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ON THE CONSTRUCTION OF TANGENTIAL CONTACT CIRCULAR CHARTS

by

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1. Introduction

In the preceding article [1] the authors researched a method of constructing the general tangential contact charts of the functional relation $F_{123}(t_1, t_2, t_3) = 0$, where F_{123} is a real continuous function of three real variables t_1, t_2 and t_3.

In this paper we treat, as a natural extension of the above tangential contact charts, the tangential contact circular charts of the general functional relation of four variables F_{1234} $(t_1, t_2, t_3, t_4) = 0$, in which the curve of solution is a circle, that is tangent to the respective curve t_i $(i = 1, 2, 3, 4)$.

Furthermore, a double tangential contact circular chart of six variables of which functional relation is represented by $F_{12\ldots6}$ $(t_1, t_2, \ldots, t_6) = 0$ is also studied.

2. General Theory of Tangential Contact Circular Charts of Four Variables

Given the general functional relation of four variables

$$F_{1234}(t_1, t_2, t_3, t_4) = 0, \tag{1}$$

where F_{1234} is a real continuous function of four real variables t_1, t_2, t_3, and t_4.

Now we try to construct the general tangential contact circular charts, where a circle of solution C tangent to curves t_i $(i = 1, 2, 3)$ of the adequately given families of curves (t_i) $(i = 1, 2, 3)$, respectively, is also tangent to a curve t_4 of the fourth family of curves (t_4), and where thus obtained value t_4 is a required solution of the given functional relation (1), t_1, t_2 and t_3 being known.

Consider adequately the following three pairs of parametric equations, involving the parameters t_i and α_i $(i = 1, 2, 3)$:

$$(2) \qquad (t_i):\ x_i = x_i(t_i, \alpha_i),\ y_i = y_i(t_i, \alpha_i) \quad (i = 1, 2, 3),$$

where α_i represent the curve lengths of t_i-curves, respectively; and we assume that $x_i(t_i, \alpha_i)$ and $y_i(t_i, \alpha_i)$ are continuous functions of t_i $(i = 1, 2, 3)$ respectively; and also that they are of class C^2 with respect to α_i $(i = 1, 2, 3)$.

Now let the equation of circle of solution C be

$$(3) \qquad K(x, y; t_1, t_2, t_3) \equiv x^2 + y^2 + 2g(t_1, t_2, t_3)x + 2f(t_1, t_2, t_3)y + c(t_1, t_2, t_3) = 0,$$

where g, f and c are undetermined functions of t_1, t_2 and t_3.

Then as, in our tangential contact circular charts, respective curve t_i of the family of curves (t_i) expressed by (2) contacts with the circle of

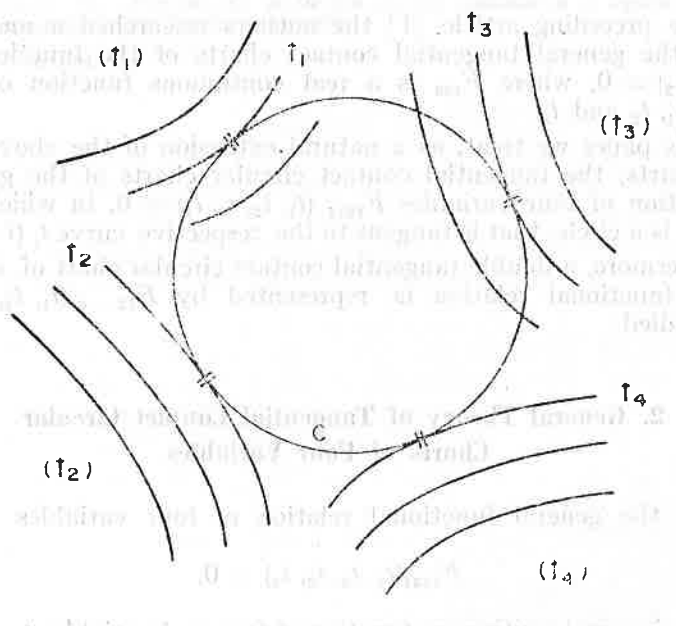

Fig. 1

solution expressed by (3), by the contact of the 1-st order, we have the following three conditions according to the theory of contact of two plane curves in the theory of classical differential geometry:

$$(4) \qquad \Psi(t_1, t_2, t_3; \alpha_i) = K(x_i(t_i, \alpha_i), y_i(t_i, \alpha_i); t_1, t_2, t_3) = x_i^2(t_i, \alpha_i) + y_i^2(t_i, \alpha_i) + 2g(t_1, t_2, t_3)x_i(t_i, \alpha_i) + 2f(t_1, t_2, t_3)y_i(t_i, \alpha_i) + c(t_1, t_2, t_3) = 0 \quad (i = 1, 2, 3),$$

$$(5) \qquad \frac{1}{2}\frac{\partial \Psi(t_1, t_2, t_3; \alpha_i)}{\partial \alpha_i} = x_i(t_i, \alpha_i)\frac{\partial x_i}{\partial \alpha_i} + y_i(t_i, \alpha_i)\frac{\partial y_i}{\partial \alpha_i} + g(t_1, t_2, t_3)\frac{\partial x_i}{\partial \alpha_i} + f(t_1, t_2, t_3)\frac{\partial y_i}{\partial \alpha_i} = 0 \quad (i = 1, 2, 3),$$

$$(6) \qquad \frac{1}{2}\frac{\partial^2 \Psi(t_1, t_2, t_3; \alpha_i)}{\partial \alpha_i^2} = \left(\frac{\partial x_i}{\partial \alpha_i}\right)^2 + \left(\frac{\partial y_i}{\partial \alpha_i}\right)^2 + x_i\frac{\partial^2 x_i}{\partial \alpha_i^2} + y_i\frac{\partial^2 y_i}{\partial \alpha_i^2} + g(t_1, t_2, t_3)\frac{\partial^2 x_i}{\partial \alpha_i^2} + f(t_1, t_2, t_3)\frac{\partial^2 y_i}{\partial \alpha_i^2} = 0 \quad (i = 1, 2, 3).$$

Solving the above system (4) and (5) with the unknowns g, f, c; α_i $(i = 1, 2, 3)$, we shall have the forms of the following solutions:

$$(7) \qquad g = g(t_1, t_2, t_3), \quad f = f(t_1, t_2, t_3), \quad c = c(t_1, t_2, t_3);$$

$$(8) \qquad \alpha_i = \alpha_i(t_1, t_2, t_3) \quad (i = 1, 2, 3).$$

Then the equation (3) with the coefficients (7) is the required equation of the circle of solution C tangent to the curve t_i $(i = 1, 2, 3)$ of the family of curves (t_i) $(i = 1, 2, 3)$, respectively.

Now assuming that

$$(9) \qquad \frac{\partial F_{1234}(t_1, t_2, t_3, t_4)}{\partial t_3} \neq 0$$

in (1), and solving (1) with respect to t_3, we get

$$(10) \qquad t_3 = \Phi(t_1, t_2, t_4);$$

and substituting this expression into (3) with (7), we have again the equation of the circle of solution, of which new coefficients g^*, f^* and c^* consist of independent variables t_1, t_2 and t_4, respectively, i.e.,

$$(11) \qquad K^*(x, y; t_1, t_2, t_4) = x^2 + y^2 + 2g^*(t_1, t_2, t_4)x + 2f^*(t_1, t_2, t_4)y + c^*(t_1, t_2, t_4) = 0,$$

Putting $t_1 = t_1^{(0)}(= \text{const.})$, $t_4 = t_4^{(0)}(= \text{const.})$, we obtain

$$(12) \qquad K^*(x, y\,;\, t_1^{(0)}, t_2, t_4^{(0)}) \equiv x^2 + y^2 + 2g^*(t_1^{(0)}, t_2, t_4^{(0)})\, x + 2f^*(t_1^{(0)}, t_2, t_4^{(0)})\, y + c^*(t_1^{(0)}, t_2, t_4^{(0)}) = 0,$$

which represents a family of circles of solutions, with a parameter t_2, tangent to the fixed curves $t_1^{(0)}$ and $t_4^{(0)}$, respectively; therefore, $t_1^{(0)}$-curve and $t_4^{(0)}$-curve must be the envelopes of this family of circles, respectively.

In fact, assuming that the above family of circles (12) has two envelopes $t_1^{(0)}$ and $t_4^{(0)}$, and that the function K^* is of class C^2, we differentiate (12) partially with respect to t_2, that is,

$$(13) \qquad 2\frac{\partial g^*}{\partial t_2}\, x + 2\frac{\partial f^*}{\partial t_2}\, y + \frac{\partial C^*}{\partial t_2} = 0,$$

then both (12) and (13) express the parametric equations, with a parameter t_2, of the envelopes of circles of solutions, representing both $t_1^{(0)}$-curve and $t_4^{(0)}$-curve. (The sufficient condition for the existence of envelopes is well-known [2]).

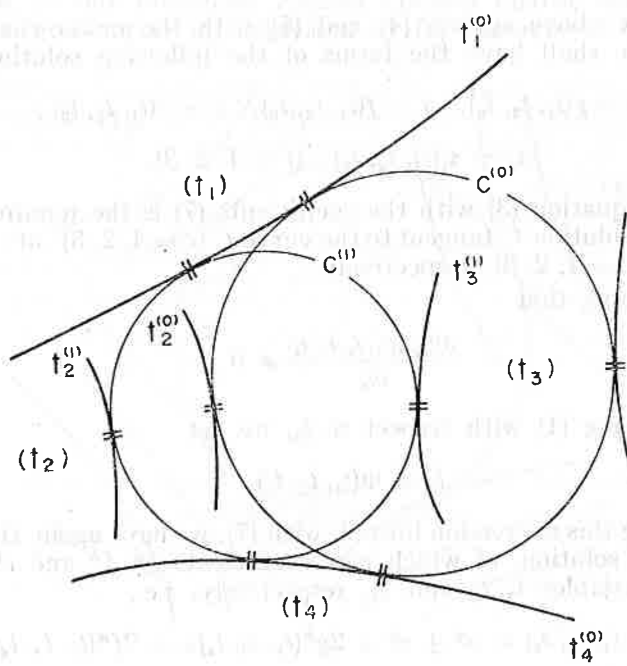

Fig. 2

Hence eliminating t_2 from (12) and (13), we have an equation of the form

$$(14) \qquad E(x, y\,;\, t_1^{(0)}, t_4^{(0)}) = 0,$$

which may represent $t_1^{(0)}$-curve and $t_4^{(0)}$-curve as the envelopes of circles of solutions.

Assuming that E is a function of C^2 class with respect to $t_1^{(0)}$ and $t_4^{(0)}$, and that

$$(15) \qquad \frac{\partial^2 \ln E(x, y\,;\, t_1^{(0)}, t_4^{(0)})}{\partial t_1^{(0)}\, \partial t_4^{(0)}} = 0,$$

we can factorize (14) as a product of two functions $\psi_1(x, y\,;\, t_1^{(0)})$, $\psi_2(x, y\,;\, t_4^{(0)})$, that is,

$$(16) \qquad E(x, y\,;\, t_1^{(0)}, t_4^{(0)}) = \psi_1(x, y\,;\, t_1^{(0)}) \cdot \psi_2(x, y\,;\, t_4^{(0)}).$$

Therefore, we have

$$(17) \qquad (t_1^{(0)}): \quad \psi_1(x, y\,;\, t_1^{(0)}) = 0,$$

$$(18) \qquad (t_4^{(0)}): \quad \psi_2(x, y\,;\, t_4^{(0)}) = 0.$$

The curve, expressed by (17), must be the same to a curve $t_1^{(0)}$, expressed by (2) for $i = 1$, and hence (17) may be rewritten as

$$(19) \qquad \psi_1(x, y\,;\, t_1^{(0)}) = \{x - x_1(t_1^{(0)}, \alpha_1)\}\{y - y_1(t_1^{(0)}, \alpha_1)\}.$$

Next, writing as t_4 for $t_4^{(0)}$ in (18), we get

$$(20) \qquad (t_4): \quad \psi_2(x, y\,;\, t_4) = 0,$$

and this is nothing but the required equation of the fourth family of curves, and may be expressed, by the parametric representation, as

$$(21) \qquad (t_4): \quad x_4 = x_4(t_4, \alpha_4),\ y_4 = y_4(t_4, \alpha_4),$$

where a parameter α_4 represents the curve length of t_4-curves.

Finally, we have obtained the following four pairs of parametric equations, representing our tangential contact circular charts of four variables:

$$(22) \qquad (t_i): \quad x_i = x_i(t_i, \alpha_i),\ y_i = y_i(t_i, \alpha_i) \quad (i = 1, 2, 3, 4),$$

where those expressions for $i = 1, 2, 3$ are the same to (2), and one for $i = 4$ to (21).

Remark 1. Four special cases of tangential contact circular charts, when n families of curves are degenerated into n curvilinear supports, where $n = 1, 2, 3, 4$, respectively, are easily obtained from our theory of this section (Fig. 3 -- Fig. 9); and the case for $n = 4$ is nothing but the Ogura's concircular chart (Fig. 9) [3].

E x a m p l e 1. Construct a tangential contact circular chart of the relation

$$t_1 - t_2 = t_3 - t_4. \tag{23}$$

Let the parametric equations of (t_1)-, (t_2)- and (t_3)-curves be

$$(t_i): \; x_i = \alpha_i/\sqrt{2}, \;\; y_i = \alpha_i/\sqrt{2} + t_i \quad (i = 1, 2); \tag{24}$$

$$(t_3): \; x_3 = \alpha_3/\sqrt{2}, \;\; y_3 = -\alpha_4/\sqrt{2} + t_3, \tag{25}$$

where α_i $(i = 1, 2, 3)$ are the curve lengths of (t_i)-lines $(i = 1, 2, 3)$, respectively.

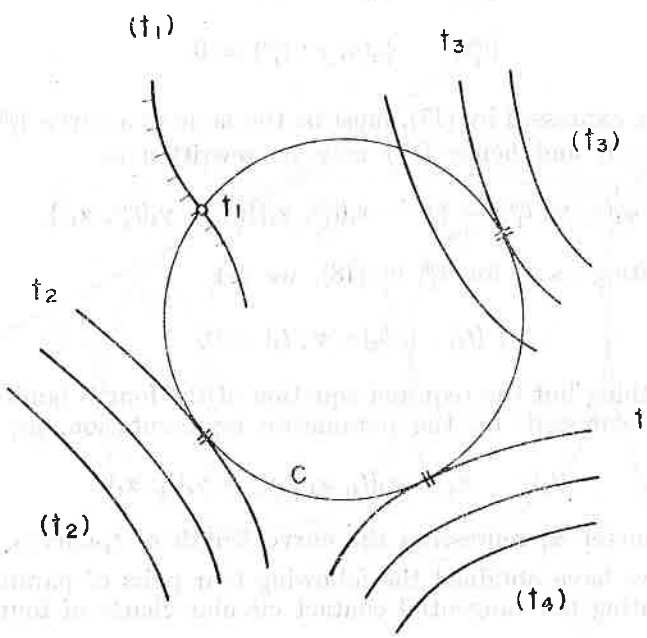

Fig. 3

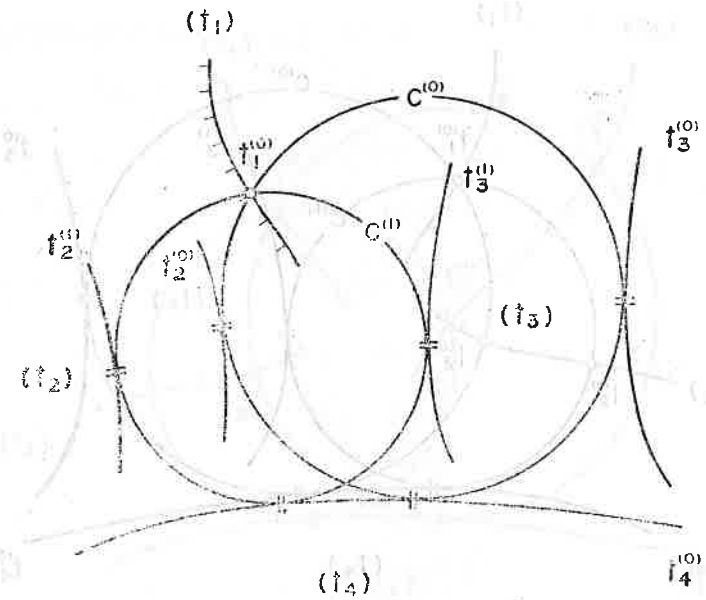

Fig. 4

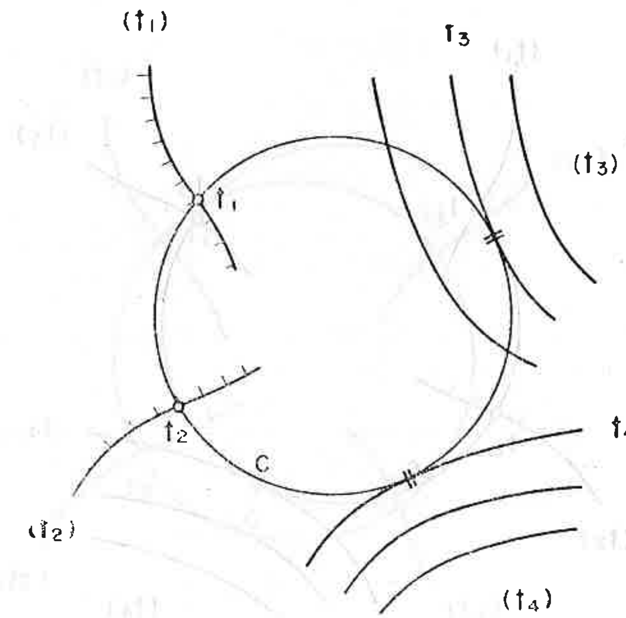

Fig. 5

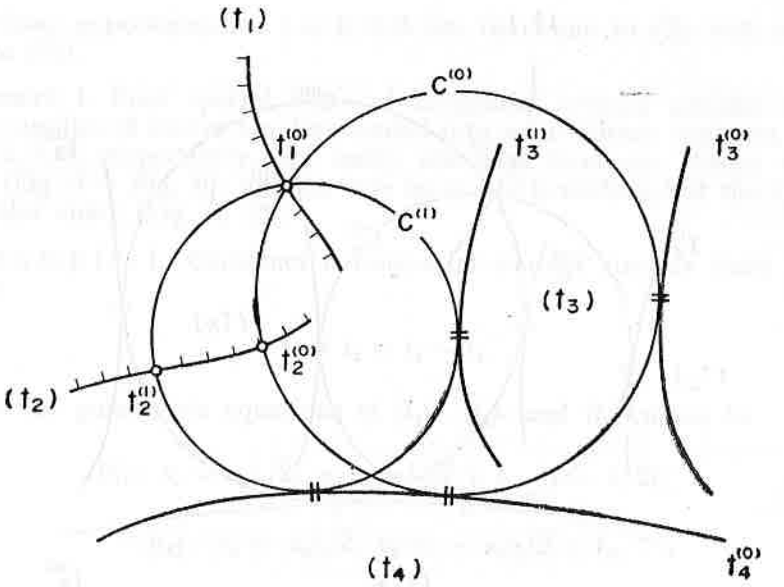

Fig. 6

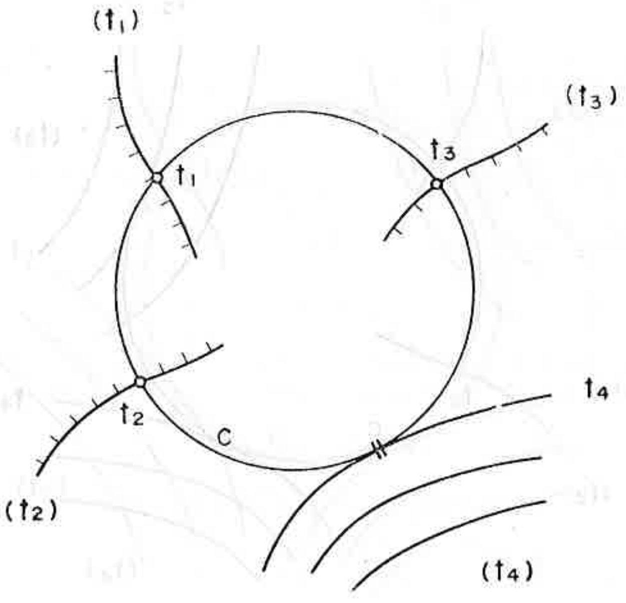

Fig. 7

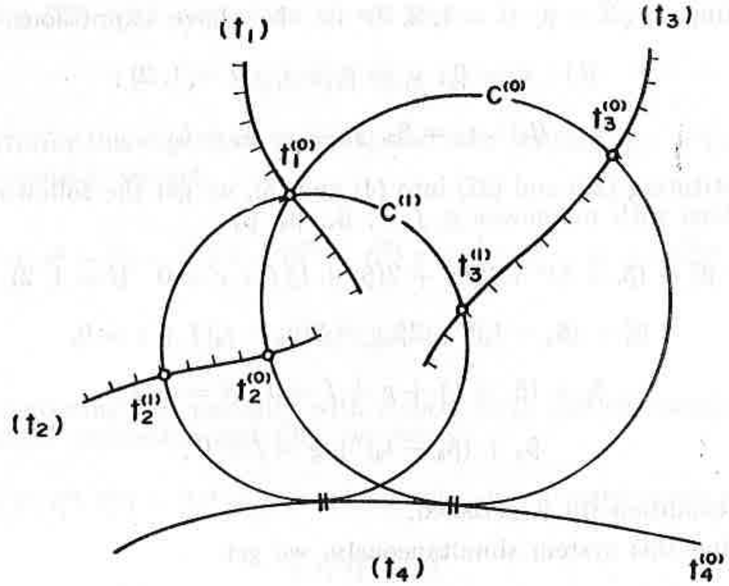

Fig. 8

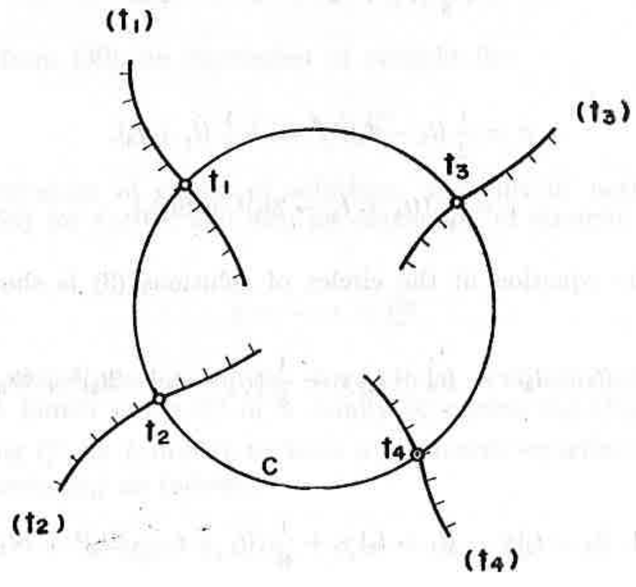

Fig. 9

Putting $\alpha_i/\sqrt{2} = \beta_i$ $(i = 1, 2, 3)$ in the above expressions, we have

(26) $$(t_i): \; x_i = \beta_i, \; y_i = \beta_i + t_i \quad (i = 1, 2);$$

(27) $$(t_3): \; x_3 = \beta_3, \; y_3 = -\beta_3 + t_3.$$

Substituting (26) and (27) into (4) and (5), we get the following system of equations with unknowns $g, f, c; \beta_1, \beta_2, \beta_3$:

(28) $$\beta_i^2 + (\beta_i + t_i)^2 + 2\beta_i g + 2(\beta_i + t_i) f + c = 0 \quad (i = 1, 2),$$

(29) $$\beta_3^2 + (\beta_3 - t_3)^2 + 2\beta_3 g - 2(\beta_3 - t_3) f + c = 0,$$

(30) $$\beta_i + (\beta_i + t_i) + g + f = 0 \quad (i = 1, 2),$$

(31) $$\beta_3 + (\beta_3 - t_3) + g - f = 0;$$

and the condition (6) is satisfied.

Solving this system simultaneously, we get

(32) $$g = \frac{1}{2}(t_1 - t_2), \; f = -\frac{1}{2}(t_2 + t_3), \quad c = \frac{1}{8}\{(t_1 + t_2 - 2t_3)^2 + 8t_2 t_3\},$$

or,

(33) $$g = \frac{1}{2}(t_2 - t_3), \; f = -\frac{1}{2}(t_1 + t_3), \quad c = \frac{1}{8}\{(t_1 + t_2 - 2t_3)^2 + 8t_1 t_3\}.$$

Hence the equation of the circles of solutions (3) is shown by

(34) $$x^2 + y^2 + (t_1 - t_3)x - (t_2 + t_3)y + \frac{1}{8}\{(t_1 + t_2 - 2t_3)^2 + 8t_2 t_3\} = 0,$$

or

(35) $$x^2 + y^2 + (t_2 - t_3)x - (t_1 + t_3)y + \frac{1}{8}\{(t_1 + t_2 - 2t_3)^2 + 8t_1 t_3\} = 0.$$

Firstly, we treat the first expression (34).

Solving (23) with respect to t_3, we have

(36) $$t_3 = t_1 - t_2 + t_4,$$

and substituting this expression into (34), and then putting $t_1 = t_1^{(0)}(= \text{const.})$, $t_4 = t_4^{(0)}(= \text{const.})$, we get

(37) $$x^2 + y^2 + (t_2 - t_4^{(0)})x - (t_1^{(0)} + t_4^{(0)})y + \frac{1}{8}\{(t_1^{(0)} - t_2 + 2t_4^{(0)})^2 + 4t_1^{(0)} t_2\} = 0.$$

Differentiating (37) partially with respect to t_2, and eliminating t_2 from thus obtained expression and (37), we have

(38) $$E(x, y; t_1^{(0)}, t_4^{(0)}) = 3x^2 + y^2 - (4x + t_1^{(0)} - t_4^{(0)})x - (t_1^{(0)} + t_4^{(0)})y + t_1^{(0)} t_4^{(0)} = 0.$$

It is easily seen that the above expression is rewritten as

(39) $$E = (y - x - t_1^{(0)})(y + x - t_4^{(0)}) = 0.$$

Hence, from (39), an expression of straight line

(40) $$y = x + t_1^{(0)}$$

shows one envelope of circles of solutions, and this is nothing but the expression (24) for $i = 1$; and also an expression of straight line obtained from (39)

(41) $$y = -x + t_4^{(0)}$$

represents another envelope of the same circles of solutions, which is just our required fourth curve $t_4^{(0)}$ of a family of curves (t_4) (Fig. 10).

Replacing $t_4^{(0)}$ by t_4 in (41), we have a parametric equation of the fourth family of curves (t_4) as follows:

(42) $$(t_4): \; x_4 = \beta_4, \; y_4 = -\beta_4 + t_4,$$

where $\beta_4 = \alpha_4/\sqrt{2}$, and α_4 represents the curve length of t_4-line.

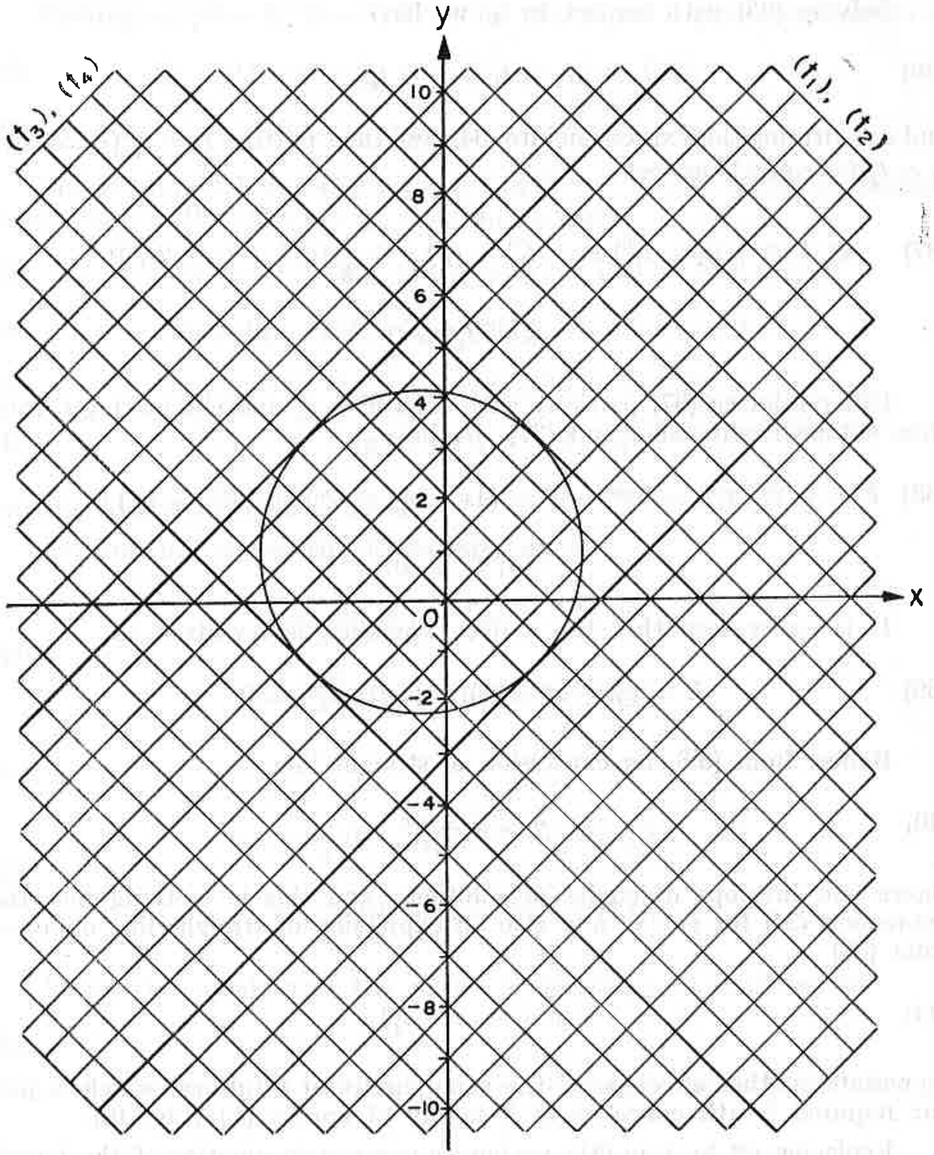

Fig. 10

Hence our four families of curves consisting of tangential contact circular charts are shown by

(26) $(t_i): \ x_i = \beta_i, \quad y_i = \beta_i + t_i \qquad (i = 1, 2);$

(43) $(t_j): \ x_j = \beta_j, \quad y_j = -\beta_j + t_j \qquad (j = 3, 4),$

where again $\beta_i = \alpha_i/\sqrt{2}$, and α_i represents the curve length of t_i-line, respectively $(i = 1, 2, 3, 4)$.

We must remark here that the second expression (35) is not satisfied by the condition (15), and therefore we cannot obtain the second tangential contact circular chart.

3. Tangential Contact Circular Charts of Six Variables

Given the general functional relation

(44) $$F_{12\dots56}(t_1, t_2, \dots, t_5, t_6) = 0,$$

where $F_{12\dots56}$ is a real continuous function of six real variables $t_1, t_2, \dots, t_5, t_6$.

Let us assume that the above expression is separable into the following two equations

$(45)_1$ $$f_{1230}(t_1, t_2, t_3, t_0) = 0,$$

$(45)_2$ $$g_{4560}(t_4, t_5, t_6, t_0) = 0,$$

where t_0 is a parameter.

Firstly, for the first expression $(45)_1$, we assume the following three pairs of parametric equations adequately, as similar as the case of (2):

(46) $$(t_i): \ x_i = x_i(t_i, \alpha_i), \quad y_i = y_i(t_i, \alpha_i) \quad (i = 1, 2, 3).$$

Then, from the theory of the preceding section, we have the following fourth family of curves (t_0), which need not have the values of indices:

(47) $$(t_0): \ x_0 = x_0(t_0, \alpha_0), \quad y_0 = y_0(t_0, \alpha_0).$$

Thus the first partial chart has been drawn.

And then we again assume the following three pairs of equations adequately for the second expression $(45)_2$:

(48) $$(t_i): \ x_i = x_i(t_i, \alpha_i), \quad y_i = y_i(t_i, \alpha_i) \quad (i = 0, 4, 5),$$

where the above expression for $i = 0$ is, of course, identical with (47).

Hence, again from the theory of the preceding section, we finally obtain the equation of the sixth family of curves (t_6) as follows:

(49) $$(t_6): \ x_6 = x_6(t_6, \alpha_6), \quad y_6 = y_6(t_6, \alpha_6).$$

Expressions (48) together with (49) is the ones of the second partial chart, and both the first and second partial charts consist of our required tangential contact circular charts of six variables representing the given functional relation (44).

Method of solution of this double tangential contact circular chart is easily seen in Fig. 11.

Example 2. Construct a double tangential contact circular chart of the relation

$$t_1 - t_2 - t_3 + t_4 = t_5 - t_6. \tag{50}$$

Rewriting this expression, we have

$$(51)_1 \qquad t_1 - t_2 = t_3 - t_0,$$

$$(51)_2 \qquad t_4 - t_5 = t_0 - t_6,$$

where t_0 is a parameter.

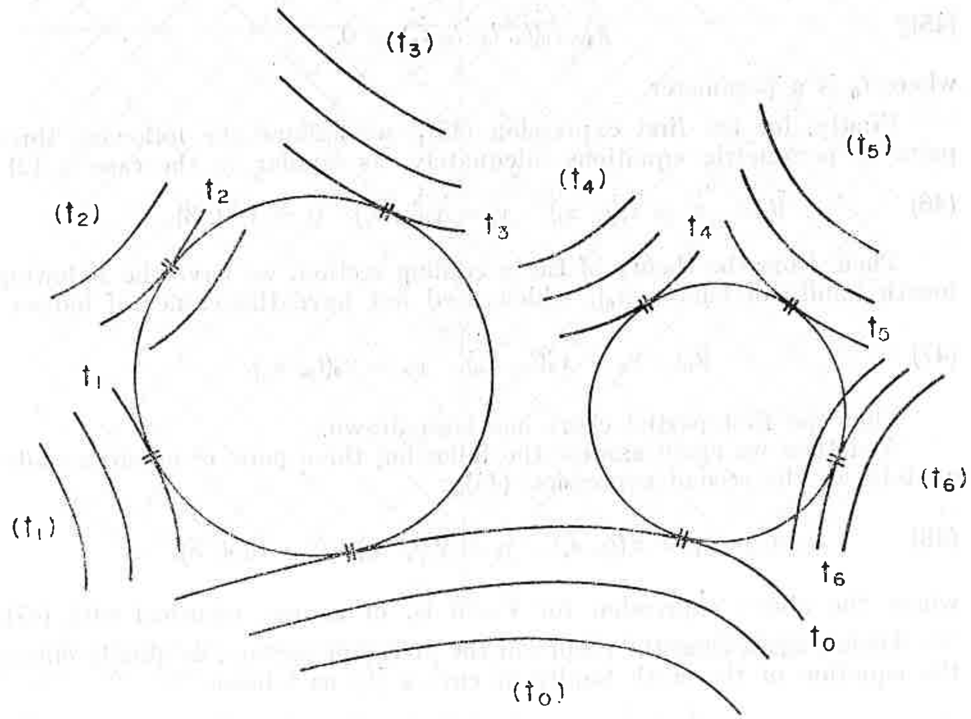

Fig 11

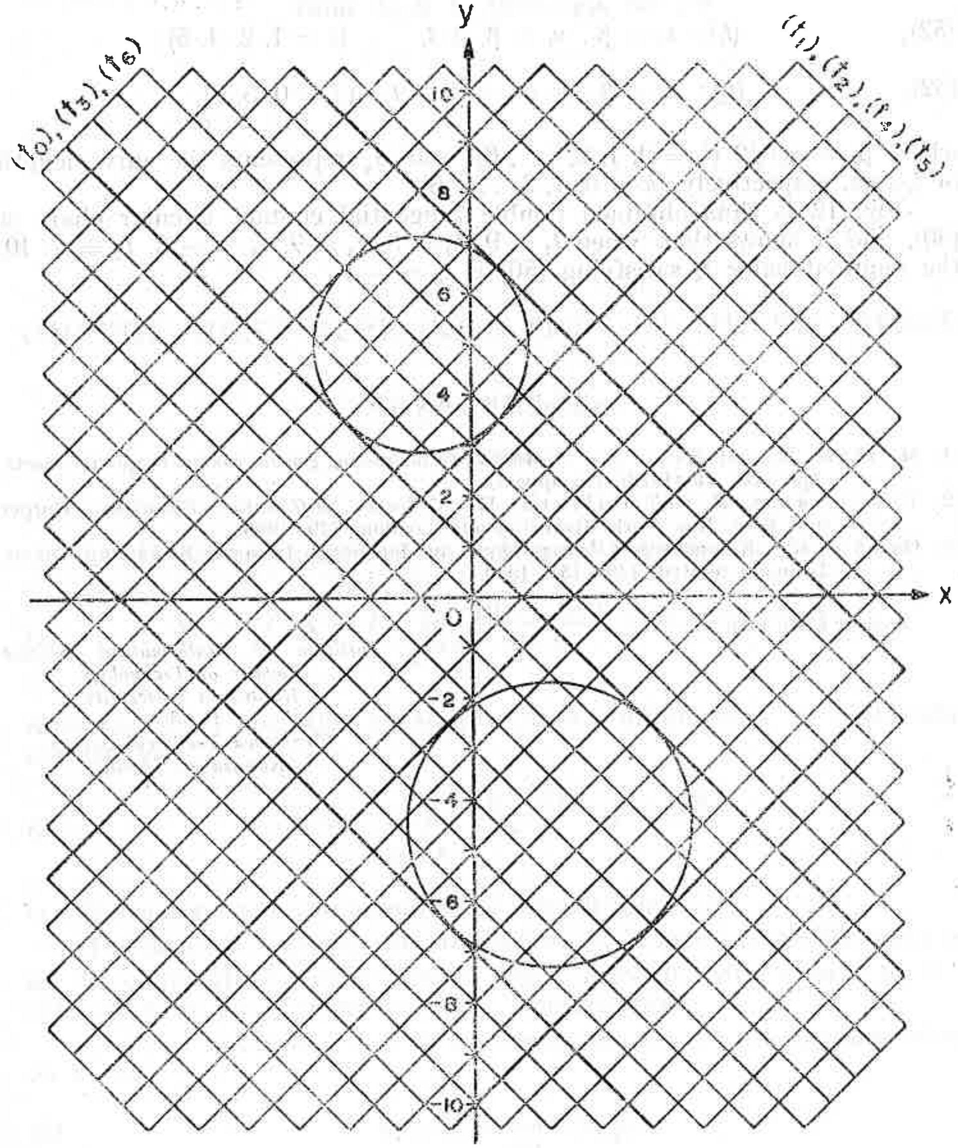

Fig. 12

As these expressions are similar to the expression (23) discussed in Example 1, we easily have the following seven pairs of parametric equations:

$$(52)_1 \qquad (t_i): x_i = \beta_i, \ y_i = \beta_i + t_i \qquad (i = 1, 2, 4, 5);$$

$$(52)_2 \qquad (t_j): x_j = \beta_j, \ y_j = -\beta_j + t_j \quad (j = 0, 3, 6),$$

where $\beta_k = \alpha_k/\sqrt{2}$ $(k = 0, 1, 2, \ldots, 6)$, and α_k represents the curve length of t_k-line, respectively $(k = 0, 1, 2, \ldots, 6)$.

Fig. 12 is thus obtained double tangential contact circular chart of (50), and it shows that when $t_1 = 9$, $t_2 = 3$, $t_3 = 7$, $t_4 = -2$, $t_5 = -10$, the required value t_6 satisfying (50) is $t_6 = -7$.

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