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TIME-ENTIRE DYNAMICAL SYSTEMS

by

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The aim of this paper is to prove that every dynamical system on a locally compact separable metric space is embeddable into a time-entire dynamical system.

Definition 1. *A dynamical system is a pair* (X, f), *where* X *is a topological space and* f *is a continuous function from the product space* $X \times R$ *into the space* X *satisfying the following axioms :*

(1) $$f(x, 0) = x \text{ for all } x \text{ in } X$$

(2) $$f(f(x, t), s) = f(x, t + s) \text{ for every } x \text{ in } X \text{ and } t, s \text{ in } R.$$

Remark 1. We shall also use the expression : f is a dynamical system on the topological space X.

Definition 2. *Let* L *be a Banach space. The dynamical system* f *on* L *is time-entire if* f *has an expansion in series*

(3) $$f(p_0, t) = p_0 + p_1 t + p_0 t^2 + \ldots + p_n t^n + \ldots$$

which is convergent in norm for every real number t, *where* p_n (*for* $n = 1, 2, 3, \ldots$) *are dependant only on* p_0, *which may be chosen arbitrarily in* L.

Definition 3. *The dynamical system* (X, f) *is embeddable into the dynamical system* (Y, g) *if there is an homeomorphism* F *of* X *in* Y *so that :*

(4) $$F(f(x, t)) = g(F(x), t) \quad \text{for every } x \text{ in } X \text{ and } t \text{ in } R.$$

Remark 2. In this case, if we denote by $g' = g|_{F(X) \times R}$, (X, f) is isomorphic with $(F(X), g')$, which is a subsystem of (Y, g) because $g'(F(X), R) = F(X)$.

Using a theorem of BEBUTOV, E. A. BARBASHIN and F. A. SHOLOHOVICH have proved in [1] that a dynamical system defined on a compact metric space and having at most one critical point is embeddable into a time-entire dynamical system. Our goal is to apply a generalization of Bebutov's theorem given by us in [8] for the demonstration of the following:

THEOREM. *Every dynamical system defined on a locally compact separable metric space is embeddable into a time-entire dynamical system.*

Proof. We remind that the shift (or Bebutov) system on the topological space Y is the system f_0 defined on $C(R, Y)$ (with the compact-open topology) by $f_0(x, t) = x_t$, where $x_t(s) = x(t + s)$. In [8] we have demonstrated the following property of universality of the shift systems: every dynamical system definite on a locally compact separable metric space is embeddable into the shift system on l^2. To prove this, for a system f definite on X we have taken his prolongation $\hat{f}$ to the one point compactification $\hat{x}$ of X and then have embedded f_0 into the shift system on l^2. After all f is embedded into a compact subsystem (that is defined on a compact space) of the shift system on l^2. Taking this into account, for the demonstration of the theorem it is enough to prove the following.

L e m m a. *Every compact subsystem of the shift system f_0 on a Banach space Y is embeddable into a time-entire dynamical system.*

Proof of the lemma. Y being a Banach space, the compact-open topology on $C(R, Y)$ is generated by the metrics:

$$d(x, y) = \sup_{T<0} \min \left\{ \max_{|t| \leqslant T} \|x(t) - y(t)\|_Y ;\ \frac{1}{T} \right\} \tag{5}$$

Let X be a compact, invariant (that is $f_0(x, t) \in X$ for every x in X and t in R) subset of $C(R, Y)$ and denote the restriction of f_0 to $X \times R$ again by f_0. All functions from X are bounded because, X being compact, there is a constant $m > 0$ so that for every $x \in X : d(0, x) \leqslant m$, therefore $\|x(t)\|_Y = \|x_t(0)\|_Y \leqslant d(0, x_t) \leqslant m$. We shall use a transformation like Fourier's used in [1] but for functions from $C(R, Y)$. For this we need to use complex numbers so that we replace Y (if it is real) by the complex Banach space Y_c obtained from Y by the extention of the field of scalars ([7], pp. 5–6). Let the set of continuous functions $\varphi : R \to Y_c$ for which the Riemann integral $\int_{-\infty}^{\infty} \|\varphi(t)\|_{Y_c}^2 e^{-t^2} dt$ is finit. If we define in the natural way the addition and the multiplication by complex numbers, while the norm we define by:

$$\|\varphi\| = \left[\int_{-\infty}^{\infty} \|\varphi(t)\|_{Y_c}^2 \, e^{-t^2} dt \right]^{1/2} \tag{6}$$

we get a Banach space that we denote by N. To obtain a map Φ from X

in N we put for a $x \in X$:

$$(\Phi(x))(t) = \int_{-\infty}^{\infty} x(s) \, e^{-s^2} e^{ist} \, ds. \tag{7}$$

We have

$$\|(\Phi(x))(t)\|_{Y_c} \leqslant \int_{-\infty}^{\infty} \|x(s)\|_{Y_c} \, e^{-s^2} ds \leqslant m \int_{-\infty}^{\infty} e^{-s^2} ds = m \sqrt{\pi} \tag{8}$$

so that $\Phi(x) \in N$, that is Φ is well defined. To prove that Φ is continuous, we consider a sequence $\{x_n\}$ which tends to x in X and an $\varepsilon > 0$. We may choose an $\eta > \frac{2\pi}{\varepsilon}$ so that $\int_{\eta}^{\infty} e^{-s^2} ds < \frac{\varepsilon}{8m \sqrt[4]{\pi}}$. Because $x_n \to x$ there is a natural n_0 so that for every $n > n_0 : d(x_n, x) < \frac{1}{\eta}$, that is for $|t| \leqslant \eta$: $\|x_n(t) - x(t)\|_{Y_c} < \frac{1}{\eta}$. For such a n:

$$\left\| \int_{-\infty}^{\infty} [x_n(x) - x(s)] \, e^{-s^2} e^{ist} \, ds \right\|_{Y_c} \leqslant \int_{-\infty}^{\infty} \|x_n(s) - x(s)\|_{Y_c} \, e^{-s^2} ds \leqslant \tag{9}$$

$$\leqslant \frac{1}{\eta} \int_{-\eta}^{\eta} e^{-s^2} ds + 2 \cdot 2m \int_{\eta}^{\infty} e^{-s^2} ds \leqslant \frac{\varepsilon}{2 \sqrt[4]{\pi^3}} + 4m \cdot \frac{\varepsilon}{8m \sqrt[4]{\pi}} = \frac{\varepsilon}{\sqrt[4]{\pi}}$$

hence

$$\|\Phi(x_n) - \Phi(x)\|_N \leqslant \varepsilon. \tag{10}$$

Assume now that for an $x \in X$:

$$\int_{-\infty}^{\infty} x(s) \, e^{ist} \, ds = 0. \tag{11}$$

Then for every λ, t in R:

$$\int_{-\infty}^{\infty} x(s + \lambda) e^{ist} \, ds = 0. \tag{12}$$

Fix $a < b$ and put:

$$\varphi(s) = \int_a^b x(s + \lambda) d\lambda \tag{13}$$

We have

(14) $$\int_a^b \int_{-\infty}^{\infty} x(s+\lambda) e^{ist}\, ds \cdot d\lambda = 0$$

or applying Fubini's theorem [6]:

(15) $$\int_{-\infty}^{\infty} \varphi(s)\, e^{ist}\, ds = 0 \text{ for every } t \text{ in } R.$$

It may be deduced from (15) as in [4] (p. 398) that $\varphi(s) \equiv 0$. But a and b being arbitrary it follows that $x = 0$ because if we suppose $x(t_0) \neq 0$ for a $t_0 \in R$, by the continuity of x, for $r = \|x(t_0)\|$ there is a $\delta > 0$ so that $\|x(t) - x(t_0)\| \leqslant \frac{r}{2}$ for every $t \in (t_0 - \delta, t_0 + \delta)$. Then $\frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} x(t) dt \in$ $\in \overline{S\left(x(t_0), \frac{r}{2}\right)}$ so that $\int_{t_0-\delta}^{t_0+\delta} x(t) dt \neq 0$, which is impossible. After all $\Phi(x) = 0$ implies $x = 0$, hence Φ is an injection. X being a compact space and Φ a continuous injection it follows that Φ is a homeomorphism from X on $\Phi(X) \subset N$.

We define $f: \Phi(X) \times R \to \Phi(X)$ by:

(16) $$f[\Phi(x), t] = \Phi[f_0(x, t)].$$

It is easy to prove that f is a dynamical system on $\Phi(X)$, isomorphic (by (16)) with the shift system on X. Let us prove that f is time-entire, that is the function

(17) $$g(s, t) = F[\Phi(x), t](s)$$

is entire with respect to t. But

(18) $$g(s, t) = \int_{-\infty}^{\infty} x(t+u) e^{-u^2} e^{ius}\, du = \int_{-\infty}^{\infty} x(u) e^{-(u-t)^2} e^{i(u-t)s}\, du.$$

The function $x(u)\, e^{-(u-t)^2} e^{i(u-t)s}$ being analytic with respect to t for every u, continuous with respect to u for every t, and bounded, it follows that g is analytical with respect to t (for the ordinary case this property is proved in [5] pp. 295—296; for functions with values in a Banach space it is necessary to verify a lot of other theorems; the starting one is Cauchy's integral formula which is true in this case [3]). That is:

(19) $$g(s, t) = \sum_{k=0}^{\infty} \varphi_k(s)\, t^k.$$

It must be shown that $\varphi_k \in N$ and that the series converges in the norm of N. Firstly $\varphi_0 = \Phi(x) \in N$. Then $\varphi_1(s) = g_t'(s, 0)$, that is

(20) $$\varphi_1(s) = \int_{-\infty}^{\infty} x(u)\, e^{-u^2} e^{ius} (2u - is)\, du$$

and so

(21) $$\|\varphi_1(s)\|_{Y_c} \leqslant m \left[2 \int_{-\infty}^{\infty} |u|\, e^{-u^2} du + |s| \int_{-\infty}^{\infty} e^{-u^2} du \right] = m(2 + \sqrt{\pi} \cdot |s|).$$

Generally

(22) $$\varphi_n(s) = \sum_{k=0}^{n} p_k(s) \int_{-\infty}^{\infty} x(u)\, e^{-u^2} e^{ius} u^k\, du$$

that is

(23) $$\|\varphi_n(s)\| \leqslant 2m \sum_{k=0}^{n} |p_k|(|s|) \int_0^{\infty} u^k e^{-u^2} du = 2m\, P_n(|s|)$$

because

(24) $$\int_0^{\infty} u^k e^{-u^2} du = \begin{cases} \dfrac{p!}{2} & \text{if } k = 2p + 1 \\ \dfrac{\sqrt{\pi}}{2^{p+1}} (2p-1)!! & \text{if } k = 2p \end{cases}$$

where by p_k and P_n we have denoted polynomials and by $|p_k|$ the polynomial obtained from p_k taking all the coefficients in absolute value. After all, φ_n belongs to N for every natural n. The proof of the convergence of the series (19) is the same as that of [1] (by formal changes).

Remark 3. Using the generalization of Vinograd's theorem given by D. H. CARLSON in [2] it follows a similar result for local dynamical systems. The lemma may be applied also to systems for which an embedding theorem in a compact subsystem of the shift system is proved.

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