}(x) D^\beta \varphi(x) dx + \mathcal{G}_i(\varphi) + (-1)^{\tau_i+n} D^{\beta-\gamma-1} \varphi(x^*),$$

where

$$\mathcal{G}_i(\varphi) = (-1)^{\tau_i} \sum_{p=0}^{\gamma} \frac{(x^{(k)} - v_i^{(k)})^{\gamma-p}}{(\gamma-p)!} D^{\beta-p-1} \varphi(v_i^{(k)}) +$$

(12)
$$+ (-1)^{\tau_i+i} \sum_{q=1}^{n} \sum_{\substack{p=0 \\ p_q=\gamma_q}}^{\gamma} \frac{(x^{(k)} - v_i^{(k)})^{\gamma-p}}{(\gamma-p)!} D^{\beta-p-1} \varphi(v_{iq}^{(k)}) + \ldots +$$

$$+ (-1)^{\tau_i+n-1} \sum_{q=1}^{n} \sum_{p_r=0}^{\gamma_r} \frac{(x^{(k)} - v_{ir}^{(k)})^{\gamma_r-p_r}}{(\gamma_r-p_r)!} D^{\beta-p-1} \varphi(v_{iq}).$$

We deduce immediately

$$D^{\beta-\gamma-1}\varphi(x^*) = 2^{-n} \sum_{i=1}^{2^n} (-1)^{\tau_i+n+1} \Big\{ G_i(\varphi) +$$

$$+ \int_{S_i^*} \sum_{x^{(k)} \in S_i^*} P_k(x^{(k)} - x) H_i^{(k)}(x) D^\beta \varphi(x) dx \Big\} +$$

$$+ \sum_{i=1}^{2^n} (-1)^{\tau_i+n} \int_{S_i^*} \Phi(x) D^\beta \varphi(x) dx.$$

As before, *we will consider*

(13)
$$D^{\beta-\gamma-1}\varphi(x^*) \equiv 2^{-n} \sum_{i=1}^{2^n} (-1)^{\tau_i+n+1} \Big[G_i(\varphi) + \int_{S_i^*} \sum_{x^{(k)} \in S_i^*} P_k(x^{(k)} - x) H_i^{(k)}(x) D^\beta \varphi(x) dx \Big],$$

as an approximation formula for $D^{\beta-\gamma-1}\varphi(x)$, *the remainder of which is*

(14)
$$R = \sum_{i=1}^{2^n} (-1)^{\tau_i+n} \int_{S_i^*} \Phi(x) D^\beta \varphi(x) dx,$$

if this remainder can be done sufficiently small. Or, as in the preceding paragraphs, we can choose, — in each set S_i^* — the polynomials $Q_k(x) = \sum_{l\geq k} P_l(x^{(l)} - x)$, such that $|R| \leq 2^n M\, O\,(\|h\|^{m+1})$.

For $\beta = \gamma - 1$, (13) is an interpolation formula, for $\beta > \gamma - 1$, the same formula is a numerical derivation one.

REFERENCE

[1] Adolf Haimovici — *Sur une certaine approximation des distributions, généralisant des formules d'intégration, de dérivation numérique et d'interpolation*, Revue Roumaine de Math. pures et appl. T. **XV** *1415—1420* (1970).

Received 14. X. 1974.