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A GEOMETRICAL APPROACH TO CONJUGATE POINT CLASSIFICATION FOR LINEAR DIFFERENTIAL EQUATIONS

by

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0. Definitions and results

If V is a vector space and $v_i \in V$, $i = 1, \ldots, m$, we shall denote by sp $(v_1, \ldots, v_m)$ the subspace of V spanned by the elements $v_1, \ldots, v_m$.

Denote by $C^n(J)$ the vector space over $\mathbf{R}$ of the real valued functions with continuous n-th derivatives on the (open, half closed or closed) interval J of the real axis.

Definition 1. *The n-dimensional linear subspace L_n in $C^n(J)$ will be said to be a Chebyshev space (CSp) if any nonzero element in L_n has at most $n-1$ distinct zeros in J. A basis of a CSp is called a Chebyshev system (CS).*

Definition 2. *The n-dimensional linear subspace L_n in $C^n(J)$ will be said to form an unrestricted Chebyshev space (UCSp) if any nonzero element of its has at most $n-1$ distinct zeros in J counting multiplicities and a basis of L_n is called an unrestricted Chebyshev system (UCS).*

Consider the differential equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_{n-1}(t)x' + p_n(t)x = 0, \tag{1}$$

where p_i are continuous real valued functions defined on $\mathbf{R}$.

D e f i n i t i o n 3. *It is said that the points $a, b \in \mathbf{R}$, $a < b$ are conjugate for the differential equation* (1), *if the space L_n of solutions of* (1) *forms an UCSp of dimension n on $[a, b)$, but this property fails on $[a, b]$.*

Remarks. (i) For a given differential equation (1), each point $a \in \mathbf{R}$ has a neighbourhood in which there are no conjugate points (see [5] p. 81, Proposition 1.).

(ii) If a and b are conjugate points for the differential equation (1), then the space L_n of solutions of it forms an UCSp also on the interval $(a, b]$ (see [5], p. 102, Theorem 8).

In all which follows we shall consider that $a = 0$ and $b = 1$ are conjugate points for the differential equation (1).

From the disconjugacy theory for the linear differential equations it follows (see for ex. [5] p. 89, Proposition 5 and pp. 98—99) that the points 0 and 1 are conjugate points for the differential equation (1) if and only if there exists a solution of (1) with a zero of multiplicity $\geqq n - k$ at 0 and a zero of multiplicity $\geqq k$ at 1 for some k, $1 \leqq k \leqq n - 1$, and there are no solutions with a similar property for two points a, b in $[0, 1)$ (or, by Remark (ii), for two points $a, b \in (0, 1]$).

D e f i n i t i o n 4. *The conjugate points 0 and 1 for* (1) *are said to be of type k $(1 \leqq k \leqq n - 1)$, if* (1) *has a solution with a zero of multiplicity $\geqq k$ at 1 and a zero of multiplicity $\geqq n - k$ at 0, and there are no solution with a similar property for $l < k$.*

THEOREM 1. *Suppose that $n \geqq 3$ and that 0 and 1 are conjugate points of the type k for the differential equation* (1). *Suppose that there exists a single solution (up to a multiplicative constant) with zero of multiplicity $n - k - 1$ at 0 and zero of multiplicity k at 1. Then the space L_n of the solutions of* (1) *form a Chebyshev space on $[0, 1]$, whose domain of definition can be extended with at most $n - 3$ distinct points with the preserving of the property of L_n to be a Chebyshev space.*

A geometrical version of a strengthened form of this theorem will be stated in the paragraph 4 of our paper.

In a recent note [14], starting from a result in [13] we have estabilished some properties of the space L_n of solutions of (1) on the closed interval $[0, 1]$, in the case when 0 and 1 are conjugate points for the differential equation (1). In [14] we have given between others a Chebyshev space of the dimension 4 defined on a closed interval, whose domain of definition can be extended exactly with a single point (with the preserving of the property to be a Chebyshev space). Our above Theorem 1 constitutes an extension of the Proposition 4 in [14] and is a contribution to the study of conjugate points, a study initiated by PH. HARTMAN [7] and A. YU. LEVIN [11]. The results in the mentioned papers concerns investigations about the analytical properties of the conjugate points [11], and a classification of the solutions in the neighbourhood of the conjugate points [7, 11], while our conjugate point classification is a contribution to the theory of Chebyshev spaces, which in the last time obtains an advance by the results of V. I. VOLKOV [17, 18], S. KARLIN and W. STUDDEN [9], R. ZIELKE [19, 20, 21], P. HADELER [8], YU. G. ABAKUMOV [1, 2], M. G. KREIN, and A. A. NUDEL'MAN [10] and the author [12, 13]. By the Theorem 1 and 2 in our paper becomes possible the constructions of some Chebyshev spaces defined on closed intervals and eventually a finite set of points outside this intervals, whose domain of definition can be extended with no point. This property implies other structural properties of the respective spaces (see [12]). (From these properties it follows that the recent constructions of R. ZIELKE [21] furnish examples of Chebyshev spaces defined on closed and halfclosed intervals, whose domain of definition cannot be extended with any point.)

It remains open the problem if the extension of the domain of definition of the CSp-s in our theorems is evermore actually possible or not.

The method applied by us requires some results from the disconjugacy theory due to G. PÓLYA [16], PH. HARTMAN [6], O. ARAMĂ [4], A. YU. LEVIN [11], for which we have already used the reference monography of W. A. COPPEL [5]. But the essential step is the use of a classical geometrical machinery which essentially is the same as that of F. NEUMAN [15] and YU. A. ABAKUMOV [1, 2] and more concretly is the extension of our method in [13] to the differentiable case.

1. Geometrical auxiliaries

The geometrical method which we use is in fact the classical theory of the differentiable curves. We shall particularize in this paragraph some aspects of this theory for our special purposes.

1. Let $x_1, \ldots, x_n$ be elements of $C^n[0, 1]$. Consider the mapping $\Phi : [0, 1] \to \mathbf{R}^n$ given by

$$\Phi(t) = (x_1(t), \ldots, x_n(t)). \tag{2}$$

In all which follows we shall consider only the case when the Wronskian $W(x_1, \ldots, x_n; t)$ is different from zero for any t in $[0, 1]$. In this case $\Phi([0, 1])$ will be the curve in $\mathbf{R}^n$ given in the parametric form

$$x^i = x_i(t),\ i = 1, \ldots, n. \tag{3}$$

This curve will be said to be the *characteristic curve* of the space sp $(x_1, \ldots, x_n)$ in $C^n[0, 1]$. It is obviously uniquelly determined up to a linear, nonsingular transformation.

Denote by $\mathbf{R}^k$ a subspace of dimension k in $\mathbf{R}^n$. We say that the curve (3) (or $\Phi([0, 1])$) has an intersection point of multiplicity l with $\mathbf{R}^k$ at the point $\Phi(t_0)$, if

$$\Phi^{(j)}(t_0) = (x_1^{(j)}(t_0), \ldots, x_n^{(j)}(t_0)) \in \mathbf{R}^k,\ j = 0, \ldots, l - 1$$

and

$$\Phi^{(l)}(t_0) = (x_1^{(l)}(t_0), \ldots, x_n^{(l)}(t_0)) \notin \mathbf{R}^k.$$

If we denote by $a_i = (a_i^1, \ldots, a_i^n)$, $i = 1, \ldots, n-k$ a basis of $\mathbf{R}^{n-k}$, the orthogonal complement of $\mathbf{R}^k$ in $\mathbf{R}^n$, then the condition to have (3) an intersection point of multiplicity l with $\mathbf{R}^k$ at $\Phi(t_0)$ may be interpreted analytically as follows:

The subspace of the space $L_n = \mathrm{sp}\,(x_1, \ldots, x_n)$ in $C^n[0, 1]$ spanned by the elements

$$a_i^1 x_1 + \ldots + a_i^n x_n, \quad i = 1, \ldots, n-k \tag{4}$$

has the property that any its element has at t_0 a zero of multiplicity at least k and it contains an element with zero of multiplicity at most k at t_0.

From this in particular it follows that the space L_n is an UCSp on $[0, 1]$ if and only if no subspace $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$ has with the curve $\Phi\,([0, 1])$ more than $n-1$ intersection points, counting their multiplicities (see also [15]).

It follows also that the space of all elements in L_n which have a zero of multiplicity at least k at t_0 is the space spanned by the elements (4), where $a_1, \ldots, a_{n-k}$ spans the space $\mathbf{R}^{n-k}$, the orthogonal complement of the space

$$\mathbf{R}^k = \mathrm{sp}\,(\Phi(t_0), \ldots, \Phi^{(k-1)}(t_0)).$$

Then if we want to determine the space of all elements in L_n which have zeros of multiplicity k_i at t_i, $i = 1, \ldots, m$, we have to consider the vectors $\Phi^{(j)}(t_i)$, $i = 1, \ldots, m$, $j = 0, \ldots, k_i - 1$, the space $\mathbf{R}^\nu$ spanned by them, the orthogonal complement $\mathbf{R}^{n-\nu}$ of this space in $\mathbf{R}^n$, a basis $a_i = (a_i^1, \ldots, a_i^n)$, $i = 1, \ldots, n-\nu$ of this space, and to consider the space spanned by the elements

$$a_i^1 x_1 + \ldots + a_i^n x_n, \quad i = 1, \ldots, n-\nu. \tag{5}$$

2. Let us consider the representation of the vector $\Phi(t)$ in the form $\Phi(t) = \Phi_1(t) + \Phi_2(t)$, where $\Phi_1(t) \in \mathbf{R}^\nu$ and $\Phi_2(t) \in \mathbf{R}^{n-\nu}$. Then, if a_i $i = 1, \ldots, n-\nu$ are the vectors determined above, we have

$$a_i^1 x_1(t) + \ldots + a_i^n x_n(t) = (a_i, \Phi(t)) = (a_i, \Phi_2(t)),\ i = 1, \ldots n-\nu, \tag{6}$$

Suppose now that a_i, $i = 1, \ldots, n-\nu$ form an orthonormal basis in $\mathbf{R}^{n-\nu}$. After a rotation (and a respective change of the basis in L_n) we may suppose that

$$a_i = (\delta_i^1, \ldots, \delta_i^n), \quad i = 1, \ldots, n-\nu, \tag{7}$$

where δ_i^j is the Kronecker-symbol.

Let us denote

$$\tilde{x}_i(t) = (a_i, \Phi_2(t)). \tag{8}$$

In what follows we shall need the characteristic curve spanned by the functions (5). From (6) and our above assumptions (7) on the vectors a_i, it follows that the characteristic curve $\Psi([0,1])$ of these functions is the projection of the curve $\Phi([0, 1])$ into $\mathbf{R}^{n-\nu}$.

3. We need also the following simple fact:

If the curve $\Phi([0, 1])$ has with $\mathbf{R}^k$ an intersection point of multiplicity l at $\Phi(t_0)$ and $\mathbf{R}^s$ is a subspace of $\mathbf{R}^k$ spanned by the vectors $\Phi(t_0), \ldots, \Phi^{(s-1)}(t_0)$, $s < l < k$, $\mathbf{R}^{n-s}$ is the orthogonal complement of $\mathbf{R}^s$ in $\mathbf{R}^n$, p denotes the projection of $\mathbf{R}^n$ onto $\mathbf{R}^{n-s}$, $\mathbf{R}^{k-s} = p(\mathbf{R}^k)$, then the tangent vector to the arc $p(\Phi([0, 1]))$ in the point $0 = p(\Phi(t_0))$ is contained in $\mathbf{R}^{k-s}$.

To verify this we consider the Taylor formula for the vector function $\Phi(t)$ in t_0 until the term $l-1$:

$$\Phi(t) = \Phi(t_0) + \frac{t-t_0}{1!}\Phi'(t_0) + \ldots + \frac{(t-t_0)^{s-1}}{(s-1)!}\Phi^{(s-1)}(t_0) +$$
$$+ \frac{(t-t_0)^s}{s!}\Phi^{(s)}(t_0) + \ldots + \frac{(t-t_0)^{l-1}}{(l-1)!}\Phi^{(l-1)}(t_0) + o_0(t-t_0)^{l-1},$$

where $o_0(t-t_0)^{l-1}$ denotes a vector with all the components functions of orders $o(t-t_0)^{l-1}$. After the application of the projector p we get

$$p\Phi(t) = \frac{(t-t_0)^s}{s!}p\Phi^{(s)}(t_0) + \ldots + \frac{(t-t_0)^{l-1}}{(l-1)!}p\Phi^{(l-1)}(t_0) + po_0(t-t_0)^{l-1}.$$

From this formula it follows that the arc $p(\Phi([0, 1]))$ has at $t = t_0$ (in the point 0) a nonessential singular point. The tangent vector to this arc in the point 0 is the first derivative vector which is different from zero, i.e., in our case will be the vector $p\Phi^{(s)}(t_0)$. Because $\Phi^{(s)}(t_0) \in \mathbf{R}^k$ we have $p\Phi^{(s)}(t_0) \in p(\mathbf{R}^k) = \mathbf{R}^{k-s}$. We observe also that $p\Phi^{(s)}(t_0)$ cannot be the zero vector.

4. We shall say that a sequence of subspaces in $\mathbf{R}^n$, of the dimension m tends to a subspace $\mathbf{R}^m$ of the same dimension, if there exist bases of each subspace in the sequence such that the sequence of the corresponding elements of the bases are tending to the elements of a basis in $\mathbf{R}^m$. The same terminology will be used, when the notion of the convergence of sequences is changed in the notion of convergence of functions. Using this terminology we have the assertion:

Let be $t_0, t_1, \ldots, t_r$, $t_i \neq t_j$, $i \neq j$, $i, j = 1, \ldots, r$, $r < n-1$ points in $[0, 1]$. Then the space

$$\mathrm{sp}\,(\Phi(t_0), \Phi(t_1), \ldots, \Phi(t_r))$$

is tending to the space

$$\operatorname{sp}\,(\Phi(t_0),\ \Phi'(t_0),\ \ldots,\ \Phi^{(r)}(t_0))$$

as $\sup\limits_{1\leqq i\leqq r} |t_i - t_0| \to 0.$

For verification let us consider the Taylor formula for $\Phi(t)$

$$\Phi(t_i) - o_i(t_i - t_0)^r = \Phi(t_0) + \frac{t_i - t_0}{1!}\,\Phi'(t_0) + \ldots + \frac{(t_i - t_0)^r}{r!}\,\Phi(r),$$

$i = 1, \ldots, r$, where $o_i(t_i - t_0)$ denotes a vector with all the components functions of order $o(t_i - t_0)^r$. Considering $\Phi^{(j)}(t)$ column vectors, we have the identity

$$\|\,\Phi(t_0),\ \Phi(t_1) - o_1(t_1 - t_0)^r, \ldots, \Phi(t_r) - o_r(t_r - t_0)^r\,\| \times$$

$$\times \begin{Vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & \frac{t_1 - t_0}{1!} & \frac{t_2 - t_0}{1!} & \ldots & \frac{t_r - t_0}{1!} \\ \vdots & \vdots & & & \\ 0 & \frac{(t_1 - t_0)^r}{r!} & \frac{(t_2 - t_0)^r}{r!} & & \frac{(t_r - t_0)^r}{r!} \end{Vmatrix} =$$

$$= \|\,\Phi(t_0),\ \Phi'(t_0),\ \ldots,\ \Phi^{(r)}(t_0)\,\|.$$

This means that

$$\operatorname{sp}\,(\Phi(t_0),\ \Phi(t_1) - o_1(t_1 - t_0)^r, \ldots, \Phi(t_r) - o_r(t_r - t_0)^r) =$$
$$= \operatorname{sp}\,(\Phi(t_0),\ \Phi'(t_0),\ \ldots,\ \Phi^{(r)}(t_0)),$$

which proves our assertion.

5. Suppose that the subspaces $\mathbf{R}_\nu^m$ tend for $\nu \to \infty$ to the subspace $\mathbf{R}_0^m$ in the above sense, and denote by $\mathbf{R}_\nu^{n-m}$, $\nu = 0, 1, \ldots$, the respective orthogonal complements. Then $\mathbf{R}_\nu^{n-m}$ tends to $\mathbf{R}_0^{n-m}$. If p_ν denotes the orthogonal projection onto $\mathbf{R}_\nu^{n-m}$, then for any $a \in \mathbf{R}^n$ we have $p_\nu a \to p_0 a$ for $\nu \to \infty$.

Let be $\mathbf{R}_\nu^m = \operatorname{sp}\,(a_{\nu 1}, \ldots, a_{\nu m})$, $\nu = 0, 1, \ldots$, and $a_{\nu i} \to a_{0i}$ for $\nu \to \infty$, $i = 1, \ldots, m$. Suppose that $a_{m+1}, \ldots, a_n$ are vectors in $\mathbf{R}^n$ such that $\mathbf{R}^n = \operatorname{sp}\,(a_{01}, \ldots, a_{0m}, a_{m+1}, \ldots, a_n)$. Then for sufficiently great ν we have also

$$\mathbf{R}^n = \operatorname{sp}\,(a_{\nu 1}, \ldots, a_{\nu m}, a_{m+1}, \ldots, a_n). \tag{9}$$

Denote by p^ν the orthogonal projection onto $\mathbf{R}_\nu^m$, $\nu = 0, 1, \ldots$. Then $p^\nu a \to p_0 a$, if $\nu \to \infty$. We have

$$id_{\mathbf{R}^n} = p^\nu + p_\nu,\quad \nu = 0, 1, \ldots, \tag{10}$$

from which it follows that

$$p_\nu a \to p_0 a \text{ for } \nu \to \infty. \tag{11}$$

For ν sfficiently great $p_\nu a_i$, $i = m+1, \ldots, n$ will be a basis of $\mathbf{R}_\nu^{n-m}$ according (9) and (10), which, together with (11) proves that $\mathbf{R}_\nu^{n-m}$ tends to $\mathbf{R}_0^{n-m}$ as $\nu \to \infty$.

2. Two dimensional Chebyshev spaces with special properties

Consider the functions x_1 and x_2 in $C^2\,[0, 1]$.

1. Suppose that $L_2 = \operatorname{sp}\,(x_1, x_2)$ is a CSp of dimension 2 on $(0, 1]$, and

(i) $x_1(0) = x_2(0) = 0$;

(ii) the tangent line in the point 0 to the characteristic curve of L_2 coincides with the axis $0x^2$;

(iii) $x_2(1) \neq 0$, $x_1(1) = 0$, i.e., the characteristic curve of L_2 meets $0x^2$ for $t = 1$.

A CSp with the properties (i), (ii) and (iii) above has the property that its domain cannot be extended with any point.

Suppose that $x_2(1) > 0$ and $x_1(t) > 0$ for $t \in (0, 1)$. Consider the function $\varphi(t) = \arctan x_2(t)/x_1(t)$. Then we have

$$\varphi(1) = \pi/2 \text{ and } \varphi(0) = \lim_{t\to 0} \varphi(t) = -\pi/2. \tag{12}$$

The first relation in (12) is obvious. To prove the second, we observe that by (ii) only the cases $\varphi(0) = \pm\pi/2$ are possible. From $x_1(t) \geqq 0$ it follows also that $-\pi/2 \leqq \varphi(t) \leqq \pi/2$ for any t in $[0, 1]$. If $\varphi(0) = \pi/2$, suppose that t_0 is the minimum point for φ. We have $-\pi/2 < \varphi(t_0) < \pi/2$, because in the case of $\varphi(t_0) = \pm\pi/2$ it would follow that the vectors $(x_1(1), x_2(1))$ and $(x_1(t_0), x_2(t_0))$ are colinear, which contradicts the fact that x_1, x_2 form a CS on $(0, 1]$. From the continuity of $\varphi(t)$ it follows that for any $t_1 \in (0, t_0)$ there exists a $t_2 \in [t_0, 1)$ such that $\varphi(t_1) = \varphi(t_2)$. This means that the vectors $(x_1(t_1), x_2(t_1))$ and $(x_1(t_2), x_2(t_2))$ are colinear for $t_1 \neq t_2$, $t_1, t_2 \in (0, 1]$, which is a contradiction (see Fig. 1. *a*). This proves the second relation in (12), i.e., the characteristic curve has the form *b* in Fig. 1.

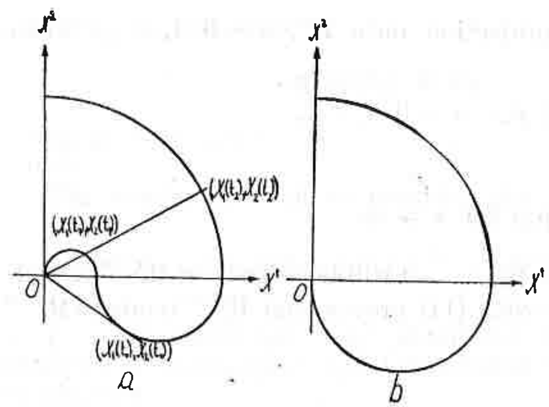

Fig. 1

Since $\varphi(0) = -\pi/2$, it follows that any straight line passing through the origin intersects the characteristic curve of L_2 in a point.

Consider any extension of the domain of definition of L_2 with a point, and the characteristic curve of the space with the extended domain. From the above conclusion it follows that there exists a line passing through the origin, which intersects the characteristic curve for two distinct values of t, i.e., the extended space cannot form a CSp.

2. Suppose that $L_2 = \mathrm{sp}\,(x_1, x_2)$ is a CSp on (0, 1) and

(i) $x_1(0) = x_2(0) = x_1(1) = x_2(1) = 0$;

(ii) the tangent lines for the characteristic curve for $t = 0$ and $t = 1$ coincide with $0x^2$.

A CSp with the properties (i) and (ii) above has the property that its domain of definition can be extended with a single point α and as extensions of x_1 and x_2 can be set $x_1(\alpha) = 0$, $x_2(\alpha) = 1$.

Suppose that $x_1(t') = 0$ for some t' in (0, 1) and that t' is the minimal value of t with this property. We have then $x_2(t') \neq 0$ and by 2.1 above it follows that L_2 forms a CSp on $(0, t']$ whose domain cannot be extended. From this contradiction it follows that $x_1(t) \neq 0$, say $x_1(t) > 0$ for $t \in (0, 1)$. The function

$$\varphi(t) = \arctan x_2(t)/x_1(t)$$

is then well defined and continuous on (0, 1) and $-\pi/2 \leqq \varphi(t) \leqq \pi/2$ $t \in (0, 1)$. By a similar argument as in 2.1 we deduce that

$$\varphi(0) = \lim_{t\to 0} \varphi(t) = -\lim_{t\to 1} \varphi(t) = -\varphi(1) = \pm\pi/2.$$

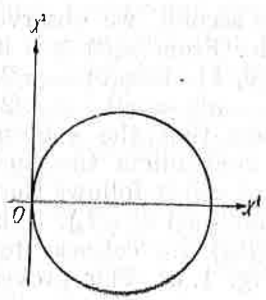

Fig. 2

From the continuity of $\varphi(t)$ it follows that any straight line passing through the origin, except $0x^2$ intersects the characteristic curve of L_2 in a point (see Fig. 2). It follows also that we may extend the domain of definition of L_2 setting for x_1 and x_2 in the point $\alpha \notin [0, 1]$ values such that $(x_1(\alpha), x_2(\alpha))$ be on $0x^2$, and the domain of the CSp obtained in this form cannot be extended.

3. Preparatory lemmas

L e m m a 1. *Suppose that* 0 *and* 1 *are conjugate points of type* k *for the differential equation* (1). *Suppose that there exists a solution* $x_1 \in L_n$ *with a zero of multiplicity* i, $i \leqq n - k$ *at the point* 0 *and a zero of multiplicity* $j \leqq k$ *at* 1, *and has* m *distinct zeros at* $t_1, \ldots, t_m \in (0, 1)$, *then there exists an other solution* $x_2 \in L_2$ *with zero of multiplicity* i *at* 0 *and zero of multiplicity* j *at* 1, *which has* m *distinct zeros in* (0, 1) *and which changes its sign passing through these zeros.*

Proof. By a result of A. YU. LEVIN (see [5], Proposition 11, p. 99), there exists an element x_0 of L with a zero of multiplicity $n - k$ at 0 and a zero of multiplicity k at 1, which is positive in (0, 1). Suppose that $t_{q1}, \ldots, t_{ql}$ are the zeros at which x_1 does not change the sign. Then there exist $[(l + 1)/2]$ zeros in the neighbourhood of which x_1 is of the same sign, say positive. Consider the solution $x = x_1 - \varepsilon x_0$, $\varepsilon > 0$. For ε sufficiently small x will have $m - l + 2[(l + 1)/2] \geqq m$ zeros in (0, 1) at which it changes the sign.

L e m m a 2. *If* 0 *and* 1 *are conjugate points of type* k *for* (1), *then the vectors*

$$\Phi(0), \ldots, \Phi^{(n-k-1)}(0), \Phi(1), \ldots, \Phi^{(k-1)}(1) \tag{13}$$

are linearly dependent, while the vectors

$$\Phi(0), \ldots, \Phi^{(n-k-1)}(0), \Phi(1), \ldots, \Phi^{(k-2)}(1) \tag{14}$$

are lineraly independent, where $\Phi^{(j)}(t) = (x_1^{(j)}(t), \ldots, x_n^{(j)}(t))$, $x_1, \ldots, x_n$ *being a fundamental system of solutions of* (1).

Proof. From the definition of the conjugate points of type k, there exists an element $x \neq 0$ in L_n with zero of multiplicity $\geqq n - k$ at 0 and with zero of multiplicity $\geqq k$ at 1 and k is the minimal number for which there exists a such x.

Let be

$$x = a^1 x_1 + \ldots + a^n x_n. \tag{15}$$

By our geometrical interpretation (see 1.1) it follows that the vectors (13) are all orthogonal to $a = (a^1, \ldots, a^n) \neq 0$, which proves the first part of the lemma.

If the vectors (14) would be linearly dependent, then the system of vectors obtained adding to (14) the vector $\Phi^{(n-k)}(0)$ would be also linearly dependent and would have a span H, a space of dimension $\leqq n - 1$. Let

$a = (a^1, \ldots, a^n)$ be a nonzero vector which is orthogonal to H. Then the element x of the form (15) would have a zero of multiplicity $\geqq n - k + 1$ at 0 and a zero of multiplicity $k - 1$ at 1, which is a contradiction.

Lemma 3. *Suppose that* 0 *and* 1 *are conjugate points of type* k *for the differential equation* (1). *Then the set of functions* x *in* L_n, *the space of solutions of* (1), *with the property that*

$$x(0) = \ldots = x^{(i-1)}(0) = x(1) = \ldots = x^{(j-1)}(1) = 0, \tag{16}$$

with $i \leqq n - k$, $j < k$ *form a* CSp L^{n-i-j} *of the dimension* $n - i - j$ *on the interval* $(0, 1)$.

Proof. We prove the lemma by contradiction. Suppose that i, $i \leqq n-k$ is the minimal number for which there exists a $j < k$ and the distinct points $t_1, \ldots, t_{n-i-j}$ in $(0, 1)$, such that there exists a nonzero solution $x = a^1x_1 + \ldots + a^nx_n$, which has a zero of multiplicity i at 0, a zero of multiplicity j at 1 and zeros at the distinct points $t_1, \ldots, t_{n-i-j}$ in $(0, 1)$. By the Lemma 1 we may suppose that x changes the sign passing through $t_1, \ldots, t_{n-i-j}$. Geometrically the existence of an x with these properties (see 1.1) means that: the vectors

$$\Phi(0), \ldots, \Phi^{(i-1)}(0), \tag{17}$$

$$\Phi(1), \ldots, \Phi^{(j-1)}(1), \tag{18}$$

$$\Phi(t_1), \ldots, \Phi(t_{n-i-j}) \tag{19}$$

are linearly dependent and the curve $\Phi([0, 1])$ is passing trough the hyperplane

$$\mathbf{R}^{n-1} = \{\xi \in \mathbf{R}^n : a^1\xi^1 + \ldots + a^n\xi^n = 0\} \tag{20}$$

at the points (19).

We shall prove that for any m, $1 \leqq m \leqq n - i - j$ the system of the vectors (17), (18) and

$$\Phi(t_1), \ldots, \Phi(\hat{t}_m), \ldots, \Phi(t_{n-i-j}) \tag{21}$$

(the symbol $\wedge$ above a term of a sequence means that the respective term is omitted) cannot be linearly independent. Suppose the contrary. Then for t' close to 1, $t' \neq t_q$, $q = 1, \ldots, n - i - j$ the system of vectors

$$\Phi(t'), \ldots, \Phi^{(j-1)}(t') \tag{22}$$

can be arbitrarily close to the system of vectors (18) and the hyperplane determined by (17), (22), (21) is arbitrarily close to the hyperplane (20)

(see 1.4 for this notion), which according our hypothesis is spanned by (17), (18) and (21). Because the hyperplane (20) has an intersection point with the arc $\Phi([0, 1])$ at $\Phi(t_m)$ and this arc pass through (20) in this point, it follows that the hyperplane through the origin spanned by (17), (22), (21) will have, for t' sufficiently close to 1, an intersection point $\Phi(t'')$ with the curve $\Phi([0, 1])$ such that $t'' \in (0, 1) \setminus \{t_1, \ldots, \hat{t}_m, \ldots, t_{n-i-j}\}$. But this means that the element

$$x_1 = b^1x_1 + \ldots + b^nx,$$

where $b = (b^1, \ldots, b^n)$ is a normal vector of the hyperplane containing the vectors (17), (22) and (21), will have a zero of the multiplicity i at 0, a zero of multiplicity j at t' and zeros at the distinct points $t_1, \ldots, \hat{t}_m, \ldots, t_{n-i-j}, t''$, that is, n zeros in $[0, 1)$, which contradicts the fact that 0 and 1 are conjugate points.

We have proved that the system of vectors (17), (18) and (21) is linearly dependent for any m, $1 \leqq m \leqq n - i - j$. For $n - i - j = 1$ this contradicts the hypothesis. Suppose $n - i - j \geqq 2$ and let be $1 \leqq m_1 < m_2 \leqq n - i - j$. Then from the linear dependence of the respective systems (17), (18), (21) of the vectors for $m = m_1$ and $m = m_2$, it follows that there exist the constants c_q^r, $r = 1, 2$ such that

$$\sum_{q=0}^{i-1} c_q^r \Phi^{(q)}(0) + \sum_{q=1}^{m_r-1} c_{q+i-1}^r \Phi(t_q) + \sum_{q=m_r+1}^{n-i-j} c_{q+i-2}^r \Phi(t_q) + \sum_{q=n-j-1}^{n-2} c_q^r \Phi^{(q-n+j+1)}(1) = 0, \tag{23$_r$}$$

where $\sum_{q=0}^{n-1} |c_q^r| \neq 0$, $r = 1, 2$. We observe that $c_{i-1}^r \neq 0$, $r = 1, 2$, because the minimality of i. Really, if contrary, say $c_{i-1}^1 = 0$, we would have that the system of the $n - 2$ vectors

$$\Phi(0), \ldots, \Phi^{(i-2)}(0), \text{ (18) and (21)} \tag{24}$$

is linearly dependent, i.e., it span a subspace of the dimension $\leqq n - 3$. Complete this system by two vectors: $\Phi(t')$ and $\Phi(t'')$, $t', t'' \neq t_q$, $q = 1, \ldots, \hat{m}, \ldots, n - i - j$. The obtained system of vectors is contained in a subspace $\mathbf{R}^{n-1}$ of dimension $n - 1$. Let $c = (c^1, \ldots, c^n)$ be a normal vector to $\mathbf{R}^{n-1}$. Then the element

$$x_2 = c^1x_1 + \ldots + c^nx_n$$

will have a zero of the multiplicity $i - 1$ at 0, a zero of multiplicity j at 1 and $n - i - j + 1$ zeros in $(0, 1)$, which is the desired contradiction.

Now, multiplying (23_2) by $-c_{i-1}^1/c_{i-1}^2$ and adding to (23_1), in the case of $|c_{i-1+m_1}^2| + |c_{i-2+m_2}^1| \neq 0$ we conclude that the system (24) of vectors is linearly dependent which yield a contradiction as above. Hence $c^2_{i-1+m_1}=0$ for any m_1, $1 \leqq m_1 < n-i-j$. This, together with (23_2) means that the system of vectors (17) and (18) is linearly dependent, $i \leqq n-k$, $j<k$. But this contradicts the Lemma 2. This last contradiction proves the lemma.

4. Proof of the theorems

In the conditions of the theorem the vectors

$$\Phi(0), \ldots, \Phi^{(n-k-1)}(0), \Phi(1), \ldots, \Phi^{(k-1)}(1) \tag{25}$$

are linearly dependent and the vectors

$$\Phi(0), \ldots, \Phi^{(n-k-1)}(0), \Phi(1), \ldots, \Phi^{(k-2)}(1) \tag{26}$$

are linearly independent (Lemma 2). From the condition that there exists a single function (up to a scalar factor) in L_n with zero of multiplicity $n-k-1$ at 0 and a zero of multiplicity k at 1, it follows also that

$$\Phi(0), \ldots, \Phi^{(n-k-2)}(0), \Phi(1), \ldots, \Phi^{(k-1)}(1) \tag{27}$$

are linearly independent vectors.

We observe that according Proposition 1 in [14] or Theorem 1 in [13], it follows by the linear independence of the system of vectors (26) or (27) that L_n is actually a Chebyshev space on [0, 1].

In what follows we shall consider two cases.

1. *The case* $k \geqq 2$. Denote

$$\mathbf{R}^{n-2} = \operatorname{sp}(\Phi(0), \ldots, \Phi^{(n-k-2)}(0), \Phi(1), \ldots, \Phi^{(k-2)}(1)) \tag{28}$$

and let be $\mathbf{R}^2$ the orthogonal complement of $\mathbf{R}^{n-2}$ in $\mathbf{R}^n$. The set of the elements in L_n which have a zero of multiplicity $n-k-1$ at 0 and a zero of multiplicity $k-1$ at 1 form, according Lemma 3, a 2-dimensional CSp on (0, 1). The characteristic curve of this CSp may be obtained according 1.2 by projection of $\Phi((0, 1))$ into $\mathbf{R}^2$. Let us consider the space

$$\mathbf{R}^{n-1} = \operatorname{sp}(\Phi(0), \ldots, \Phi^{(n-k-1)}(0), \Phi(1), \ldots, \Phi^{(k-1)}(1)). \tag{29}$$

This space will be projected by the projection p on $\mathbf{R}^2$ in the line $\mathbf{R}^1$. Because of the linear independence of the vectors (26) and (27), $\Phi^{(n-k-1)}(0)$ and $\Phi^{(k-1)}(1)$ will be projected in nonzero vectors in $\mathbf{R}^1$. This means by 1.3 that the projected curve $p\Phi((0, 1))$ has the line $\mathbf{R}^1$ as tangent line for $t=0$ and $t=1$. Because $p\Phi(0) = p\Phi(1) = 0$, it follows that L_2, the CSp on (0, 1) of the functions in L_n having zero of multiplicity $n-k-1$ at 0 and zero of multiplicity $k-1$ at 1, is in fact a CSp of the type 2.2.

Suppose that the domain of definition of the CSp defined L_n on [0, 1] may be extended with $n-2$ distinct points, say $\alpha_1, \ldots, \alpha_{n-2}$. Then the vectors $\Phi(0), \Phi(1), \Phi(\alpha_1), \ldots, \Phi(\alpha_{n-2})$ must be linearly independent and therefore at least one of them, say $\Phi(\alpha_1)$ cannot be contained in the space $\mathbf{R}^{n-1}$ defined by (29). Then the projection $p\Phi(\alpha_1)$ cannot be in $\mathbf{R}^1$, the projection in $\mathbf{R}^2$ of $\mathbf{R}^{n-1}$. Then $\operatorname{sp}(p\Phi(\alpha_1))$ will intersect the characteristic curve $\Psi((0, 1)) = p\Phi((0, 1))$ of the subspace L_2 in a point $\Psi(t_0)$, $t_0 \in (0, 1)$. According 1.4 we may choose the distinct points $t'_1, \ldots, t'_{n-k-2}$ in (0, 1) in the neighbourhood of 0 and the distinct points $t''_1, \ldots, t''_{k-2}$ in (0, 1) in the neighbourhood of 1 such that the subspace $\mathbf{R}^{n-2}_1 = \operatorname{sp}(\Phi(0), \Phi(t'_1), \ldots, \Phi(t'_{n-k-2}), \Phi(1), \Phi(t''_1), \ldots, \Phi(t''_{k-2}))$ be arbitrarily close to the subspace $\mathbf{R}^{n-2}$ given by (28). Suppose $t'_0 > t'_i$, $i = 1, \ldots, n-k-2$, $t''_0 < t''_i$, $i = 1, \ldots, k-2$, and $t_0 \in (t'_0, t''_0)$. It follows from 1.5 that the projection of $\Phi((t'_0, t''_0))$ into $\mathbf{R}^2_1$, the ortogonal complement of $\mathbf{R}^{n-2}_1$ will be arbitrarily close to the projection of $\Phi((t'_0, t''_0))$ in $\mathbf{R}^2$ and the same is true for the line $\operatorname{sp}(\Phi(\alpha_1))$. This means that we may realise that $\operatorname{sp}(\Phi(\alpha_1))$ proijected in $\mathbf{R}^2_1$ intersects the projection $p'\Phi((t'_0, t''_0))$ in $\mathbf{R}^2_1$ in a point $p'\Phi(\bar{t})$. But then $\mathbf{R}^{n-1} = p'^{-1}(p' \operatorname{sp}(\Phi(\alpha_1)))$ will contain the vectors $\Phi(\bar{t}), \Phi(\alpha_1), \Phi(0), \Phi(t'_1), \ldots, \Phi(t'_{n-k-2}), \Phi(1), \Phi(t''_1), \ldots, \Phi(t''_{k-2})$, that is, n vectors. But this contradicts the fact that L_n extended to $[0, 1] \cup \{\alpha_1\}$ is a CSp.

2. *The case* $k=1$. Let us denote $\mathbf{R}^{n-2} = \operatorname{sp}(\Phi(0), \ldots, \Phi^{n-3}(0))$ and let be $\mathbf{R}^2$ the orthogonal complement of $\mathbf{R}^{n-2}$ in $\mathbf{R}^n$. The space L_2 of all solutions of (1) with zero of multiplicity $n-2$ at 0 form a CSp of dimension 2 on (0, 1). By the linear independence of the vectors (27) (for $k=1$), it follows that the solutions having a zero of multiplicity $n=2$ at 0 cannot all vanish in the point 1. Then by Theorem 2 in [13], L_2 forms a CSp also on (0, 1]. According 1.2 the characteristic curve of L_2 can be obtained by projection of the curve $\Phi((0, 1])$ into $\mathbf{R}^2$. Let be

$$\mathbf{R}^{n-1} = \operatorname{sp}(\Phi(0), \ldots, \Phi^{(n-2)}(0), \Phi(1)). \tag{30}$$

From 1.3. and from the linear independence of the vectors $\Phi(0), \ldots, \Phi^{(n-2)}(0)$, it follows that by the projection p onto $\mathbf{R}^2$ the vector $\Phi^{(n-2)}(0)$ becomes a tangent vector to $p(\Phi((0, 1]))$ in the point 0. The support of this tangent vector is $\mathbf{R}^1 = p(\mathbf{R}^{n-1})$. But $\mathbf{R}^1$ contains also the vector $p\Phi(1)$ which cannot be zero by the linear independence of the vectors (27) for $k=1$. This means that the space L_2 defined on (0, 1] is a Chebyshev space of the type 2.1.

Suppose that the domain of definition of the CSp L_n defined on [0, 1] can be extended with $n-2$ distinct points, say $\alpha_1, \ldots, \alpha_{n-2}$. Then the vectors $\Phi(0), \Phi(1), \Phi(\alpha_1), \ldots, \Phi(\alpha_{n-2})$ must be linearly independent, and therefore at least one of them, say $\Phi(\alpha_1)$ cannot be contained in $\mathbf{R}^{n-1}$ given by (30). This means that $\operatorname{sp}(p\Phi(\alpha_1))$ will be a line passing through the origin, which is different from $\mathbf{R}^1 = p(\mathbf{R}^{n-1})$. According 2.1 this line will intersect the characteristic curve $\Psi(0, 1) = p\Phi((0, 1])$ in a point $\Psi(t_0)$ for $t_0 \in (0, 1)$, $\Psi(t_0) = p\Phi(t_0)$. Repeating a similar argument as in the case $k \geq 2$ we obtain a contradiction with the hypothesis that L_n is a CSp on $[0, 1] \cup \{\alpha_1\}$. This completes the proof.

We observe that the above method of proof works also for the following generalised form of our theorem:

TEOREMA 2. *Suppose that* 0 *and* 1 *are conjugate points of type* k *for* (1) *and that* $\Phi(0)$ *and* $\Phi(1)$ *are linearly independent. Suppose that there exists an* s, $s \geq 0$ *such that*

$$\operatorname{rank} \| \Phi(0), \ldots, \Phi^{(n-k-1)}(0), \Phi(1), \ldots, \Phi^{(k+s-1)}(1) \| =$$

$$= \operatorname{rank} \| \Phi(0), \ldots, \Phi^{(n-k-2)}(0), \Phi(1), \ldots, \Phi^{(k+s-1)}(1) \| = n - 1$$

and

$$\operatorname{rank} \| \Phi(0), \ldots, \Phi^{(n-k-2)}(0), \Phi(1), \ldots, \Phi^{(k+s-2)}(0) \| = n - 2.$$

Then the space L_n *of the solutions of* (1) *forms a* CSp *on* $[0, 1]$, *whose domain of definition can be extended with at most* $n - 3$ *distinct points.*

5. Examples

The difficulty to give concrete examples of differential equations verifying the conditions in Theorem 1 or 2 have two aspects: (i) the theorems are not of qualitative character and (ii) even in the case when we have the explicite form of the solutions, the determination of the conjugate points may be difficult. In what follows we shall give examples only in the class of equations with constant coefficients and shall illustrate how is possible in some cases to evit the concrete determination of the conjugate points.

1. Let us consider the differential equation

$$\frac{d^2}{dt^2}\left(\frac{d^2}{dt^2} + 1^2\right)\left(\frac{d^2}{dt^2} + 2^2\right) \cdots \left(\frac{d^2}{dt^2} + m^2\right) x = 0. \tag{31}$$

A fundamental system of this differential equation is the following:

$$1, t, \sin t, \cos t, \ldots, \sin mt, \cos mt. \tag{32}$$

By a result of V. I. ANDREEV [3] (see also [10], problem II. 4.1, p. 67) the system of functions (32) form a CS on $[0, 2\pi]$ whose domain of definition cannot be extended to an interval containing this closed interval as a proper subset. From this it follows that 0 and 2π are conjugate points for (31). A direct verification gives that the type of these conjugate points is $k = 2$.

For $m = 1$ we are in the conditions of the Theorem 1. Then it follows that $L_4 = \operatorname{sp}(1, t, \sin t, \cos t)$ form CSp on $[0, 2\pi]$, whose domain of definition can be extended with at most a single point. In [14] we have shown that this extension is actually possible.

For $m > 1$ we are in the conditions of the Theorem 2. Really, we have $\Phi^{(j)}(0) = \Phi^{(j)}(2\pi)$ for $j = 1, 2, \ldots$, and therefore

$$2m + 1 = \operatorname{rank} \| \Phi(0), \Phi'(0), \ldots, \Phi^{(2m-1)}(0), \Phi(2\pi) \| =$$

$$= \operatorname{rank} \| \Phi(0), \Phi'(0), \ldots, \Phi^{(2m-1)}(0), \Phi(2\pi), \Phi'(2\pi), \ldots, \Phi^{(2m-1)}(2\pi) \| =$$

$$= \operatorname{rank} \| \Phi(0), \Phi'(0), \ldots, \Phi^{(2m-2)}(0), \Phi(2\pi), \Phi'(2\pi), \ldots, \Phi^{(2m-1)}(2\pi) \|,$$

and

$$\operatorname{rank} \| \Phi(0), \Phi'(0), \ldots, \Phi^{(2m-2)}(0), \Phi(2\pi), \Phi'(2\pi), \ldots, \Phi^{(2m-2)}(2\pi) \| = 2m,$$

i.e., we have the conditions in Theorem 2 for $n = 2m + 2$, $k = 2$, $s = 2m - 2$. We conclude then that:

The space $L_{2m+2} = \operatorname{sp}(1, t, \sin t, \cos t, \ldots, \sin mt, \cos mt)$, $m \geq 1$, is a CSp on $[0, 2\pi]$ whose domain of definition can be extended with $2m - 1$ points at most. (For $m = 1$ this extension is effectively possible.)

2. Suppose that we have a differential equation (1) defined on $[0, \infty)$ for which we know that 0 has a conjugate point $< \infty$. Then, if we can verify that for any t in some neighbourhood of this conjugate point (the exact value of which isn't known) we have

$$\operatorname{rank} \| \Phi(0), \ldots, \Phi^{(i-1)}(0), \Phi(t), \ldots, \Phi^{(n-i-2)}(t) \| = n - 1 \tag{33}$$

for $i = 1, \ldots, n - 2$, then the conditions in Theorem 1 are automatically verified.

For illustration we consider the differential equation

$$\frac{d^2}{dt^2}\left(\frac{d^2}{dt^2} + 1\right)\left(\frac{d}{dt} - 1\right) x = 0. \tag{34}$$

The differential equation which corresponds to the first two factors in (34) is in fact (31) for $m = 1$, and has 0 and 2π as conjugate points. The differential equation corresponding to the third factor is disconjugate on the whole real line. This means according Proposition 8 p. 94 in [5], that (34) has two conjugate points: 0 and a point $\eta \geq 2\pi$. Because $1, t, e^t, \sin t, \cos t$ is a fundamental system of solutions, it follows that $\eta < \infty$. For $2\pi \leq t < \infty$ we can verify the conditions of the type (33).

Then from Theorem 1 it follows that there exists an η, $2\pi \leq \eta < \infty$ such that $L_5 = \mathrm{sp}\,(1, t, e^t, \sin t, \cos t)$ is a CSp on $[0, \eta]$ whose domain of definition can be extended with two distinct points at most.

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