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A GENERALIZATION OF A THEOREM OF MAMEDOV

by

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1. Introduction

In the theory of positive linear operators the theorems of Bohman-Korovkin and Mamedov about the convergence of a sequence of operators to the identity operator are important.

THEOREM OF KOROVKIN:

Let (L_n) $(n = 1, 2, \ldots)$ *be a sequence of positive linear operators mapping* $C[a, b]$ *into* $C[a, b]$ *and let* $\{f_0, f_1, f_2\}$ *be a Tschebyscheff-system* (T-*system*) *on* $[a, b]$. *Let this sequence satisfy the following conditions*:

$$(1.1)\qquad (L_n f_i)(x) = f_i(x) + o(1) \quad (i = 0, 1, 2\,;\ n \to \infty\,;\ x \in [a, b]).$$

Then $(L_n f)(x)$ *converges to* $f(x)$ *if* $n \to \infty$ *for each* $f \in C[a, b]$.

If the convergence in (1.1) is uniform on $[a, b]$, then the convergence of $(L_n f)(x)$ to $f(x)$ is also uniform on $[a, b]$. The theorem of Mamedov gives asymptotically the speed of convergence of $(L_n f)(x)$ to $f(x)$ if the speed of convergence is known for the testfunctions e_0, e_1 and e_2 defined by $e_i(x) = x^2$ $(i = 0, 1, 2)$ forming a special T-system. To the best of my knowledge up to now asymptotic formulas for the speed of convergence in the case that the testfunctions form an arbitrary T-system do not exist. In this paper we investigate this case. We give a solution to the problem by using a special differential calculus defined with regard to the testfunctions f_0, f_1 and f_2. In section 2 we develop this differential calculus only in so far as is needed for the proof of the main theorem in section 3. Many other extensions are left out of consideration here. In section 4 we give some applications of the theorem. For an explanation about T-systems and complete T-systems (CT-systems) or Markov-systems we refer to [2].

2. T-systems, positive linear operators and differential operators

Let $\{f_0, f_1, f_2\}$ be a T-system on a closed interval $[a, b]$ $(-\infty < a < b < \infty)$ and let $(\mathfrak{L}_n)$ $(n = 1, 2, \ldots)$ be a sequence of positive linear operators mapping $C[a, b]$ into $C[a, b]$ satisfying the conditions (1.1) where x is a point of $[a, b]$.

We assume that

$$(\mathfrak{L}_n f_i)(x) = f_i(x) + \frac{\xi_i(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right) \quad (i = 0, 1, 2) \tag{2.1}$$

where $\varphi(n)$ is independent of x, $\varphi(n) \neq 0$ for each n and $\varphi(n) \to \infty$ if $n \to \infty$. Without proof we give some lemmas and we show that the operators $\mathfrak{L}_n$ and the conditions (2.1) can be modified to other operators and conditions which are simpler with regard to the theorem of Mamedov.

L e m m a 1. *There exists a linear combination* $g = \sum_{i=0}^{2} \alpha_i f_i$ *which is strictly positive on* $[a, b]$.

At least one of the numbers $\alpha_0, \alpha_1, \alpha_2$ is then not equal to zero. We assume that $\alpha_0 \neq 0$. Then we have

L e m m a 2. $\{g, f_1, f_2\}$ *is a T-system on* $[a, b]$.

We define the functions u_0, u_1 and u_2 on $[a, b]$ by $u_0(x) = 1$,

$$u_1 = \frac{f_1}{g} \text{ and } u_2 = \frac{f_2}{g}.$$

L e m m a 3. $\{u_0, u_1, u_2\}$ *is a T-system on X.*

Now we define the positive linear operators $L_n : C[a, b] \to C[a, b]$ by

$$(L_n f)(x) = \frac{1}{\sum_{i=0}^{2} \alpha_i \left(f_i(x) + \frac{\xi_i(x)}{\varphi(n)}\right)} (\mathfrak{L}_n fg)(x) \tag{2.2}$$

for $n = 1, 2, \ldots$, excluding the values of n for which the denominator in might be equal to zero.

Then we have

$$\begin{aligned} (L_n u_0)(x) &= 1 + o\left(\frac{1}{\varphi(n)}\right) \\ (L_n u_1)(x) &= u_1(x) + \frac{\Psi_1(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right) \\ (L_n u_2)(x) &= u_2(x) + \frac{\Psi_2(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right) \end{aligned} \tag{2.3}$$

where

$$\Psi_k(x) = u_k(x) \frac{\sum_{i=0}^{2} \alpha_i (f_i - \xi_i)}{\sum_{i=0}^{2} \alpha_i f_i} \qquad (k = 1, 2).$$

L e m m a 4. *The functions u_1 and u_2 do not both have an extreme value at the same inner point $x \in [a, b]$.*

L e m m a 5. *If $x \in [a, b]$ and u_1 does not have an extreme value at x, then there exists a closed neighbourhood Ω of x where $\{u_0, u_1, u_2\}$ is a complete T-system (CT-system).*

Let $\{v_0, v_1, v_2\}$ be a CT-system on the closed interval $[a, b]$.

For each function $f \in C[a, b]$ we define the operator D^0 by

$$(D^0 f)(x) = \frac{f(x)}{v_0(x)} \qquad x \in [a, b]. \tag{2.4}$$

Of course, if $v_0 = u_0$ we have $D^0 f = f$. Further we define the differential operators $D^k (k = 1, 2)$ by

$$(D^k f)(x) = \lim_{\substack{h \to 0 \\ x+h \in [a,b]}} \frac{(D^{k-1}f)(x+h) - (D^{k-1}f)(x)}{(D^{k-1}v_k)(x+h) - (D^{k-1}v_k)(x)} \qquad x \in [a, b] \tag{2.5}$$

for each $f \in C[a, b]$ for which this limit exists.

We call f k-times differentiable at $x \in [a, b]$ $(k = 1, 2)$ with regard to the set of functions $\{v_0, v_1, v_2\}$ if $(D^k f)(x)$ exists.

Because of lemmas 4 and 5 we can speak about differentiability of a function f at the point x considered above with regard to the T-system $\{u_0, u_1, u_2\}$.

L e m m a 6. *Let $f \in C[a, b]$ be twice differentiable at $x \in [a, b]$ with regard to the T-system $\{u_0, u_1, u_2\}$ with $u_0(x) = 1$ on $[a, b]$. Then there* *exists a neighbourhood* $\Omega \subset [a, b]$ *of* x *and a continuous function* h *defined on* Ω *with* $h(x) = 0$ *such that*

$$(2.6)\qquad f(t) = f(x) + \gamma_{1,x}(t)(D^1 f)(x) + \gamma_{2,x}(t)((D^2 f)(x) + h(t)),$$

where
$$\gamma_{1,x}(t) = v_1(t) - v_1(x)$$
$$\gamma_{2,x}(t) = v_2(t) - v_2(x) - (v_1(t) - v_1(x))(D^1(x))(D^1 v_2)(x)$$

and with $v_1 = u_1$ *and* $v_2 = u_2$ *or* $v_1 = u_2$ *and* $v_2 = u_1$ *and*

$$(D^1 v_2)(x) = \lim_{h \to 0} \frac{v_2(x + h) - v_2(x)}{v_1(x + h) - v_1(x)}.$$

We remark that only the existence is assumed of the second derivative at the single point x.

3. Generalized theorem of Mamedov

An important property of a number of sequences of positive linear operators is the so-called Voronowskaya-property (V-property), in detail studied by G. MÜHLBACH [5].

Definition. *Let* $x \in [a, b]$ *and let* $l \in C[a, b]$ *be a function with* $l(x) = 0$ *and* $l(t) > 0$ *if* $t \neq x$. *Let* (L_n) $(n = 1, 2, \ldots)$ *be a sequence of positive linear operators,* $L_n : C[a, b] \to C[a, b]$ *then it is said that* (L_n) *possesses the* V*-property with respect to* l *if for each function* $g \in C[a, b]$ *with* $g(x) = 0$

$$(3.1)\qquad (L_n lg)(x) = 0((L_n l)(x)), \quad n \to \infty.$$

holds.

Now we arrive at the main theorem of this paper.

THEOREM. *Let* $[a, b]$ *be a compact interval of the real axis and let* $\{u_0, u_1, u_2\}$ *be a Tschebyscheff-system on* $[a, b]$ *with* $f_0(x) = 1$ *on* $[a, b]$. *Morevoer, let* (L_n) $(n = 1, 2, \ldots)$ *be a sequence of positive linear operators* $L_n : C[a, b] \to C[a, b]$ *and let this sequence satisfy the conditions*

$$(3.2)\qquad \begin{aligned}(L_n u_0)(x) &= 1 + o\left(\frac{1}{\varphi(n)}\right)\\ (L_n u_i)(x) &= u_i(x) + \frac{\Psi_i(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right), \quad (i = 1, 2)\end{aligned}$$

where $x \in [a, b]$, $\varphi(n)$ *is independent of* x, $\varphi(n) \neq 0$ *for each* n *and* $\varphi(n) \to \infty$ *if* $n \to \infty$. *Finally we suppose that* u_2 *is differentiable with respect to* $\{u_0, u_1\}$ *at* x *and that the sequence* (L_n) $(n = 1, 2, \ldots)$ *possesses the* V*-property with respect to the function* $\gamma_{2,x}$. *Then we have for each function* $f \in C[a, b]$ *for which the second derivative with regard to the Tschebyscheff-system* $\{u_0, u_1, u_2\}$ *at* x *exists*

$$(3.3)\qquad \begin{aligned}\varphi(n)\{(L_n f)(x) - f(x)\} &= \Psi_1(x)(D^1 f)(x) +\\ &+ \{\psi_2(x) - \psi_1(x)(D^1 f_2)(x)\}(D^2 f)(x) + o(1) \quad (n \to \infty).\end{aligned}$$

Proof. Since f is twice differentiable at $x \in [a, b]$ we can write, according to lemma 6,

$$(3.4)\qquad \begin{aligned}f(t) &= f(x) + D^1 f(x)(u_1(t) - u_1(x)) +\\ &+ (D^2 f(x) + h_x(t))(u_2(t) - u_2(x) - D^1 u_2(x)(u_1(t) - u_1(x)))\end{aligned}$$

where the function h is continuous on $[a, b]$ if we define $h(x) = 0$. We apply the operators L_n to (3.4). Taking (3.2) into account we obtain

$$(3.5)\qquad \begin{aligned}(L_n f)(x) - f(x) &= D^1 f(x)\frac{\Psi_1(x)}{\varphi(n)} + D^2 f(x)\left(\frac{\Psi_2(x)}{\varphi(n)} - D^1 u_2(x) \cdot \frac{\Psi_1(x)}{\varphi(n)}\right) +\\ &+ (L_n(u_2 - u_2(x)u_0 - (u_1 - u_1(x)u_0)D^1 u_2(x))h_x)(x) + o\left(\frac{1}{\varphi(n)}\right).\end{aligned}$$

Since the sequence (L_n) $(n = 1, 2, \ldots)$ possesses the V-property with respect to the function $\gamma_{2,x}$ we have

$$(3.6)\qquad \begin{aligned}(L_n(u_2 - u_2(x)u_0 - (u_1 - u_1(x))D^1 u_2(x))h)(x) &= (L_n \gamma_{2,x} h)(x)\\ &= o((L_n \gamma_{2,x})(x))\\ &= o\left(\frac{1}{\varphi(n)}\right).\end{aligned}$$

From (3.5) and (3.6) we obtain (3.3).

Remark. With the functions $u_i(x) = x^i$ $(i = 0, 1, 2)$ the theorem reduces to the theorem of Mamedov (with a very little change) and (3.3) reduces to

$$\varphi(n)((L_n f)(x) - f(x)) = \Psi_1(x) f'(x) + \{\Psi_2(x) - 2x\Psi_1(x)\}\frac{f''(x)}{2} + o(1) \quad (n \to \infty).$$

4. Applications

1. The trigonometric case.

If $[a, b]$ is a closed interval of length less than 2π the system $\{u_0, u_1, u_2\}$ with $u_0(x) = 1$, $u_1(x) = \sin x$ and $u_2(x) = \cos x$ $(x \in [a, b])$ is a T-system.

If $x \in [a, b]$, $x \neq \left(j + \frac{1}{2}\right)\pi$, j integer, and $f \in C[a, b]$ is twice differentiable at x, then we have for each $t \in [a, b]$ by lemma 6

$$(4.1)\qquad f(t) = f(x) + (\sin t - \sin x)\frac{f'(x)}{\cos x} - (\cos t - \cos x +$$
$$+ (\sin t - \sin x)\operatorname{tg} x)(f''(x)\cos x + f(x)\cdot \sin x) + h(t)(\cos t - \cos x +$$
$$+ (\sin t - \sin x)\operatorname{tg} x)$$

where $h(x) = 0$ and $\lim\limits_{t\to x} h(t) = 0$.

Let $(L_n)(n = 1, 2, \ldots)$ be a sequence of positive linear operators having the V-property with respect to the function $\gamma_{2,x}$ where

$$\gamma_{2,x}(t) = \cos t - \cos x + (\sin t - \sin x)\operatorname{tg} x,$$

and satisfying the conditions of the theorem of Korovkin and let (L_n) $(n = 1, 2, \ldots)$ have the properties

$$(L_n u_0)(x) = 1 \qquad + o\left(\frac{1}{\varphi(n)}\right)$$
$$(4.2)\qquad (L_n u_1)(x) = \sin x + \frac{\Psi_s(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right)$$
$$(L_n u_2)(x) = \cos x + \frac{\Psi_c(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right) \qquad (n \to \infty)$$

where $\varphi(n)$ is independent of x, $\varphi(n) \neq 0$ for each n and $\varphi(n) \to \infty$ if $n \to \infty$.

With the theorem we find

$$(4.3)\qquad \varphi(n)((L_n f)(x) - f(x)) = \Psi_s(x)\cos x - \Psi_c(x)\sin x)f'(x) -$$
$$- (\Psi_c(x)\cos x + \Psi_s(x)\sin x)f''(x) + o(1)\ (n \to \infty).$$

If $x \in [a, b]$, $x = \left(j + \frac{1}{2}\right)\pi$, j integer, we consider the T-system $u_0(x) = 1$, $u_1(x) = \cos x$ and $u_2(x) = \sin x$. From lemma 6 we get an expression analogous to (4.1) and with the theorem we find (4.3) too.

Therefore, expresion (4.3) is valid for each $x \in [a, b]$. (4.3) is simpler than the corresponding formula found by SCHURER [6]. Schurer used five testfunctions: u_0, u_1, u_2, and further u_3 and u_4 with $u_3(x) = \sin 2x$ and $u_4(x) = \cos 2x$. In the right-hand side of (4.3) he got

$$(\Psi_s(x)\cos x - \Psi_c(x)\sin x)f'(x) - \frac{1}{4}(\Psi_{cc}(x)\cos 2x + \Psi_{ss}(x)\sin 2x)f''(x) + o\left(\frac{1}{(n)}\right)$$

where Ψ_{cc} and Ψ_{ss} are defined by

$$\Psi_{ss}(x) = \lim_{n\to\infty} \varphi(n)((L_n u_3)(x) - u_3(x))$$
$$\Psi_{cc}(x) = \lim_{n\to\infty} \varphi(n)((L_n u_4)(x) - u_4(x)).$$

2. We apply the theorem to the sequence of positive linear operators $P_n (n = 1, 2, \ldots)$ $P_n : C[0, 1] \to C[0, 1]$ defined by

$$(4.4)\qquad (P_n f)(x) = (1 - x)^{n+1}\exp\left(\frac{tx}{1 - x}\right)\sum_{k=0}^{\infty} f\left(\frac{k}{n + k}\right) L_k^{(n)}(t)x^k,$$

where t is a non-positive constant and $L_k^{(n)}(t)$ is a Laguerre polynomial of degree k. These operators were introduced by E. W. CHENEY and A. SHARMA [1] and also studied by S. WATANABE and Y. SUZUKI [7]. We consider on the interval $[0, 1]$ the T-system $\{u_0, u_1, u_2\}$ where $u_0(x) = 1$, $u_1(x) = x^2$ and $u_2(x) = x^4$. Then we find

$$\Psi_1(x) = x(1 - x)^2 - 2tx$$
$$\Psi_2(x) = 6x^3(1 - x)^2 - 4tx \text{ with } \varphi(n) = n + 1.$$

The sequence (P_n) possess the V-property with respect to the function $\gamma_{2,x}$ defined by $\gamma_{2,x}(t) = (t^2 - x^2)^2$. Using this result our theorem gives

$$(4.5)\qquad (n + 1)\{(P_n f)(x) - f(x)\} =$$
$$= -tx\cdot f'(x) + \frac{x(1 - x)^2}{2}f''(x) + o(1) \qquad (n \to \infty)$$

for each $f \in 0[0, 1]$ which is twice differentiable at $x \in [a, b]$. This result is due to S. WATANABE and Y. SUZUKI [7].

3. If the functions u_1, u_2 and f are twice differentiable at $x \in [a, b]$ we can simplify formula (3.3).

In this case we have

$$(D^1 f)(x) = \frac{f'(x)}{u_1'(x)}$$

and

$$(D^2 f)(x) = \frac{u_1'(x) \cdot f''(x) - u_1''(x) \cdot f'(x)}{u_1'(x) \cdot u_2''(x) - u_1''(x) \cdot u_2(x)}$$

Under conditions of the theorem in section 3, (3.3) then reads

$$\varphi(n)\{(L_n f)(x) - f(x)\} = \tag{4.6}$$

$$= \frac{(\Psi_1(x)u_2''(x) - \Psi_2(x)u_1''(x))f'(x) + (\Psi_2(x)u_1'(x) - \Psi_1(x)u_2'(x))f(x)}{u_1'(x)u_2''(x) - u_1''(x)u_2'(x)} + o(1) \qquad (n \to \infty)$$

Remark : Since $\{u_0, u_1, u_2\}$ is a T-system, the denominator in the right-hand side of (4.6) is not equal to zero.

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