MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 4, N° 2, 1975, pp. 153 – 156

APPROXIMATE SOLUTION OF A CAUCHY PROBLEM BY MEANS OF THE COMPUTER

by

G. PAVEL

(Cluj-Napoca)

In this paper, we intend to give a method for the approximate solution of the following CAUCHY problem:

$$(1)\qquad \begin{cases} y'' + py' + qy = 0 \\ y(a) = y_0,\ y'(a) = y \end{cases}$$

where $p,\ q \in C\ [a,\ b]$

The idea consists in attaching to problem (1) several CAUCHY problems, in which the differential equations are with constant coefficients.

We consider a partition of the interval $[a, b]$ with the knots:

$$x_0 = a,\ x_1 = a + h,\ \ldots,\ x_n = a + nh = b$$

On the interval $[x_0, x_1]$ we formulate the problem:

$$\begin{cases} y'' + p\left(\frac{x_0 + x_1}{2}\right)y' + q\left(\frac{x_0 + x_1}{2}\right)y = 0 \\ y(a) = y_0,\ y'(a) = y_0' \end{cases}$$

This problem has a unique solution which can be written effectively. We denote this solution by y_{n_1}.

In the interval $[x_1, x_2]$, we consider the following CAUCHY problem:

$$\begin{cases} y'' + p\left(\frac{x_1 + x_2}{2}\right)y' + q\left(\frac{x_1 + x_2}{2}\right)y = 0 \\ y(x_1) = y_{n_1}(x_1),\quad y'(x_1) = y_{n_1}'(x_1) \end{cases}$$

The solution of this problem exists, is unique and it can be written effectively. We denote it by y_{n_2}.

We proceed in the same way for the intervals $[x_2, x_3], \ldots, [x_{n-1}, x_n]$ and we get the corresponding functions $y_{n_3}, \ldots, y_{n_n}$.

We denote with y_n the function defined on the interval $[a, b]$ having the property that its restriction on the interval $[x_{k-1}, x_k]$ is equal to y_{n_k}, $k = 1, 2, \ldots, n$.

The function y_n, is an approximate solution of problem (1).

In what follows, we intend to estimate the error which occurs in this method of approximation.

For simplicity we make the following notations

$$L = \max \{1 + \|p\|, \|q\|\}$$

$$P_k = p\left(\frac{x_k + x_{k-1}}{2}\right)$$

$$Q_k = q\left(\frac{x_k + x_{k-1}}{2}\right)$$

P and Q are step functions defined on the interval $[a, b]$ with respect to the partition given by means of the numbers P_k and respectively Q_k.

Using the estimation given in ([1], p. 86), for each partition of the interval, we get the estimation.

$$|y(x) - y_{n_1}(x)| + |y'(x) - y'_{n_1}(x)| \leqslant \varepsilon \cdot h \cdot e^{Lh}$$

$$|y(x) - y_{n_2}(x)| + |y'(x) - y'_{n_2}(x)| \leqslant \varepsilon \cdot h \cdot e^{Lh}(1 + e^{Lh})$$

$$|y(x) - y_{n_n}(x)| + |y'(x) - y'_{n_n}(x)| \leqslant \varepsilon \cdot h \cdot e^{Lh}(1 + e^{Lh}) + \ldots + e^{(n-1)Lh})$$

where

$$\varepsilon = \max \{\|p - P_k\|_{C[a,b]}, \|q - Q_k\|_{C[a,b]}\} \cdot \|y\|_{C^1[a,b]}.$$

From the above estimation it results that

$$\|y - y_n\|_{C^1[a,b]} \leqslant \varepsilon \cdot h \cdot e^{Lh} \frac{1 - e^{nhL}}{1 - e^{hL}}.$$

We notice that the error tends to zero when h tends to zero.

E x a m p l e

We consider the CAUCHY problem

$$\begin{cases} y'' - xy = 0 \\ y(-1) = 1 \\ y'(-1) = 0 \end{cases}$$

and we construct an approximate solution corresponding to the knots -1; $-0{,}5$; 0; $0{,}5$; 1.

The computation algoritm of the approximate solution is described in the programming language FORTRAN. Next, we give this program and the corresponding approximate solution which was obtained using the computer FELIX C−256.

The solution is:

```
      DIMENSION X(20), A(20),B(20)
      P(U) = 0
      Q(U) = U
      READ(105,1)M,Y,DY,T,S
 1    FORMAT (I2,F10.5)
      MM1=M-1
      DO 60 K=1,M
 60   X(K)=T+(S-T)*(K-1)/MM1
      DO 70 K=1,MM1
      A(K)=P((X(K)+X(K+1))/2)
 70   B(K)=Q((X(K)+X(K+1))/2)
      WRITE(108,3)
 3    FORMAT(1H1,19X,2O(1H*))
      WRITE(108,4)
 4    FORMAT(1H1,19X,2OH*Y''+P(X)Y'+Q(X)Y=O*/20X,
     A 2O(1H*)/20X,1H*,6X,8HY(XO)=YO,8X,1H*/20X,
     B 1H*,6X,10HY'(XO)=Y'0,6X,1H*/20X,20/1H*))
      WRITE(108,5)
 5    FORMAT(///,18X, 'SE CONSIDERA CAZUL PARTICULAR'/18X,
     A 29(1H*)/25X,14H* Y''-XY = 0 */25X,14(1H*)/25X,
     B 11H* Y(-1)=1 */25X,11H*Y'(-1)=0 */25X,14(1H*))
      DO 10 K=1,MM1
      WRITE(108,6) A(K),B(K),K,Y,K,DY
 6    FORMAT(1H1,19X,31(1H*)/20X,5H*Y''+,F7.5,4H Y'+,F7.5,4HY=O*/20X
     A ,31(1H*)/20X,1H*,8X,3HY(X,I2,2H)=,F10.5,4X,1H*/20X,31(1H*)/20X,
     B 1H*,8X,4HY'(X,I2,2H)=,F10.5,3X,1H*/20X,31(1H*))
      D=A(K)*A(K)=4*B(K)
      IF(ABS(D).LT.0.00001) GO TO 30
      IF(D.LT.-0.00001) GO TO 20
 40   DR=SQRT(D)
      R1=(DR-A(K))/2
      R2=-(DR+A(K))/2
      ER1=EXP(R1*X(K))
      ER2=EXP(R2*X(K))
      C1=(DY-Y*R2)/(DR*ER1)
      C2=(Y*R1-DY)/(DR*ER2)
      KP1=K+1
      ER3=EXP(R1*X(KP1))
      ER4=EXP(R2*X(KP1))
      Y=C1*ER3+C2*ER4
      DY=R1*C1*ER1+R2*C2*ER4
      WRITE(108,7) C1,R1,C2,R2,KP1,Y
 7    FORMAT(1H1,19X,57(1H*)/2OX,7H*Y(X)=,F1O.5,4HEXP(,F7.5,3HX)+,
     A F10.5,3HEXP(,F7.5,4HX) */2OX,1H*,18X,3HY(X,I2,2H=,E15.9,
     B 14X,1H*/20X,57(1H*))
      GO TO 1O
 30   IF(ABS(A(K)).LT.0.00001.AND.ABS(B(K)).LT.0.00001) GO TO 50
      GO TO 40
 50   C1=DY
      C2=Y-C1*X(k)
      KP1=K+1
```

```
      Y=C1*X(KP1)+C2
      DY=C1
      WRITE(108,8) C1, C2,KP1,Y
    8 FORMAT(1H1,19X,33(1H*)/20X,6H*Y(X)=,F10.5,2HX+,HF10.5,1H*/20X,
    A 1H*,3X,3HY(X,12,4H) = ,E15.9,5X,1H*/20X,33(1H*))
      GO TO 10
   20 ALFA=-A(K)/2
      BETA=SQRT(-D)/2
      SB=SIN(BETA*X(K))
      CB=COS(BETA*X(K))
      EPA=EXP(ALFA*X(K))
   C1-((ALFA*Y-DY)*SB+BETA*Y*CB)/(BETA*EPA)
      C2=(BETA*Y*SB-(ALFA*Y-DY)1/2CB)/(BETA*EPA)
      KP1=K+1
      EPA1=EXP(ALFA*X(KP1))
      SB1=SIN(BETA*X(KP1))
      CB1=COS(BETA*X(KP1))
      Y=EPA1*(C1*CB1+C2*SB1)
      DY=EPA1*(C1*(ALFA*CB1=BETA*SB1)+C2*(ALFA*SB1+BETA*CB1))
      WRITE(108,9) ALFA,C1,BETA,C2,BETA,KP1,Y
    9 FORMAT(1H1,18X,71(1H*)/19X,10H*Y(X)=EXP(,F7.5,3HX)(,E11.6,
    A 4HCOS(,F7.5,3HX)+,E11.6,4HSIN(,F7.5,3HX)) */19X,1H*,23X,
    B 3HY(X,I2,4H) =,E15.9,22X,1H*/19X,71(1H*))
   10 CONTINUE
      STOP
      END
```

The solution is:

$$y(x) = \begin{cases} 1.18872\, e^{0.86603x} + 0.21031\, e^{-0.8663x}, & x \in (-1, -0.5) \\ 0.89855\, e^{0.5x} + 0.30796\, e^{-0.5x}, & x \in (-0.5, 0) \\ 1.20651 \cos (0.5x) + 0.39183 \sin (0.5x), & x \in (0, 0.5) \\ 1.12945 \cos (0.86603x) + 0.57373 \sin (0.86603x), & x \in (0.5, 1) \end{cases}$$

BIBLIOGRAFIE

[1] P. Pavel, A. I. Rus, *Ecuații diferențiale și integrale*. Cluj, (1973).
[2] Șt. Niculescu, *Inițiere în* FORTRAN. Edit. tehnică, București, (1972).

Received 14. III. 1975.