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ON THE BEST APPROXIMATION IN METRIC SPACES

by

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Let (X, d) be a metric space and x_0 a fixed point in X. The set

(1) $$X_0^{\#} = \left\{f : X \to R,\ \sup_{\substack{x \neq y \\ x, y \in X}} \frac{|f(x) - f(y)|}{d(x, y)} < \infty,\ f(x_0) = 0\right\},$$

with the usual operation of addition and multiplication by real scalars, normed by

(2) $$\|f\|_X = \sup_{\substack{x \neq y \\ x, y \in X}} \frac{|f(x) - f(y)|}{d(x, y)},\ f \in X_0^{\#},$$

is a Banach space (even a conjugate Banach space [2]).

The space $X_0^{\#}$ plays, with respect to X, in many ways, the same role as the conjugate E^* of a normed linear space E, with respect to E. In this paper we give further details in this direction.

For $\emptyset \neq Y \subseteq X$ and $x \in X$ we denote by $d(x, Y)$ the distance from x to Y, i.e.

(3) $$d(x, Y) = \inf_{y \in Y} d(x, y).$$

Proposition 1. *Let $Y \subset X$, $x_0 \in Y$ and $x_1 \in X - Y$ such that*

(4) $$d(x_1, Y) = q > 0.$$

Then there is $f \in X_0^{\#}$ such that

(5) $$f|_Y = 0,\ f(x_1) = 1,\ \|f\|_X = \frac{1}{q}.$$

Proof. We will show that a function with the required properties is given by

(6) $$f(x) = \frac{1}{q}\, d(x, Y).$$

Indeed, $f(x_0) = 0$, because $x_0 \in Y$. For $x, z \in X$ we have

$$|d(x, Y) - d(z, Y)| \leqslant d(x, z),$$

and by the definition of f, it follows that

(7) $$\|f\|_X \leqq \frac{1}{q} < \infty.$$

This means that $f \in X_0^{\#}$.

Evidently, $f|_Y = 0$ and $f(x_1) = 1$. Since $d(x_1, Y) = q > 0$, then there is a sequence $(y_n)_{n\in N} \subset Y$ such that $d(x_1, y_n) \to d(x_1, Y)$ when $n \to \infty$. It follows that we can find an increasing sequence of natural numbers $\{n_k\}$ such that $d(x_1, y_{n_k}) \leqq d(x_1, Y) + \frac{1}{k}$. Then

$$\frac{|d(y_{n_k}, Y) - d(x_1, Y)|}{d(y_{n_k}, x_1)} = \frac{d(x_1, Y)}{d(y_{n_k}, x_1)} \geqq \frac{d(x_1, y_{n_k}) - \frac{1}{k}}{d(x_1, y_{n_k})} \to 1.$$

From the above inequality we obtain

(8) $$\|f\|_X \geqq \frac{1}{q}.$$

By (7) and (8) it follows that $\|f\|_X = \frac{1}{q}$, which completes the proof of the proposition.

For $f \in X_0^{\#}$ we denote

(9) $$f^{(-1)}(0) = \{x \in X,\ f(x) = 0\}.$$

Proposition 2. *Let $f \in X_0^{\#} - \{\theta\}$. Then*

(10) $$d(x, f^{(-1)}(0)) \geqq \frac{|f(x)|}{\|f\|_X},$$

for every $x \in X$.

Proof. For every $y \in f^{(-1)}(0)$ and $x \in X$, $|f(x)| = |f(x) - f(y)| \leqq \|f\|_X d(x, y)$.
Therefore

$$d(x, f^{(-1)}(0)) \geqq \frac{|f(x)|}{\|f\|_X},$$

and the proposition is proved.

Definition 1. *A subset Y of the metric space X is called proximinal if for every $x \in X$ there is an element $y_0 \in Y$ such that*

(11) $$d(x, y_0) = d(x, Y).$$

If, for all $x \in X$ the element $y_0 \in Y$ verifying (11) is unique, then the set Y is called chebyshevian. An element $y_0 \in Y$, verifying (11) is called element of best approximation of x by elements of Y.

Proposition 3. *Let $f \in X_0^{\#} - \{\theta\}$. If for every $x \in X - f^{(-1)}(0)$ there is an element $y_x \in f^{(-1)}(0)$ such that*

(12) $$|f(x) - f(y_x)| = \|f\|_X d(x, y_x),$$

then $f^{(-1)}(0)$ is proximinal.

Proof. Let $x \in X - f^{(-1)}(0)$. Since $f^{(-1)}(0)$ is closed it follows that $0 < d(x, f^{(-1)}(0)) \leqq d(x, y)$, for all $y \in f^{(-1)}(0)$. Now, let y_x be an element of $f^{(-1)}(0)$ for which (12) holds. Then for every $y \in f^{(-1)}(0)$,

$$\|f\|_X = \frac{|f(x) - f(y_x)|}{d(x, y_x)} \geqq \frac{|f(x) - f(y)|}{d(x, y)},$$

that is

$$\frac{|f(x)|}{d(x, y_x)} \geqq \frac{|f(x)|}{d(x, y)}.$$

Therefore, $d(x, y_x) \leqq d(x, y)$ and, taking the infimum relatively to y we get $d(x, y_x) = d(x, f^{(-1)}(0))$.

In the following proposition we give a characterization of the elements of best approximation.

Proposition 4. *Let Y be a subset of X such that $x_0 \in Y$, and let $x \in X - Y$. Then $y_0 \in Y$ is an element of best approximation for x by elements of Y, if and only if there is an $f \in X_0^{\#}$ such that*

1) $\|f\|_X = 1$
2) $f|_Y = 0$
3) $|f(x) - f(y_0)| = d(x, y_0)$.

Proof. If $x \in X - Y$ and $y_0 \in Y$ is an element of best approximation for x by elements of Y, then from the proof of proposition 1 it follows that the function

(13) $$f(x) = d(x, Y)$$

has all the required properties.

Conversely, if $f \in X_0^{\#}$ is such that the conditions 1), 2), 3) hold, then for every $y \in Y$,

$$d(x, y_0) = |f(x) - f(y_0)| = |f(x) - f(y)| \leq \|f\|_X d(x, y) = d(x, y),$$

which completes the proof of the proposition.

Proposition 5. *Let Y be a proximinal subset of X, $x_0 \in Y$, and $x \in X - Y$. Let $y_0 \in Y$ be an element of best approximation of x by elements of Y. The following conditions are equivalent:*

i) *$y_0 \in Y$ is the only element of best approximation of x.*

ii) *There is no $y \in Y$, $y \neq y_0$ and $f \in X_0^{\#}$ such that*

a) $\|f\|_X = 1$

b) $f(y_0) = f(y)$

c) $|f(x) - f(y)| = d(x, y)$.

Proof. Let us suppose that i) holds and that there is $y \in Y$, $y \neq y_0$ and $f \in X_0^{\#}$ such that a), b), c) hold. Then

$$d(x, y) = |f(x) - f(y)| \leq |f(x) - f(y_0)| + |f(y_0) - f(y)| = |f(x) - f(y_0)| = d(x, y_0).$$

Therefore, y is also an element of best approximation of x, which condradicts i).

Now, let suppose that the condition i) is not accomplished. Then there is $y \in Y$, $y \neq y_0$ such that

$$d(x, y) = d(x, y_0) = d(x, Y).$$

By proposition 4, there is $f \in X_0^{\#}$ such that $\|f\|_X = 1$, $f|_Y = 0$ and $|f(x) - f(y)| = d(x, y)$. From $f|_Y = 0$ it follows that $f(y_0) = 0 = f(y)$. Therefore the condition a), b), c) hold.

Let $Y \subset X$ and $x_0 \in Y$. Let us denote

$$Y^{\perp} = \{f \mid f \in X_0^{\#},\ f|_Y = 0\}. \tag{14}$$

For $x, y \in X$ we denote

$$d_{Y^{\perp}}(x, y) = \sup_{f \in Y^{\perp} - \{0\}} \frac{|f(x) - f(y)|}{\|f\|_X}. \tag{15}$$

We have the following inequality:

$$d_{Y^{\perp}}(x, y) \leq d(x, y). \tag{16}$$

Indeed, for all $f \in X_0^{\#}$ and for all $x, y \in X$,

$$|f(x) - f(y)| \leq \|f\|_X d(x, y),$$

so that, for $f \neq \theta$,

$$\frac{|f(x) - f(y)|}{\|f\|_X} \leq d(x, y),$$

and

$$\sup_{f \in X_0^{\#} - \{0\}} \frac{|f(x) - f(y)|}{\|f\|_X} \leq d(x, y).$$

Therefore

$$d_{Y^{\perp}}(x, y) = \sup_{f \in Y^{\perp} - \{0\}} \frac{|f(x) - f(x)|}{\|f\|_X} \leq \sup_{f \in X_0^{\#} - \{0\}} \frac{|f(x) - f(y)|}{\|f\|_X} \leq d(x, y).$$

Proposition 6. *Let $Y \subset X$ and $y_0 \in Y$, $x \in X - Y$. Then, $y_0 \in Y$ is an element of best approximation for x by elements of Y if and only if*

$$d_{Y^{\perp}}(x, y_0) = d(x, y_0). \tag{17}$$

Proof. Let $y_0 \in Y$ be an element of best approximation for x. Then, by Proposition 4 it follows that there exist an element $f \in Y^{\perp}$ such that $\|f\|_X = 1$ and $|f(x) - f(y)| = d(x, y_0)$. We have

$$d_{Y^{\perp}}(x, y_0) = \sup_{g \in Y^{\perp} - \{0\}} \frac{|g(x) - g(y_0)|}{\|g\|_X} \geq \frac{|f(x) - f(y_0)|}{\|f\|_X} = d(x, y_0),$$

and, because of $d_{Y^{\perp}}(x, y_0) \leq d(x, y_0)$ we have (17).

Conversely, if (17) holds, then for all $y \in Y$ we have:

$$d(x, y_0) = d_{Y^{\perp}}(x, y_0) = \sup_{f \in Y^{\perp} - \{0\}} \frac{|f(x) - f(y_0)|}{\|f\|_X} = \sup_{f \in Y^{\perp} - \{0\}} \frac{|f(x) - f(y)|}{\|f\|_X} = $$

$$= d_{Y^{\perp}}(x, y) \leq d(x, y).$$

Hence $y_0 \in Y$ is an element of best approximation for x by elements of Y.

Remarks.

1°. Let (X, d) be a linear metric space, the metric d being translation invariant, and $x_0 = 0 \in X$. If Y is a subspace of X, then one can choose the function f in Proposition 1 such that $f \in C_X$, where C_X denotes the cone of subadditive function in $X_0^{\#}$ [4]. The subadditivity of function f follows from the proof of Proposition 2.1 [4]. If X is a normed linear space, then $X_0^{\#} \supset C_X \supset X^*$. If Y is a subspace of X, then Proposition 1 holds with $f \in X^*$([1], Lemma 12, p. 64).

2°. Simple example show that the inequality (10) in Proposition 2 can be strict. Let $X = [-1, 10] \subset R$ with the usual metric $d(x, y) = |x - y|$, $x_0 = 0 \in R$ and

$$f(x) = \begin{cases} 0 & x \in [-1, 0] \\ x & x \in (0, 1] \\ 1 & x \in (1, 10] \end{cases}$$

Then $f^{(-1)}(0) = [-1, 0]$ and for all $x \in [2,10]$,

$$d(x, [-1, 0]) > 1 = \frac{|f(x)|}{\|f\|_X}.$$

If X is a metric space, Y a closed subset of X, $x_0 \in Y$, then for every function $f \in X_0^{\#}$ of the form $f(x) = \lambda d(x, Y)$, $\lambda \in R$, the relation (10) holds with the sign „=".

If X is a normed linear space and $f \in X^*$, then (10) holds with the sign „=". (Ascoli's Theorem [6]).

3°. If X is a normed linear space and $f \in X^*$, then the condition (12) is equivalent to:

(J). There is $x_0 \in X$ such that $|f(x_0)| = \|f\| \cdot \|x_0\|$.

Indeed, since $X = f^{(-1)}(0) \dotplus Rx_0$, for every $x \in X - f^{(-1)}(0)$ there is $\lambda \in R$ and $y_x \in f^{(-1)}(0)$ such that $x = y_x + \lambda x_0$. Then

$$|f(x) - f(y_x)| = |f(x) - f(x - \lambda x_0)| = |\lambda| \cdot |f(x_0)| = \|f\| \cdot \|x_0\| \cdot |\lambda| =$$
$$= \|f\| \cdot \|x - (x - \lambda x_0)\| = \|f\|_X \cdot \|x - y_x\|.$$

Evidently, if $f \in X^* \subset X_0^{\#}$, the norm (2) agrees with the usual norm of linear functionals).

The converse implication is obvious.

By a theorem of R. C. JAMES [6] it follows that (12) is a necessary and sufficient condition for $f^{(-1)}(0)$ to be proximinal.

4°. Proposition 4 is analogous to Theorem 1.1, p. 16, [5] and to Proposition 2.1, [4], while Proposition 5 is analogous to the Theorem 3.1, p. 96, [5] and Proposition 4.1, [4].

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