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A REPRESENTATION FOR AN ADJOINT OPERATOR

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1. Introduction

When $T : X \to Y$ is a linear operator from one normed linear space into another, one is often interested in its adjoint operator $T' : Y' \to X'$ which maps the conjugate space of Y into that of X. The adjoint T' is defined by the equation

$$(1) \qquad \langle x, y'T' \rangle = \langle Tx, y' \rangle, \ x \in X, \ y' \in Y'.$$

A notable example is offered by the Riesz theory [5] when $X = Y$ and T is compact. Then one can relate the solutions to the inhomogeneous equation

$$(2) \qquad (I - \lambda T)x = y$$

to solutions of the homogeneous adjoint equation

$$(3) \qquad (I - \lambda T')z = 0.$$

The application of this theory to Fredholm integral equations of the second kind is well known.

It is not always possible to obtain an explicit representation for T', but it can be done in some particular cases. For example, let $X = L^2(a, b)$ denote the square summable real valued functions on $[a, b]$, with

$$(4) \qquad (Tx)(t) = \int_a^b k(t, s)x(s)ds, \qquad x \in X,$$

where $k(t, s)$ is square summable on $[a, b] \times [a, b]$. Then, $X' = X$, T is compact, and

$$(T'x')(t) = \int_a^b k(s, t)x'(s)ds, \qquad x' \in X'. \tag{5}$$

Generally X' will be a space different from X. This is the case when $X = C[a, b]$, continuous real valued functions on $[a, b]$, and $X' = NBV[a, b]$, normalized functions of bounded variation [1] on $[a, b]$. If T is an integral operator of the type in Eq. (4), with $k(t, s)$ continuous on $[a, b] \times [a, b]$, then T is compact and its adjoint is given by

$$(T'x')(t) = \int_a^t \int_a^b jk(s, \zeta)\, dx'(s)d\zeta, \qquad x' \in X', \tag{6}$$

where $\int_a^b k(s, \zeta)dx'(s)$ is a Stieltjes integral. Equation (6) ia a special case of a result [3, p. 516] which gives the adjoint for certain compact integral operators defined on $X = C(S)$, where $C(S)$ is a space of continuous scalar functions on a compact Hausdorff space S.

Here we obtain an expression for the adjoint of the integral operator

$$(Tx)(t) = \int_0^t k(t, s)\, x(s)ds \tag{7}$$

defined on the space $X = C_l = \{x \in C(R\ , R) : \lim_{t\to\infty} x(t) \text{ exists}\}$, with

(i) $k \in C(R_+ \times R_+, R)$;
(ii) $k(t, s) \to \bar{k}(s)$ uniformly on compact intervals;
(iii) $\lim_{t\to\infty} \int_0^t |k(t, s)|\, ds = \int_0^\infty |\bar{k}(s)|\, ds < \infty$.

This space has been employed by CORDUNEANU C. [2] in developing the concept of admissibility for the study of integral equations. One difference between the case treated here and the previously discussed examples is that the functions in C_l are defined on a non compact set, R_+, having infinite measure.

2. The Adjoint Operator

Let $X = C_l$, and let T be the operator defined by Eq. (7) and (i), (ii), (iii). Then $X' = NBV[0, \infty)$, where $v \in NBV[0, \infty)$ if v is of bounded variation on $[0, \infty)$, $v(0) = 0$, and v is continuous from the right. That is

$$v(t + 0) = v(t), \quad t \in [0, \infty). \tag{8}$$

Furthermore, for $v \in X'$ and $x \in X$, $\langle x, v\rangle$ is given by the Stieltjes integral

$$\langle x, v\rangle = \int_0^\infty x(s)dv(s). \tag{9}$$

We assume throughout this paper that

$$k(t, s) = 0 \text{ for } s > t. \tag{10}$$

Then the adjoint T' of T is given by

$$(T'v)(t) = \int_0^\infty \int_0^t k(s, u)du\, dv(s) = \int_0^t \int_0^\infty k(s, u)dv(s)du, \quad v \in X'. \tag{11}$$

Proof. From Eqs. (1), (7), and (9) it follows that

$$\int_0^\infty x(s)\, d(T'v)(s) = \int_0^\infty \int_0^s k(s, u)x(u)du\, dv(s), \quad x \in X,\ v \in X'. \tag{12}$$

We arbitrarily fix $v \in X'$ and seek $(T'v)(t)$. This is done by applying Eq. (12) to a convenient sequence of functions in C_l and taking the appropriate limits.

For any $\zeta \in [0, \infty)$ and $\Delta\zeta > 0$, define the sequence of functions

$$\Phi_{\zeta, \Delta\zeta, n}(s) = \begin{cases} 0 \text{ for } 0 \leqslant s \leqslant \zeta \\ (n/\Delta\zeta)(s - \zeta) \text{ for } \zeta \leqslant s \leqslant \zeta + \Delta\zeta/n \\ 1 \text{ for } \zeta + \Delta\zeta/n \leqslant s \leqslant \zeta + \Delta\zeta \\ 1 - (s - \zeta - \Delta\zeta)n/\Delta\zeta \text{ for } \zeta + \Delta\zeta \leqslant s \leqslant \zeta + \Delta\zeta/n \\ 0 \text{ for } \zeta + \Delta\zeta + \Delta\zeta/n \leqslant s < \infty. \end{cases}$$

Also define

$$\Phi_{\zeta, \Delta\zeta}(s) = \begin{cases} 0 \text{ for } 0 \leqslant s \leqslant \zeta \\ 1 \text{ for } \zeta < s \leqslant \zeta + \Delta\zeta \\ 0 \text{ for } \zeta + \Delta\zeta < s < \infty. \end{cases}$$

Note that $\Phi_{\zeta,\Delta\zeta,n} \to \Phi_{\zeta,\Delta\zeta}$ pointwise on $[0, \infty)$, but not uniformly, and $\Phi_{\zeta,\Delta\zeta} \notin C_l$. However, each $\Phi_{\zeta,\Delta\zeta,n} \in C_l$ for $n = 1, 2, \ldots$ and

$$\int_0^\infty \Phi_{\zeta,\Delta\zeta,n}(s)d(T'v)(s) = \int_0^\infty \int_0^s k(s, u)\Phi_{\zeta,\Delta\zeta,n}(u)du\, dv(s). \tag{13}$$

The left side of Eq. (13), which we denote by I_n, can be written as the sum of three integrals, I_1, I_2, and I_3, where the latter are the integrals of $\Phi_{\zeta,\Delta\zeta,n}$ over the three intervals on which it does not vanish identically. Then

$$\lim_{n\to\infty} I_n = (T'v)(\zeta+\Delta\zeta) - (T'v)(\zeta).$$

To prove this note that

$$|I_n - ((T'v)(\zeta+\Delta\zeta) - (T'v)(\zeta))| = $$
$$= |I_1 + I_3 + (T'v)(\zeta) - (T'v)(\zeta+\Delta\zeta/n)| \leqslant$$
$$\leqslant \overset{\zeta+\Delta\zeta/n}{\underset{\zeta}{\mathrm{V}}}(T'v) + \overset{\zeta+\Delta\zeta+\Delta\zeta/n}{\underset{\zeta+\Delta\zeta}{\mathrm{V}}}(T'v) + |(T'v)(\zeta) - (T'v)(\zeta+\Delta\zeta/n)|,$$

where $\overset{b}{\underset{a}{\mathrm{V}}} f$ denotes the total variation of the function f on the interval $[a, b]$. Because $(T'v)$ is continuous from the right the last term $\to 0$ as $n\to\infty$. Also, since $(T'v)$ is of bounded variation too, it follows that the function $h(x) = \overset{x}{\underset{0}{\mathrm{V}}}(T'v)$ is also continuous from the right [4]. Hence, the first two terms in the above inequality $\to 0$ as $n\to\infty$. This completes the proof that the left side of Eq. (13) has the limit $(T'v)(\zeta+\Delta\zeta) - (T'v)(\zeta)$ as $n\to\infty$.

To take $\lim_{n\to\infty}$ of the right side of Eq. (13) we use a limit theorem for Stieltjes integrals [4, p. 232]. Let

$$\eta_{\zeta,\Delta\zeta,n}(s) = \int_0^s k(s,u)\Phi_{\zeta,\Delta\zeta,n}(u)du$$

and

$$\eta_{\zeta,\Delta\zeta}(s) = \int_0^s k(s,u)\Phi_{\zeta,\Delta\zeta}(u)du.$$

We show $\eta_{\zeta,\Delta\zeta,n} \to \eta_{\zeta,\Delta\zeta}$ uniformly on $[0, \infty)$, which will allow the $\lim_{n\to\infty}$ to be taken inside the first integral in the right side of Eq. (13).

$$|\eta_{\zeta,\Delta\zeta,n}(s) - \eta_{\zeta,\Delta\zeta}(s)| \leqslant \int_0^s |k(s,u)||\Phi_{\zeta,\Delta\zeta,n}(u) - \Phi_{\zeta,\Delta\zeta}(u)|\,du \leqslant$$
$$\leqslant \int_0^s |k(u)||\Phi_{\zeta,\Delta\zeta,n}(u) - \Phi_{\zeta,\Delta\zeta}(u)|\,du +$$
$$+ \int_0^s |k(s,u) - k(u)||\Phi_{\zeta,\Delta\zeta,n}(u) - \Phi_{\zeta,\Delta\zeta}(u)|\,du.$$

Let $M_1 = \sup_{u\in[0,\zeta+2\Delta\zeta]} |k(u)|$. Since $k(s,u)\to k(u)$ uniformly on compact intervals, there exists $M_2 > 0$ such that $|k(s,u) - k(u)| < M_2$ on $[0,\infty)\times[0,\zeta+2\Delta\zeta]$. Hence,

$$|\eta_{\zeta,\Delta\zeta,n}(s) - \eta_{\zeta,\Delta\zeta}(s)| \leqslant M_1 2\Delta\zeta/n + M_2 2\Delta\zeta/n.$$

That is, the convergence is uniform as claimed. Taking the limit inside the integral gives

$$(T'v)(\zeta+\Delta\zeta) - (T'v)(\zeta) = \int_0^\infty \int_0^s k(s,u)\Phi_{\zeta,\Delta\zeta}(u)du\,dv(s). \tag{14}$$

Letting $\zeta = 0$ and $\Delta\zeta = t$ gives the first equality in Eq. (11), since the upper limit, s, in Eq. (14) can be replaced by t because $k(s, u) = 0$ for $u > s$ and $\Phi_{0,t}(u) = 0$ for $u > t$.

For the second equality we show that $T'v$ is absolutely continuous on compact intervals, which means it is the integral of its derivative on $[0,\infty)$, and the derivative (a.e.) is shown to be $\int_0^\infty k(s,u)dv(s)$. Actually, the calculations are done only for the right derivative, with the result for the left derivative following immediately.

To prove absolute continuity, note that $|k(s, u)|$ is uniformy bounded by some $M > 0$ on sets of the form $[0,\infty)\times[0,t_0]$ by (ii), where the M depends on t_0. Let $\varepsilon > 0$. For any $0 \leqslant a_1 < b_1 < a_2 < \ldots < a_n < b_n \leqslant t_0$ with $\sum_{i=1}^n (b_i - a_i) < \delta = \varepsilon/(M \overset{\infty}{\underset{0}{\mathrm{V}}} v)$, it follows that

$$\sum_{i=1}^n |(T'v)(b_i) - (T'v)(a_i)| = \sum_{i=1}^n \left| \int_0^\infty \int_{a_i}^{b_i} k(s,u)du\,dv(s)\right| \leqslant$$
$$\leqslant \overset{\infty}{\underset{0}{\mathrm{V}}} vM \sum_{i=1}^n (b_i - a_i) < \overset{\infty}{\underset{0}{\mathrm{V}}} vM\delta = \varepsilon$$

which establishes the absolute continuity.

To obtain the right derivative of $T'v$ we show that

$$\lim_{\Delta\zeta\to 0} (1/\Delta\zeta) \int_0^\infty \int_0^s k(s,u)\Phi_{\zeta,\Delta\zeta}(u)\,du\,dv(s) =$$
$$= \int_\zeta^\infty k(s,\zeta)\,dv(s) = \int_0^\infty k(s,\zeta)\,dv(s) \tag{15}$$

the last equality holding since $k(s, \zeta) = 0$ for $s < \zeta$. Let

$$\delta(\Delta\zeta) = \left| \int_0^\infty (1/\Delta\zeta) \int_0^s k(s, u) \Phi_{\zeta, \Delta\zeta}(u)\, du\, dv(s) - \int_\zeta^\infty k(s, \zeta) dv(s) \right|.$$

Since $\Phi_{\zeta, \Delta\zeta}(u) = 0$ on $[0, s]$ for $s < \zeta$,

$$\delta(\Delta\zeta) = \left| \int_\zeta^\infty \left\{ (1/\Delta\zeta) \int_0^s k(s, u) \Phi_{\zeta, \Delta\zeta}(u) du - k(s, \zeta) \right\} dv(s) \right| \leqslant \tag{16}$$

$$\leqslant \left| \int_\zeta^{\zeta+\Delta\zeta} \left\{ (1/\Delta\zeta) \int_0^s k(s, u) \Phi_{\zeta, \Delta\zeta}(u) du - k(s, \zeta) \right\} dv(s) \right| +$$

$$+ \left| \int_{\zeta+\Delta\zeta}^\infty \left\{ (1/\Delta\zeta) \int_\zeta^{\zeta+\Delta\zeta} k(s, u) du - k(s, \zeta) \right\} dv(s) \right|.$$

We can assume each $\Delta\zeta < 1$, and let

$M_4 = \sup_{[\zeta, \zeta+1] \times [0, \zeta+1]} |k(s, u)|$. Then, the first term in inequality (16) is bounded from above by $2M_4 \overset{\zeta+\Delta\zeta}{\underset{\zeta}{\mathbf{V}}} v$ which $\to 0$ as $\Delta\zeta \to 0$.

To show the second term $\to 0$ as $\Delta\zeta \to 0$, note that it is bounded by

$$\sup_{s \in [\zeta+\Delta\zeta, \infty)} \left| (1/\Delta\zeta) \int_{\zeta+\Delta\zeta}^{\zeta} k(s, u) du - k(s, \zeta) \right| \cdot \overset{\infty}{\underset{2}{\mathbf{V}}} v$$

which can be made arbitrarily small for small enough $\Delta\zeta$ becaues $\overset{\infty}{\underset{2}{\mathbf{V}}} v \leqslant$ $\leqslant \overset{\infty}{\underset{0}{\mathbf{V}}} v < \infty$, and the first factor can be made small as we now show.

$$\text{Let } \gamma(s, \Delta\zeta) = \left| (1/\Delta\zeta) \int_\zeta^{\zeta+\Delta\zeta} k(s, u) du - k(s, \zeta) \right| \leqslant$$

$$\leqslant (1/\Delta\zeta) \int_\zeta^{\zeta+\Delta\zeta} |k(s, u) - k(s, \zeta)| du.$$

Let $\varepsilon > 0$. Since $k(s, u) \to k(u)$ uniformly on $[\zeta, \zeta + 1]$, there exists $S(\varepsilon)$ such that $|k(s, u) - k(u)| < \varepsilon/3$ for all $s \geqslant S$ and $u \in [\zeta, \zeta + 1]$. Furthurmore, $k(s, u)$ is uniformly continuous on $[0, S) \times [\zeta, \zeta + 1]$, and $k(u)$ is continuous on $[\zeta, \zeta + 1]$. Hence, there exists $\theta(\varepsilon)$ such that $\Delta\zeta <$ $< \theta(\varepsilon)$ implies $|k(s, u) - k(s, \zeta)| < \varepsilon/3$ on $[0, S] \times [\zeta, \zeta + \Delta\zeta]$ and

$|k(u) - k(\zeta)| < \varepsilon$ for $u \in [\zeta, \zeta + \Delta\zeta]$. Hence, if $\Delta\zeta < \theta(\varepsilon)$, either $s \leqslant S$ and $|k(s, u) - k(s, \zeta)| < \varepsilon/3 < \varepsilon$, or $s > S$, in which case

$$|k(s, u) - k(s, \zeta)| \leqslant |k(s, u) - k(u)| + |k(u) - k(\zeta)| >$$

$$+ |k(\zeta) - k(s, \zeta)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

which gives the desired result that $\gamma(s, \Delta\zeta) \to 0$ uniformly in s as $\Delta\zeta \to 0$. This completes the proof that the right derivative of $T'v$ at ζ is

$$\int_\zeta^\infty k(s, \zeta) dv(s) = \int_0^\infty k(s, \zeta) dv(s).$$

It follows in similar fashion that the left derivative is the same. Hence

$$(T'v)(t) = \int_0^t \int_\zeta^\infty k(s, \zeta)\, dv(s)\, d\zeta = \int_0^t \int_0^\infty k(s, \zeta)\, dv(s)\, d\varepsilon.$$

Q.E.D.

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