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ON THE LENGHT OF THE INTERPOLATION INTERVAL FOR A CLASS OF LINEAR DIFFERENTIAL EQUATIONS

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1. We consider the n-th order linear homogeneous differential equation

$$x^{(n)} + \sum_{i=1}^{n} a_i(t) x^{(n-i)} = 0, \qquad n \geqslant 2, \tag{1.1}$$

where the coefficients $a_i(t) \in L^p[a, a+h]$, $(i = 1, \ldots, n)$, $p > 1$, $h > 0$. We denote by X_n the set of the solutions of E_q. (1.1), that is the set of functions $x : [a, a+h] \to R$ with the properties

a) $x^{(n-1)}(t)$ is absolutely continuous on $[a, a+h]$,
b) $x(t)$ verifies E_q. (1.1) almost everywhere on $[a, a+h]$.

The set X_n has the interpolation properties $D_n^{(\nu)}[a, a+h]$, $(\nu = 0, 1, \ldots, n-2)$ if for each two points $a \leqslant t_1 < t_2 \leqslant a+h$ and for any system of n real numbers $r_1, \ldots, r_n$ there is a unique solution $x(t) \in X_n$ satisfying the boundary value conditions

$$x^{(i)}(t_1) = r_i, \quad x^{(\nu)}(t_2) = r_n, \quad i = 0, 1, \ldots, n-2, \tag{1.2}$$

where ν is one of the numbers $0, 1, \ldots, n-2$.

We shall use the notation for $i = 1, \ldots, n$

$$\| a_i \|_p = \begin{cases} \left[\dfrac{1}{h} \displaystyle\int_a^{a+h} | a_i(t) |^p \, dt \right]^{1/p}, & \text{if } 1 < p < \infty, \\[2ex] \operatorname*{ess.sup}_{[a,a+h]} | a_i(t) |, & \text{if } p = \infty. \end{cases} \tag{1.3}$$

The purpose of this paper is to give some upper bounds for h, that is for the lenght of the interpolation interval. These upper bounds are expressed in the form of some nonlinear inequalities, namely

$$\sum_{i=1}^{n} A_i \| a_i \|_p h^i \leqslant 1, \tag{1.4}$$

where A_i, $(i = 1, \ldots, n)$ are constants.

The criterions of this type which assure an analogous properties to the $D_n^{(\nu)}[a, a+h]$ of X_n have been obtained by several authors: see for instance the references in papers O. ARAMĂ and D. RIPIANU [3], [4], O. ARAMĂ [2], D. WILLETT [9].

The sufficient conditions of the type (1.4) for the properties $D_n^{(\nu)}[a, a+h]$ obtained by the method which we use below, were given by O. ARAMĂ [1] in the case $n = 4$, $p = 2$, A. LUPAŞ [6], [7] in the cases $n \leqslant 5$, different p, and author [8] for arbitrary $n \geqslant 2$, $p = \infty$.

In this paper we give the conditions of the type (1.4) which generalize to arbitrary $n \geqslant 2$, $p > 1$ results of [1], [6], [7] and may be regarded as an improvement of theorems in [1], [6], [7].

2. We need the following result of D. W. BOYD [5] (see [6]) which we formulate as

Lemma 2.1. *If $y : [a, b] \to R$ is absolutely continuous on $[a, b]$ and $y(a) = 0$, then*

(2.1)

$$\int_a^b | y(t) |^p | y'(t) |^q dt \leqslant (b-a)^{(rp-p-q+r)/r} K(p, q, r) \left[\int_a^b | y'(t) |^r dt \right]^{\frac{p+q}{r}},$$

for $r \geqslant 1$, $p > 0$, $0 \leqslant q \leqslant 2$. The best constant $K(p, q, r)$ is given by the following expressions

a) if $p > 0$, $r > 1$, $0 \leqslant q < r$, *then*

$$K(p, q, r) = \frac{(r-q)p^p}{(r-1)(p+q)} \beta^{p+q-r} I(p, q, r)^{-p}, \tag{2.2}$$

where

$$\beta = \left[\frac{p(r-1) + (r-q)}{(r-1)(p+q)} \right]^{1/r}$$

and

$$I(p, q, r) = \int_0^1 \left[1 + \frac{r(q-1)}{r-q} t \right]^{-(q+p+rp)/rp} [1 + (q-1)t] t^{1/p-1} dt.$$

b) *If $r = 1$, then*

$$K(p, q, r) = \begin{cases} q^q (p+q)^{-q}, & q > 0 \\ 1, & q = 0. \end{cases}$$

c) *If $q = r$, then*

$$K(p, r, r) = \frac{rp^p}{p+r} \left(\frac{p}{p+r} \right)^{p/r} B(1/r + 1, 1/p)^{-p}. \tag{2.3}$$

From Lemma 2.1 we obtain the following two inequalities.

Lemma 2.2. *If $w : [a, b] \to R$, $w(t) \in C^1(a, b)$ and $w(a) = w(b) = 0$, then for $0 < q \leqslant 2$ holds the inequality*

$$\int_a^b | w(t) w'(t) |^q dt \leqslant \frac{K(q, q, 2)}{2^q} (b-a) \left[\int_a^b w'^2(t) dt \right]^q, \tag{2.4}$$

where $K(q, q, 2)$ is given by (2.2) or (2.3).

Lemma 2.3. *If $w : [a, b] \to R$ is absolutely continuous on $[a, b]$ and $w(a) = w(b) = 0$, then for $p \geqslant 2$ holds*

$$\int_a^b | w(t) |^p dt \leqslant K(p, 0, 2) \left(\frac{b-a}{2} \right)^{p/2+1} \left[\int_a^b w'^2(t) dt \right]^{p/2}, \tag{2.5}$$

where $K(p, 0, 2)$ is given by (2.2).

Proof of Lemma 2.2. Let $u : [a, b] \to R$ be any function absolutely continous on $[a, b]$. Then $w_1(s) = u(a+b-s) - u(b)$ is absolutely continuous on $[a, b]$ and $w_1(a) = 0$, therefore from (2.1) for $p = q$, $r = 2$ $(0 < p, q \leqslant 2)$ we have

$$\int_a^b | [u(a+b-s) - u(b)] u'(a+b-s) |^q ds \leqslant \tag{2.6}$$

$$\leqslant (b-a) K(q, q, 2) \left[\int_a^b | u'(a+b-s) |^2 ds \right]^q.$$

Making the substitution $t = a + b - s$ in (2.6) we obtain

$$\int_a^b | [u(t) - u(b)] u'(t) |^q dt \leqslant (b-a) K(q, q, 2) \left[\int_a^b u'^2(t) dt \right]^q,$$

thus for $w_2(t) = u(t) - u(b)$ absolutely continuous on $[a, b]$ and $w_2(b) = 0$, holds inequality

$$\int_a^b | w_2(t) w_2'(t) |^q \, dt \leqslant (b-a) K(q, q, 2) \left[\int_a^b w_2'^2(t) dt \right]^q, \quad 0 < q \leqslant 2. \tag{2.7}$$

By the inequalities (2.1), (2.7) for $c \in (a, b)$ we obtain

$$\int_a^b | w(t) w'(t) |^q \, dt \leqslant$$
$$\leqslant (c-a) K(q, q, 2) \left[\int_a^c w'^2(t) dt \right]^q + (b-c) K(q, q, 2) \left[\int_c^b w'^2(t) dt \right]^q. \tag{2.8}$$

If we choose c such that

$$\int_a^c w'^2(t) dt = \int_c^b w'^2(t) dt = \frac{1}{2} \int_a^b w'^2(t) dt,$$

then the inequality (2.4) follows from (2.8).

The proof of Lemma 2.3 is analogous to the previous one (cf. [6]), therefore it is omitted.

Remark 2.4. One can easily verify that, if $p > 2$ and $w(t) \not\equiv 0$, $t \in [a, b]$, then the equality in (2.5) is impossible.

For $p = 2$ from (2.5) we obtain the Wirtinger's inequality, namely

$$\int_a^b w^2(t) dt \leqslant \frac{(b-a)^2}{\pi^2} \int_a^b w'^2(t) dt. \tag{2.9}$$

3. The main result of this paper is the following

THEOREM 3.1. *Suppose, that*

(i) $a_1(t) \in L^{p_1}(a, a+h)$, $(2 \leqslant p_1 < \infty)$, $a_i(t) \in L^{p_2}(a, a+h)$
$(i = 2, \ldots, n;\ 1 < p_2 < \infty)$,

(ii) *h satisfies the following inequality*

$$\frac{K(q_1, q_1, 2)^{1/q_1}}{2} ||a_1||_{p_1} h + \frac{K(q_2, q_2, 2)^{1/q_2}}{\pi^2} ||a_3||_{p_2} h^3 +$$
$$+ \frac{1}{2} \left[\frac{K(2q_2, 0, 2)}{2} \right]^{1/q_2} \left(||a_2||_{p_2} h^2 + A(q_2) \sum_{i=4}^{n} \frac{1}{(i-2)!} \left[\frac{i-2}{2q_2(i-2)-1} \right]^{1-1/2q_2} ||a_i||_{p_2} h^i \right) \leqslant 1^{*)}, \tag{3.1}$$

where

$$\frac{1}{p_k} + \frac{1}{q_k} = 1, \ (k = 1, 2), \quad A(q_2) = \frac{[4q_2(2q_2 - 1)]^{1-1/2q_2}}{4q_2}, \tag{3.2}$$

and

$||a_1||_{p_1}$, $||a_i||_{p_2}$, $(i = 2, \ldots, n)$ *are defined by* (1.3),

$K(q_k, q_k, 2)$, $(k = 1, 2)$, $K(2q_2, 0, 2)$ *are given by* (2.2) *or* (2.3).

Then X_n has the interpolation properties $D_n^{(\nu)}[a, a+h]$, $(\nu = 0, 1, \ldots, n-2)$.

Proof. It suffices to prove that X_n has property $D_n^{(n-2)}[a, a+h]$. Let us suppose that X_n has not property $D_n^{(n-2)}[a, a+h]$. Then there is a nontrivial solution $x(t) \in X_n$ and there are the points t_1, t_2 $(a \leqslant t_1 < t_2 \leqslant a+h)$, such that

$$x^{(i)}(t_1) = x^{(n-2)}(t_2) = 0, \quad i = 0, 1, \ldots, n-2. \tag{3.3}$$

We denote

$$x^{(n-2)}(t) = v(t). \tag{3.4}$$

From (3.3) and (3.4) we get

$$x^{(n-i)}(t) = \frac{1}{(i-3)!} \int_{t_1}^{t} (t-s)^{i-3} v(s) ds, \quad (i = 3, \ldots, n;\ n \geqslant 3). \tag{3.5}$$

Substituting (3.4) and (3.5) in E_q. (1.1) we have

$$v''(t) + a_1(t) v'(t) + a_2(t) v(t) + \sum_{i=3}^{n} a_i(t) \frac{1}{(i-3)!} \int_{t_1}^{t} (t-s)^{i-3} v(s) ds = 0.$$

*) In the cases $n = 2, 3$ the left-hand side of (3.1) we treat as the sum two or three first components, respectively.

Multiplying the above equality by $v(t)$ and integrating from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2} v'^2(t)dt = \int_{t_1}^{t_2} a_1(t)v'(t)v(t)dt + \int_{t_1}^{t_2} a_2(t)v^2(t)dt + \tag{3.6}$$

$$+\sum_{i=3}^{n}\int_{t_1}^{t_2}\frac{a_i(t)}{(i-3)!}v(t)\int_{t_1}^{t}(t-s)^{i-3}v(s)ds\,dt.$$

Applying Hölder's inequality with indices $p_1 \geqslant 2$, $q_1 = p_1/(p_1-1)$ to the first integral on the right-hand side of (3.6) and with indices $p_2 > 1$, $q_2 = p_2/(p_2-1)$ to the next integrals, one has

$$\int_{t_1}^{t_2} v'^2(t)dt \leqslant \Big[\int_{t_1}^{t_2}|a_1(t)|^{p_1}dt\Big]^{1/p_1}\Big[\int_{t_1}^{t_2}|v'(t)v(t)|^{q_1}dt\Big]^{1/q_1} +$$

$$+\Big[\int_{t_1}^{t_2}|a_2(t)|^{p_2}dt\Big]^{1/p_2}\Big[\int_{t_1}^{t_2}|v(t)|^{2q_2}dt\Big]^{1/q_2} + \tag{3.7}$$

$$+\sum_{i=3}^{n}\frac{1}{(i-3)!}\Big[\int_{t_1}^{t_2}|a_i(t)|^{p_2}dt\Big]^{1/p_2}\Big[\int_{t_1}^{t_2}|v(t)|^{q_2}\Big|\int_{t_1}^{t}(t-s)^{i-3}v(s)ds\Big|^{q_2}dt\Big]^{1/q_2}.$$

From Lemma 2.3 and Remark 2.4 we get

$$\Big[\int_{t_1}^{t_2}|v(t)|^{2q_2}dt\Big]^{1/q_2} < \frac{1}{2}\Big[\frac{K(2q_2, 0, 2)}{2}\Big]^{1/q_2}(t_2-t_1)^{1+1/q_2}J,\quad q_2 > 1, \tag{3.8}$$

and likewise from Lemma 2.2 we have

$$\Big[\int_{t_1}^{t_2}|v'(t)v(t)|^{q_1}dt\Big]^{1/q_1} \leqslant \frac{K(q_1, q_1, 2)^{1/q_1}}{2}(t_2-t_1)^{1/q_1}J,\quad 1 < q_1 \leqslant 2, \tag{3.9}$$

where

$$J = \int_{t_1}^{t_2} v'^2(t)dt.$$

According to an inequality of A. LUPAŞ [6] (see inequality (24.3)) we may write

$$\Big[\int_{t_1}^{t_2}\Big|v(t)\int_{t_1}^{t}v(s)ds\Big|^{q_2}dt\Big]^{1/q_2} \leqslant \frac{K(q_2, q_2, 2)^{1/q_2}}{\pi^2}(t_2-t_1)^{2+1/q_2}J. \tag{3.10}$$

By Hölder's inequality with indices $2q_2/(2q_2-1)$, $2q_2$ from

$$I_i(t) = \Big|\int_{t_1}^{t}(t-s)^{i-3}v(s)ds\Big|,\quad i = 4, \ldots, n,\ t \in [t_1, t_2]$$

follows

$$I_i(t) \leqslant \Big[\int_{t_1}^{t}(t-s)^{2q_2(i-3)/(2q_2-1)}ds\Big]^{1-1/2q_2}\Big[\int_{t_1}^{t}|v(s)|^{2q_2}ds\Big]^{1/2q_2} =$$

$$= C_i(q_2)(t-t_1)^{i-2-1/2q_2}\Big[\int_{t_1}^{t}|v(s)|^{2q_2}ds\Big]^{1/2q_2},$$

where

$$C_i(q_2) = \Big[\frac{2q_2-1}{2q_2(i-2)-1}\Big]^{1-1/2q_2},\quad (i = 4, \ldots, n).$$

The Schwarz's inequality yields

$$I_i = \Big[\int_{t_1}^{t_2}|v(t)|^{q_2}\,[I_i(t)]^{q_2}dt\Big]^{1/q_2} \leqslant$$

$$\leqslant C_i(q_2)\Big[\int_{t_1}^{t_2}(t-t_1)^{2q_2(i-2)-1}dt\Big]^{1/2q_2}\Big[\int_{t_1}^{t_2}|v(t)|^{2q_2}\int_{t_1}^{t}|v(s)|^{2q_2}ds\,dt\Big]^{1/2q_2}.$$

Hence, we have the inequalities

$$I_i \leqslant C_i(q_2)D_i(q_2)(t_2-t_1)^{i-2}\Big[\int_{t_1}^{t_2}|v(t)|^{2q_2}dt\Big]^{1/q_2},\quad (i = 4, \ldots, n),$$

where

$$D_i(q_2) = [4q_2(i-2)]^{-1/2q_2},\quad (i = 4, \ldots, n).$$

Finally, in view of (3.8) we obtain the estimates

$$I_i < \frac{1}{2}\left[\frac{K(2q_2, 0, 2)}{2}\right]^{1/q_2} C_i(q_2)D_i(q_2)(t_2 - t_1)^{i-1+1/q_2} J, \tag{3.11}$$

$$(i = 4, \ldots, n).$$

Now, if we insert the inequalities (3.8)—(3.11) in (3.7), after dividing with J, $J > 0$, we have

$$1 < \frac{K(q_1, q_1, 2)^{1/q_1}}{2}(t_2 - t_1)^{1/q_1}\left[\int_{t_1}^{t_2} |a_1(t)|^{p_1} dt\right]^{1/p_1} +$$

$$+ \frac{1}{2}\left[\frac{K(2q_2, 0, 2)}{2}\right]^{1/q_2}(t_2 - t_1)^{1+1/q_2}\left[\int_{t_1}^{t_2} |a_2(t)|^{p_2} dt\right]^{1/p_2} +$$

$$+ \frac{K(q_2, q_2, 2)^{1/q_2}}{\pi^2}(t_2 - t_1)^{2+1/q_2}\left[\int_{t_1}^{t_2} |a_3(t)|^{p_2} dt\right]^{1/p_2} +$$

$$+ \frac{1}{2}A(q_2)\left[\frac{K(2q_2, 0, 2)}{2}\right]^{1/q_2}\sum_{i=4}^{n}\frac{1}{(i-2)!}\left[\frac{i-2}{2q_2(i-2)-1}\right]^{1-1/2q_2}(t_2 - t_1)^{i-1+1/q_2}$$

$$\left[\int_{t_1}^{t_2} |a_i(t)|^{p_2} dt\right]^{1/p_2}.$$

We observe that

$$C_i(q_2)D_i(q_2) = A(q_2)\frac{1}{i-2}\left[\frac{i-2}{2q_2(i-2)-1}\right]^{1-1/2q_2},$$

where $A(q_2)$ is given by (3.2). Since $t_2 - t_1 \leqslant h$, then

$$(t_2 - t_1)^{1/p}\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} |a_i(t)|^p dt\right]^{1/p} \leqslant h^{1/p}\left[\frac{1}{h}\int_{a}^{a+h} |a_i(t)|^p dt\right]^{1/p}, \quad (i = 1, \ldots, n),$$

where $p = p_1$ for $i = 1$ and $p = p_2$ for $i = 2, \ldots, n$.

Therefore according to (1.3) we get the inequality

$$1 < \frac{K(q_1, q_1, 2)^{1/q_1}}{2}||a_1||_{p_1}h + \frac{1}{2}\left[\frac{K(2q_2, 0, 2)}{2}\right]^{1/q_2}||a_2||_{p_2}h^2 +$$

$$+ \frac{K(q_2, q_2, 2)^{1/q_2}}{\pi^2}||a_3||_{p_2}h^3 +$$

$$+ \frac{1}{2}A(q_2)\left[\frac{K(2q_2, 0, 2)}{2}\right]^{1/q_2}\sum_{i=4}^{n}\frac{1}{(i-2)!}\left[\frac{i-2}{2q_2(i-2)-1}\right]^{1-1/2q_2}||a_i||_{p_2}h^i$$

which is a contradiction with (3.1). This completes the proof of Theorem 3.1.

Remark 3.2. Since

$$K(2, 2, 2) = \frac{4}{\pi^2}$$

hence, in particular, for $p_1 = q_1 = 2$ the condition (3.1) takes the form

$$\frac{||a_1||_2 h}{\pi} + \frac{2||a_3||_2 h^3}{\pi^3} +$$

$$+ \frac{1}{2}\left[\frac{K(4, 0, 2)}{2}\right]^{1/2}\left(||a_2||_2 h^2 + \left(\frac{3}{2}\right)^{3/4}\sum_{i=4}^{n}\frac{1}{(i-2)!}\left[\frac{i-2}{4i-9}\right]^{3/4}||a_i||_2 h^i\right) \leqslant 1^{*}), \tag{3.12}$$

where

$$K(4, 0, 2)^{1/2} = \frac{16\pi\sqrt{3}}{\Gamma(1/4)]^4}.$$

The inequality (3.12) in case $n \leqslant 5$ may be compared with the inequalities in [6], [7].

THEOREM 3.3. *Let the coefficients* $a_i(t) \in L^\infty(a, a + h)$, $(i = 1, \ldots, n)$ *and* h *satisfies the following inequality*

$$\frac{1}{4}h||a_1||_\infty + \frac{1}{\pi^2}h^2||a_2||_\infty + \frac{1}{2\pi^2}\sum_{i=3}^{n}\frac{h^i||a_i||_\infty}{(i-3)!\sqrt{(2i-5)(i-2)}} < 1^{*}), \tag{3.13}$$

where $||a_i||_\infty$, $(i = 1, \ldots, n)$ *is defined by* (1.3).

*) Compare the reference mark in Theorem 3.1.

Then X_n has the interpolation properties $D_n^{(\nu)}$ $[a, a+h]$, $(\nu = 0, 1, \ldots, n-2)$.

Proof. We show that X_n has property $D_n^{(n-2)}$ $[a, a+h]$. Suppose, that X_n has not this property. Then taking into account that $L^\infty(a, a+h) \subset \subset L^p(a, a+h)$, we may apply Theorem 3.1. Therefore by Theorem 3.1 we may write the inequality

$$\begin{gathered}\frac{K(q, q, 2)^{1/q}}{2}||a_1||_p h + \frac{K(q, q, 2)^{1/q}}{\pi^2}||a_3||_p h^3 + \\ + \frac{1}{2}\left[\frac{K(2q, 0, 2)}{2}\right]^{1/q}\left(||a_2||_p h^2 + A(q)\sum_{i=4}^{n}\frac{1}{(i-2)!}\left[\frac{i-2}{2q(i-2)-1}\right]^{1-1/2q}||a_i||_p h^i\right) > 1,\end{gathered} \tag{3.14}$$

where $1/p + 1/q = 1$, $p > 2$, $A(q)$ is defined by (3.2).
From (2.2) we get

$$K(2q, 0, 2) = \frac{(2q)^{2q}}{q}(1 + 1/q)^{q-1} I(2q, 0, 2)^{-2q}, \tag{3.15}$$

where

$$I(2q, 0, 2) = \int_0^1 (1-t)^{-1/2} t^{1/2q-1}\, dt = \frac{\Gamma(1/2q)\Gamma(1/2)}{\Gamma(1/2q + 1/2)},$$

likewise

$$K(q, q, 2) = \frac{2-q}{2} I(q, q, 2)^{-q}, \tag{3.16}$$

where

$$I(q, q, 2) = \int_0^1 \left[1 + \frac{2(q-1)}{2-q}t\right]^{-2}[1 + (q-1)t]t^{1/q-1}\, dt.$$

Ona can easily show that

$$\lim_{q \to 1+} I(q, q, 2) = 1. \tag{3.17}$$

In view of (3.15)—(3.17), (3.2) and taking into account that for $a_i(t) \in \in L^\infty(a, a+h)$

$$||a_i||_\infty = \lim_{p \to \infty} ||a_i||_p, \quad (i = 1, \ldots, n)$$

and the fact that $\Gamma(\alpha)$, $(\alpha > 0)$ is continuous, we conclude from (3.14) that

$$\begin{gathered}\frac{1}{2}\left(\frac{||a_1||_\infty h}{2} + \frac{||a_3||_\infty h^3}{\pi^2}\right) + \\ + \frac{K(2, 0, 2)}{4}\left(||a_2||_\infty h^2 + \frac{1}{2}\sum_{i=4}^{n}\frac{1}{(i-2)!}\left[\frac{i-2}{2i-5}\right]^{1/2}||a_i||_\infty h^i\right) \geqslant 1.\end{gathered} \tag{3.18}$$

Since

$$K(2, 0, 2) = \frac{4}{\pi^2},$$

it follows from (3.18) that

$$\frac{h||a_1||_\infty}{4} + \frac{h^3||a_3||_\infty}{2\pi^2} + \frac{1}{\pi^2}\left(h^2||a_2||_\infty + \frac{1}{2}\sum_{i=4}^{n}\frac{h^i||a_i||_\infty}{(i-3)!\sqrt{(2i-5)(i-2)}}\right) \geqslant 1.$$

This contradiction completes the proof of Theorem 3.3.

Remark. The inequality (3.13) in the case $n = 5$ may be regarded as an improvement of the inequality given by Theorem 2.1 in paper [7] The condition (3.13) can be found in [8], however in this paper we gave a different proof of this theorem.

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