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INTERPOLATING SPLINE BASES

by

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(Sopot)

1. Introduction. In this paper we are working with some special systems of splines, which have been proposed by SCHONEFELD [13] to form simultaneous bases in corresponding Banach spaces of differentiable functions. For this let us recall the basic notions. The sequence $(f_n;\ n = 0, 1, \ldots) \subset X$ is said to be a (Schauder) basis of the Banach space $(X, \|\ \|)$ iff each element $f \in X$ has an expansion (convergent with respect to the norm $\|\ \|$)

$$f = \sum_{n=0}^{\infty} a_n f_n,$$

with a_n uniquely determined by the element f. We are interested in the Banach spaces $C^m(\mathbf{I}^d)$of m times continuously differentiable functions f on the d-dimensional cube $\mathbf{I}^d$, where $\mathbf{I} = \langle 0, 1\rangle$, with $m \geq 0, d \geq 1$. SCHONEFELD [13] [14] has introduced the notion of simultaneous basis for $C^m(I^d)$, i.e. such a sequence $(f_n : n = 0, 1, \ldots)$ which is a basis in each of the Banach spaces $C^k(I^d)$ for all $k = 0, 1, \ldots, m$. This notion is very useful because of the following theorem. (For details see [13] and [14], and for special cases see also [4]).

THEOREM A. *Let $F = (f_0, f_1, \ldots)$ and $G = (g_0, g_1, \ldots)$ be simultaneous bases for $C^m(I^d)$ and $C^m(I^{d'})$ respectively. Then the tensor products $(f_0 \times g_0, f_1 \times g_0, f_0 \times g_1, f_1 \times g_1, \ldots)$ suitable numbered form a* (simultaneous) *basis for $C^m(\mathbf{I}^{d+d'})$.*

The first example of a simultaneous basis for $C^1(\mathbf{I})$ was given by CIESIELSKI [2] and then used independently by CIESIELSKI [3] and SCHONEFELD [12] to give a basis in $C^1(\mathbf{I}^d)$, $d = 1, 2, \ldots$ Other examples of simultaneous bases were given by CIESIELSKI and DOMSTA [4] for $C^m(I)$ with any $m \geq 0$. One of the examples namely for $m = 1$ was investigated independently by RADECKI [8]. These systems are orthogonal with respect to the usual scalar product. They lead to orthonormal (simultaneous) bases for the spaces $C^m(I^d)$ with any value of $m \geqslant 0$, $d \geqslant 1$ and also for the corresponding Sobolev spaces $W_p^m(I^d)$, with $p \in \langle 1, \infty)$.

Moreover SCHONEFELD has constructed a simultaneous basis for the space $C^m(\mathbf{T})$, where $\mathbf{T}$ denotes the torus, for any $m \geqslant 0$ [14]. These systems are interpolating in the sense of SEMADENI [15] with respect to the diadic rationals, and they lead to interpolating (simultaneous) bases in the spaces $C^m(\mathbf{T}^d)$ with any m and d.

The last author has proposed also [13] a construction of interpolating simultaneous bases for the spaces $C^m(\mathbf{I})$. The proof given in [13] of existence of the proposed systems and of their properties depends essentially on some special sequences of matrices and was given only for $0 \leqslant m \leqslant 4$. Our purpose is to complete the proof for all values of $m \geqslant 0$, (see Theorem 1). As a consequence we may state that for any $\mathbf{M}$ of the form

$$\mathbf{M} = \mathbf{I}^d \times \mathbf{T}^{d'}$$

we have a (simultaneous) interpolating basis in $C^m(\mathbf{M})$, for $m \geqslant 0$.

The most interesting property of the considered interpolating basis in $C^{2m+2}(\mathbf{I})$ is the order of approximation of continuous functions by the nth partial sum, which is estimated from above by the modulus of smoothness of order $2m+3$, i.e. by $\omega_{2m+3}\left(f; \frac{1}{n}\right)$. It appears that a slight modification of the Schonefeld construction leads to interpolating basis of splines for $C^{2m+2}(\mathbf{I})$ which approximates with the order $\omega_{2m+4}\left(f; \frac{1}{n}\right)$. It should be noted that analogous results were obtained by SUBBOTIN [16] for the interpolating spline bases in the periodic case and by CIESIELSKI [5] for orthonormal spline bases on the interval.

Here we are interested only in the property „to be a simultaneous basis" for the systems proposed by Schonefeld and therefore their approximation properties are not considered. This will be discussed together with the modified system of interpolating splines in another paper.

The main result of this paper concerns the mentioned above family of matrices and is contained in Theorem 1. It should be noted that this theorem is very important for all investigations of the interpolating splines on the interval (non-periodic case).

2. Spline functions. Let $S = \{s_i : i \in Z\}$, with $\mathbf{Z} = \{0, \pm 1, \ldots\}$, be a set of reals numbered as an increasing sequence: $\ldots < s_{i-1} < s_i < s_{i+1} < \ldots$ without limit points. The function $f \in C^m(\mathbf{R})$ the mth derivative of which is continuous, $m \geqslant 0$, is said to be a spline function of order m (degree $m+1$) iff it is polynomial in each of the intervals $\mathbf{I}_j = \langle s_{j-1}, s_j)$. The elements of S are called knots of the splines. The set of all spline functions of order m, $m \geqslant 0$, with the same set S of knots is a linear space and is denoted here by $C_S^m(\mathbf{R})$, $\mathbf{R} = (-\infty, \infty)$. In the sequel we shall be

interested also in the spaces $C_S^m\langle a, b\rangle$, of splines restricted to the interval $\langle a, b\rangle \subset \mathbf{R}$. Obviously this space depends only on the set $S \cap (a, b)$.

The properties of splines have been investigated firstly by SCHOENBERG et al. in many of their works. We shall use only these which are contained in [1], [9—11] and [6]. The most interesting for us concern the B-splines. We define them as follows. Let $[s_0, s_1, \ldots, s_r; f(\cdot)]$ denote the divided difference of order r taken at points $s_0, s_1, \ldots, s_r$, i.e. $[s_0; f(\cdot)] = f(s_0)$, and for $r \geqslant 1$

$$[s_0, \ldots, s_r; f(\cdot)] = \frac{[s_1, \ldots, s_r; f(\cdot)] - [s_0, \ldots, s_{r-1}; f(\cdot)]}{s_r - s_0},$$

the points being different. For each $i \in \mathbf{Z}$ we define the ith B-spline of order m, corresponding to the set S of knots by the formula

$$N_i^{(m)}(t) = (s_{i+m+1} - s_{i-1})[s_{i-1}, \ldots, s_{i+m+1}; (\cdot - t)_+^{m+1}], \quad \text{for } t \in \mathbf{R} \tag{2.1}$$

where $x_+^l = (\max\{0, x\})^l$ for $l = 1, 2, \ldots$ Obviously, the fonction $N_i^{(m)}$ defined by (2.1) is a spline function for each $i \in \mathbf{Z}$ because the functions $f_s(t) = (s-t)_+^{m+1}$ are such whenever $s = s_i \in S$, for some $i \in \mathbf{Z}$.

L e m m a B (cf. [6] and also [10]). *For each $i \in \mathbf{Z}$ and each $m \geqslant 0$ we have the following relations:*

$$N_i^{(m)}(t) \geqslant 0 \text{ and } \operatorname{supp} N_i^{(m)} = \langle s_{i-1}, s_{i+m+1}\rangle, \tag{2.2}$$

$$\int_{-\infty}^{\infty} N_i^{(m)}(t)dt = (s_{i+m+1} - s_{i-1})/(m+2), \tag{2.3}$$

and for $t \in (-\infty, \infty)$ we have moreover

$$\sum_{i=-\infty}^{\infty} N_i^{(m)}(t) = 1. \tag{2.4}$$

The finite system $(N_i^{(m)};\ \operatorname{supp} N_i^{(m)} \cap (a, b) \neq \emptyset)$ is a basis in the finite dimensional space $C_S^m\langle a, b\rangle$ of splines restricted to the interval $\langle a, b\rangle \subset \mathbf{R}$.

In particular the functions $N_i^{(m)}$, $i = -m, \ldots, n$, form a basis in the $(n+m+1)$-dimensional space $C_n^m\langle s_0, s_n\rangle = C_S^m\langle s_0, s_n\rangle$.

It is obvious that the $(m+1)$ st derivative of a spline is constant in each of the intervals $I_j' = (s_{j-1}, s_j)$, and therefore we can denote by $D^{m+1}f$ the version which is continuous from the right. Let moreover D_j^{m+2} denote the jump of this derivative at s_j for $j \in \mathbf{Z}$, i.e.

$$D_j^{m+2} = D^{m+1}f(s_j) - D^{m+1}f(s_{j-1}).$$

Lemma 1. *For each $f \in C_S^m(\mathbf{R})$, the sequence D_j^{m+2} satisfies the following equations*

(2.5) $$\sum_{j\in Z} N_i^{(m)}(s_j)\, D_j^{m+2} = (m+1)\,!(s_{i+m+1} - s_{i-m-1})\,[s_{i-1}, \ldots, s_{i+m+1}\,; f(\cdot)].$$

This lemma is equivalent to the formulas (7) and (8) of [1], but it seems to have a slightly simpler form. Also the proof is very simple. For this let us notice that it is sufficient to prove it for $f(s) = f_k(s) = (s - s_k)_+^{m+1}$ for any $k \in \mathbf{Z}$. But it is trivial in view of (2.1) and the following identity

$$D_j^{m+2} f_k = (m+1)\,!\,\delta_{k,j} \quad \textit{whenever } k, j \in \mathbf{Z}.$$

The B-splines corresponding to the case of equidistant knots, i.e. whenever

$$s_j - s_{j-1} = h = \text{const.} \quad \text{for } j \in \mathbf{Z},$$

possesses very interesting properties. For completness we recall them here. Therefore we denote by $M_i^{(m)}$ the original B-splines of SCHOENBERG, i.e.

(2.6) $$M_i^{(m)}(t) = (m+2)\,[i-1, \ldots, i+m+1\,;\, (\cdot - t)_+^{m+1}],$$

for $i \in \mathbf{Z}$, $t \in \mathbf{R}$. For each $m \geqslant 0$ we have the following identity

(2.7) $$M_i^{(m)}\, t = M_{i+j}^{(m)}(t+j) = M_{m-i}^{(m)}(-t) \quad \text{for } i, j \in Z, \quad t \in \mathbf{R},$$

which denotes the translation invariance and the symmetry of the B-splines in the case of equidistant knots. The general case of $h > 0$ is connected to the case of knots at the integers with a help of the formula

(2.8) $$N_i^{(m)}(t) = M_0^{(m)}(t') \text{ with } t = s_0 + h(t' + i) \text{ for } t \in \mathbf{R}.$$

Moreover, the scalar products have the following representation (cf. [11]):

(2.9) $$\int_{-\infty}^{\infty} M_i^{(m)}(t) M_j^{(m)}(t)\,dt = M_{-m-1}^{(2m+2)}(j-i) \text{ for } i, \; j \in \mathbf{Z}.$$

According to the last three formulas and Lemma B we can state that the numbers

(2.10) $$G_l^{(m)} = M_{-m-1}^{(2m+2)}(l) \quad \text{for} \quad l \in \mathbf{Z}$$

satisfy the conditions

(2.11) $$G_{-l}^{(m)} = G_l^{(m)} \geqslant 0 \text{ and } G_l^{(m)} = 0 \text{ iff } |l| > m+1,$$

(2.12) $$\sum_{l\in Z} G_l^{(m)} = 1.$$

A non-trivial property of the numbers $G_l^{(m)}$ may be expressed by the following

Lemma C. (cf. [11], Lemma 8, cf. also [1]). *The roots $\gamma_i^{(m)}$, $i = \pm 1, \ldots, \pm(m+1)$ of the function*

(2.13) $$g^{(m)}(z) = \sum_{l=-m-1}^{m+1} G_l^{(m)} z^l \qquad \text{for } z \neq 0$$

are all simple and negative. Moreover we may enumerate them as follows

(2.14) $$\gamma_{m+1}^{(m)} < \ldots < \gamma_1^{(m)} < -1 < \gamma_{-1}^{(m)} < \ldots < \gamma_{-m-1}^{(m)} < 0.$$

In this case we have also

(2.15) $$\gamma_i^{(m)} = (\gamma_{-i}^{(m)})^{-1} \text{ for } i = \pm 1, \ldots, \pm(m+1).$$

Applying the notation (2.6) and the following ones $D_j^m = D^m f(s_j)$, $D_j^{m+1} = D^{m+1} f(s_j^-) = D^{m+1} f(s_{j-1})$, $D_j^{m+2} = D_{j+1}^{m+1} - D_j^{m+1}$, for $j \in \mathbf{Z}$ we can obtain from Lemma 1 the following

Lemma 1′. *In the case of equidistant knots,* i.e. $s_j - s_{j-1} = h$, *for $j \in \mathbf{Z}$, we have satisfied the following equations:*

(2.16) $$\sum_{j\in\mathbf{Z}} M_i^{(m)}(j) D_j^{m+2} = (m+2)\,!h\,[s_{i-1}, \ldots, s_{i+m+1}\,; f(\cdot)],$$

(2.17) $$\sum_{j\in\mathbf{Z}} M_i^{(m)}(j) D_j^{m+1} = (m+1)\,!\,[s_{i-1}, \ldots, s_{i+m}\,; f(\cdot)],$$

(2.18) $$\sum_{j\in\mathbf{Z}} M_i^{(m)}\, j \;\, D_j^m = m\,!\,[s_i, \ldots, s_{i+m}\,; f(\cdot)],$$

for all $i \in Z$ and $f \in C_S^m(\mathbf{R})$.

Proof: The equations (2.16) follow immediately from (2.5) — (2.8). For the proof of (2.17) let us restrict to the functions $f_k(s) - f_{k+1}(s)$ (see the proof of Lemma 1) for which $D_j^{m+1} = (m+1)\,!\,\delta_{j,k+1}$, i.e. the right hand side of (2.17) equals $(m+1)\,!\; M_i^{(m)}(k+1)$. On the other hand we have

$$(m+1)\,!\,[s_{i-1}, \ldots, s_{i+m}\,; (\cdot - s_k)_+^{m+1} - (\cdot - s_{k+1})_+^{m+1}] =$$

$$= (m+1)!\,(m+2)h[s_{i-1}, \ldots, s_{i+m+1}\,;\ (\cdot - s_{+1})_+^{m+1}] =$$

$$= (m+1)!\,M_i^{(m)}(k+1), \quad \text{for all } i, k \in \mathbf{Z}.$$

An analogous proof deals for (2.18).

In the sequel we shall take the order of the splines equal to $(2m+2)$, with fixed parameter $m \geqslant 0$, if nothing else is said. For this the indices m or $2m+2$ respectively will be omitted very often.

3. Some special splines. Let us notice that for $f \in C_Z^{2m+2}(\mathbf{R})$ the equations (2.17) read as follows

$$(3.1)\quad \sum_{j\in Z} G_{j-i}^{(m)} D_j^{2m+3} = (2m+3)!\,[i-m-2, \ldots, i+m+1\,;\ f(\cdot)]$$

for $i \in \mathbf{Z}$, where the notations are used

$$(3.2)\qquad D_j^{2m+3} = D^{2m+3} f(j^-) = D^{2m+3} f(j-1^+) \text{ for } j \in \mathbf{Z},$$

$$(3.3)\qquad G_l^{(m)} = M_{-m-1}^{(2m+2)}(1) = (M_0^{(m)}, M_l^{(m)}) \text{ for } l \in \mathbf{Z},$$

(cf. (2.9) and (2.10)).

Let us denote by $C_n^{2m+2} \equiv C_Z^{2m+2}\langle 0, n\rangle$ the space of splines of order $2m+2$ on the interval $\langle 0, n\rangle$, with knots at the points $0, 1, \ldots, n$, $n \geqslant 2m+3$. To investigate the Schonefeld construction of the spline basis in $C^{2m+2}(\mathbf{I})$, let us consider the subspace $C_{n,0}^{2m+2}$ of C_n^{2m+2} defined as follows. The spline f belongs to $C_{n,0}^{2m+2}$ iff the „boundary conditions" are fulfilled

$$(3.4)\qquad D_j^{2m+3} = 0 \quad \text{for} \quad j = 1, \ldots, m+1, n-m, \ldots, n.$$

It is obvious that this space $C_{n,0}$ is at least $(n+1)$-dimensional. According to (3.1) each $f \in C_{n,0}$ satisfies the equations (3.1) for $i = m+2, \ldots, n-m-1$, if we take into account (2.11) and (3.4). Let us notice that the matrix $(G_{j-i}^{(m)}\,;\ i, j = m+2, \ldots, n-m-1)$ is non-singular as the Gramm matrix for the linearly independent system of splines $(M_i^{(m)}, i = m+2, \ldots, n-m-1)$, (cf. Lemma B and (3.3)). Therefore the equations (3.1) give us a one-to-one correspondence between the values D_j^{2m+3}, $j = m+2 \ldots, n-m-1$, and the values of the differences

$$(3.5)\qquad \Delta_i^{2m+3} = [i-m-2, \ldots, i+m+1\,;\ f(\cdot)],$$

with $i = m+2, \ldots, n-m-1$.

Further, let us consider the mapping $\alpha_n : C_{n,0} \to \mathbf{R}^{n+1}$, where $(\alpha_n f)_i = f(i)$. According to the above remark the kernel $\operatorname{Ker} \alpha_n$ of this mapping is trivial. Indeed, let $f(i) = 0$ for $i = 0, 1, \ldots, n$. Then, according to (3.1) and (3.4) the $(2m+3)$rd derivative $D^{2m+3} f$ of this function equals zero

in each point of the interval $\langle 0, n\rangle$. But the unique polynomial of degree not exceeding $2m+2$ with $1+n \geqslant 2m+4$ roots $0, 1, \ldots, n$ equals zero too. As a simple corollary we obtain the following

L e m m a 2. *The space $C_{n,0}^{2m+2}$ of splines of order $2m+2$ on the interval $\langle 0, n\rangle$, which satisfy the boundary conditions (3.4) is $(n+1)$-dimensional. Each function $f \in C_{n,0}^{2m+2}$ is uniquely determined by its values $f(i)$, with $i = 0, 1, \ldots, n$, for $n \geqslant 2m+3$, $m \geqslant 0$.*

Our further purpose is to investigate the inverse matrices for the matrices $(G_{j-i}\,;\ i, j = m+2, \ldots, n-m-1)$ with any value of $n \geqslant 2m+3$. In the proof of our main result (Theorem 1 of Sec. 5) we shall deal with the notion of diagonally exponential matrices. The corresponding formalism, introduced in [7], Secs. 5 and 6, we recall here in the next section.

4. Diagonally exponential matrices. We begin with the basic dfinitions given in the following

D e f i n i t i o n D. *The family $\{M_n : n = 0, 1, \ldots\}$ of matrices $M_n = (M_{n;i,j}\,;\ i, j = 0, 1, \ldots, n)$ is said to be*

1_0. *diagonally exponential* (d.e.) *iff for all n the following estimates of the elements hold*

$$(4.1)\qquad |M_{n;i,j}| \leqslant C q^{|i-j|}, \text{ for } i, j = 0, \ldots, n,\ n \geqslant 0,$$

with some constants $C > 0$, $q \in (0, 1)$.

2_0. *of almost null rows* (a.n.r.) (or almost null columns (a.n.c.) respectively) *if for all n the numbers of elements of the following sets are uniformly estimated by a constant, i.e.*

$$(4.2')\qquad \#\ \{j : M_{n;i,j} \neq 0\} \leqslant K \quad \text{for } i = 0, \ldots, n,\ n \geqslant 0$$

(or

$$(4.2'')\qquad \#\ \{i : M_{n;i,j} \neq 0\} \leqslant K \text{ for } j = 0, \ldots, n,\ n \geqslant 0,$$

respectively), *with $K > 0$ independent of i* (or j) *and n.*

3_0. of l-shape of the second kind, with $l \in \{0, 1, \ldots\}$ iff for all $n \geqslant 0$

$$(4.3)\qquad M_{n;i,j} = \delta_{i,j} \text{ for } i = l+1, \ldots, n-l-1,\ j = 0, \ldots, n.$$

4_0. *almost diagonal* (a.d.) *iff for all n we have*

$$(4.4)\qquad M_{n;i,j} = 0 \text{ whenever } |i-j| > L,$$

ith some constant L independent of n.

Note that (4.3) implies (4.2″) and that (4.4) implies (4.2′) and (4.2″). The following lemma is not difficult to be proved (cf. [7]).

L e m m a E. *Let $\{M_n\}$ and $\{N_n\}$ be two d.e. families of matrices with the same set of indices $n = 0, 1, \ldots$ and for each n let them be of $n+1$ rows and columns, i.e. $M_n = (M_{n;i,j}\,;\, i, j = 0, \ldots, n)$ and $N_n = (N_{n;i,j}\,;\, i, j = 0, \ldots, n)$ for $n \geqslant 0$.*

(T.1) *If moreover $\{M_n\}$ is of a.n.r. or $\{N_n\}$ is of a.n.c., then the family of products $\{M_n \circ N_n : n \geqslant 0\}$, where*

$$(M_n \circ N_n)_{i,k} = \sum_{j=0}^{n} M_{n;i,j} \cdot N_{n;i,k} \text{ for } i, k = 0, \ldots, n,$$

is diagonally exponential too.

(T.2) *If the family $\{M_n\}$ is of the l-shape of the second kind, with $l \geqslant 0$, and with some positive constant $b > 0$ independent of n, the following estimate holds*

$$|\det (M_n)| \geqslant b > 0 \text{ for all } n \geqslant 0, \tag{4.5}$$

then the inverses $\{(M_n)^{-1}\}$ form a diagonally exponential family too.

The detailed proofs of the above theses are contained in [7], Lemmas 11.13 and Theorem 2.

It is useful to introduce also the notion of rotatively symmetric matrices, which means for the matrix $M = (M_{i,j}\,;\, i, j = 0, \ldots, n)$ that

$$M_{i,j} = M_{n-i,n-j} \text{ for all } i, j = 0, 1, \ldots, n. \tag{4.6}$$

Obviously, the product of two rotatively symmetric (*r.s.*) matrices is a *r.s.* matrix too. Indeed, for any pair (i, k) we have then

$$\sum_{j=0}^{n} M_{i,j} \cdot N_{j,k} = \sum_{j=0}^{n} M_{n-i,n-j} \cdot N_{n-j,n-k} \equiv (M \circ N)_{n-i,n-k},$$

whenever the matrix $N = (N_{j,k}\,;\, j, k = 0, \ldots, n)$ is also r.s.

Note also that the d.e. property (4.1) leads to the following uniform estimate of the sums

$$\sum_{j} |M_{n;i,j}| \geqslant C \cdot (1-q)^{-1} \text{ for } i = 0, \ldots, n, \; n \geqslant 0. \tag{4.7}$$

As a simple corollary of (4.7) we easily obtain the following esti ates of the matrix norms

$$\|M_n\| = \sup_{X \neq 0} \frac{\max_i \left|\sum_j M_{n;i,j} X_j\right|}{\max_i |X_i|} \leqslant C \cdot (1-q)^{-1} \tag{4.8}$$

for all $n \geqslant 0$, whenever (4.1) is satisfied.

5. The inverse matrices for $G_n^{(m)}$. Our purpose is to investigate the following inverse matrices

$$M_n^{(m)} = (G_n^{(m)})^{-1} \quad \text{for } n = 0, 1, \ldots, \tag{5.1}$$

where the matrices $G_n^{(m)}$ are given as follows

$$G_{n;i,j}^{(m)} = G_{j-i}^{(m)} = G_{i-j}^{(m)} \quad \text{for } i,j = 0, \ldots, n, \tag{5.2}$$

and the numbers $G_n^{(m)}$ are given in Sec. 2 (cf. also (3.3)).

According to the properties of the numbers $G_l^{(m)}$ given in Sec. 2 (cf. Lemma B) we can state the following

L e m m a 3. *The family $\{G_n^{(m)};\; n \geqslant 0\}$ is almost diagonal* (cf. (4.4)) *and hence of almost null columns and of almost null rows. Their elements are uniformly bounded and hence the family is diagonally exponential. Moreover each $G_n^{(m)}$, $n \geqslant 0$, is non-singular, symmetric and rotatively symmetric.*

Now we are ready to prove the main restul.

THEOREM 1. *The inverse matrices $M_n^{(m)} = (G_n^{(m)})^{-1}$ with fixed $m \geqslant 0$, form a diagonally exponential family, i.e. (cf. (4.1)) with some constants $C > 0$* $0 < q < 1$, independent of n, the following estimates hold for the inverse, matrices elements

$$|M_{n;i,j}^{(m)}| < C \cdot q^{|i-j|} \text{ for } i,j = 0, \ldots, n, \; n \geqslant 0. \tag{5.3}$$

Proof: It is obvious that the d.e. property does not depend on a finite number of elements of the family $\{M_n^{(m)}\}$. Thus it is sufficient to prove (5.3) for large enough n. For this let us take $n \geqslant 2m+1$. Let us introduce moreover the following matrices $C_n = (C_{n;j,k}\,;\, j, k = 0, \ldots, n)$, where

$$C_{n;j,k} = \begin{cases} (\gamma_{k-m-1}^{(m)})^{j} & \text{whenever } k = 0, \ldots, m, \\ H_{j,k} = H_{k-j} & \text{whenever } k = m+1, \ldots, n-m-1, \\ (\gamma_{n-m-k-1}^{(m)})^{n-j} & \text{whenever } k = n-m, \ldots, n. \end{cases} \tag{5.4}$$

In this formula $\gamma_l^{(m)}$, $l = -1, -2, \ldots, m-1$, denote the roots of the function $g^{(m)}(z)$ (see Lemma C). The numbers $H_l^{(m)}$ are the coefficients of the Laurent expansion

$$h^{(m)}(z) = \sum_{-\infty}^{\infty} H_l^{(m)} z^l \tag{5.5}$$

of the function

(5.6) $$h^{(m)}(z) = (g_{(m)}(z))^{-1},$$

defined in the annulus $r \leqslant |z| \leqslant r^{-1}$, where $r \in (\gamma_{-1}^{(m)}, 1)$ (see (2.14)).

Lemma 4. *The family of matrices $\{C_n : n \geqslant 2m+1\}$ is d.e.*

The proof of this lemma is based on the following estimate for the Laurent coefficients $H_l^{(m)}$, which may be obtained from the Cauchy formula

$$|H_l^{(m)}| \leqslant C_H \cdot q_H^{|l|}, \text{ for all } l \in \mathbf{Z},$$

with constants $C_H > 0$ and $q_H \in (0, 1)$ depending only on m.

Now let us apply the Cauchy's formula for sums of products of two absolutely convergent Laurent series. We obtain then from (5.6)

(5.7) $$\sum_{j \in \mathbf{Z}} G_{j-i}^{(m)} H_{k-j}^{(m)} = \delta_{i,k}, \text{ for any pair } i, k \in \mathbf{Z}.$$

This leads to the first part of the following

Lemma 5. *The matrices D_n defined as follows*

(5.8) $$D_n = G_n \cdot C_n \text{ for } n \geqslant 2m+1,$$

form a d.e. family, which is moreover of m-shape of the second kind (see (4.3)). *For large enough n, the determinants are estimated from below*

(5.9) $$|\det (D_n)| \geqslant b > 0,$$

by a constant $b > 0$ independent of n.

Proof: The required property (4.3), i.e.

(5.10) $$D_{n;i,k} = \delta_{i,k}$$

for $i, k = m+1, \ldots, n-m-1$ follows from (5.7) and for $i = m+1, \ldots, n-m-1$, $k = 0, \ldots, m, n-m, \ldots, n$, it follows from the defition of the roots $\gamma_l^{(m)}$ and the relations (2.15). The d.e. property of the family $\{D_n\}$ follows now from Lemmas 3,4 and (T.1) of Lemma E.

Now we are going to prove that (5.9) holds for large enough values of n. First let us notice that D_n is rotatively symmetric and therefore

(5.11) $$\det (D_n) = \det (D_{n(a;a)})^2 + O(q_D^n),$$

as $n \to \infty$ where $q_D \in (0, 1)$ is this number which expresses the d.e. property of the family $\{D_n\}$. Moreover we have used the following convention of notation for the submatrices

$$D_{n(\alpha;\beta)} = (D_{n;i,j} \; ; \; i \in \mathbf{J}_\alpha, \; j \in \mathbf{J}_\beta) \text{ for } \alpha, \beta = a, b, c,$$

with $\mathbf{J}_a = \{0, \ldots, m\}$, $\mathbf{J}_b = \{m+1, \ldots, n-m-1\}$, $\mathbf{J}_c = \{n-m, \ldots, n\}$.

It should be noted that in the detailed proof of (5.11) we have used the fact that $\{D_n\}$ is of m-shape of the second kind.

On the other hand for each $n \geqslant 2m+1$ we can state the following

(5.12) $$D_{n(a;a)} = D_{2m+1(a;a)} = G_{2m+1(a;a)}^{(m)} \circ C_{2m+1(a;a)} + G_{2m+1(a;c)}^{(m)} \circ C_{2m+1(c;a)}.$$

Thus, according to (5.11) we are needed to prove only that the submatrix $D_{2m+1(a;a)}$ is non-singular. For the convenience we shall deal without the subindices $2m+1$ to the end of this lemma.

Obviously the submatrix $C_{(a;a)}$ is non-singular as the Vandermonde determinant for (different) numbers $\gamma_{-m-1}^{(m)}, \ldots, \gamma_{-1}^{(m)}$. Also the submatrix $C_{(c;a)} = C_{2m+1(c;a)}$ is non-singular because of the formula

(5.13) $$C_{(c;a)} = E_{(c;a)} \circ C_{(a;a)} \circ \Gamma,$$

where the notationes are used

$$E_{(c;a)} = (\delta_{j-m-1,k} \; ; \; j = m+1, \ldots, 2m+1, \; k = 0, \ldots, m),$$
$$\Gamma = ((\gamma_{k-m-1}^{(m)})^{m+1} \cdot \delta_{j,k} \; ; \; j, k = 0, \ldots, m).$$

The property (see Lemma C for definition of $\gamma_i^{(m)}$)

$$\sum_{l=-m-1}^{m+1} G_l^{(m)} (\gamma_i^{(m)})^l = 0 \text{ for } i = -1, \ldots, -m-1.$$

may now be written as follows

(5.14) $$(E_{(c;a)})^T \circ G_{(c;a)} \circ C_{(a;a)} \circ (\Gamma)^{-1} + G_{(a;a)} \circ C_{(a;a)} + \\ + G_{(a;c)} \circ E_{(c;a)} \circ C_{(a;a)} \circ \Gamma = \mathbf{0},$$

where $\mathbf{0}$ denotes the null matrix $(o_{i,j} = 0 \; ; \; i, j = 0, \ldots, m)$. Indeed, the (i, j) th element of the left hand side matrix equals

$$L_{i,j} = \sum_{p,q,r} \delta_{p-m-1,i} G_{q-p} (\gamma_{r-m-1})^{q-m-1} \delta_{r,j} + \sum_q G_{q-i} (\gamma_{j-m-1})^q + \\ + \sum_{p,q,r} G_{p-i} \delta_{p-m-1,q} (\gamma_{r-m-1})^{q+m+1} \delta_{r,j} = 0,$$

for $i, j \in \mathbf{J}_a = \{0, \ldots, m\}$, the indices q, r running over $\mathbf{J}_a$ and the index p over $\mathbf{J}_c = \{m+1, \ldots, 2m+1\}$. Comparing (5.12), (5.13) and (5.14) we obtain that the submatrix $D_{n(a;a)}$ may be written as a product of non-singular matrices

$$D_{n(a;a)} = -(E_{(c;a)})^T \circ G_{(c;a)} \circ C_{(a;a)} \circ (\Gamma)^{-1},$$

because $G_{(c;a)}$ is a trigonal matrix with positive elements on the main diagonal (comp. Lemma B for $N_i^{(2m+2)}$ and (2.11)), q.e.d.

As a simple corollary of Lemma 5 and (T. 2) of Lemma E we get that for large enough values of n the inverses $(D_n)^{-1}$ exist and they form a diagonally exponential family which is moreover of the m-shape of the second kind. For these values of n we can write (cf. 5.1))

$$M_n^{(m)} = C_n \circ (G_n^{(m)} \circ C_n)^{-1} = C_n \circ (D_n)^{-1}.$$

Applying to the obtained formula the properties of the matrices C_n given in Lemma 4 and the properties of the inverses $(D_n)^{-1}$ given above we get with a help of (T.1) of Lemma E that the family $\{M_n^{(m)}\} = \{(G_{n_n}^{(m)})^{-1}\}$ is diagonally exponential too, q.e.d.

6. Construction of the bases. For each $N = 2^{\mu} \geqslant 2m + 3$ let us denote by $C_{N,0}^{2m+2}$ (**I**) the space of splines of order $2m + 2$ (degree $2m + 3$) with knots at

$$s_{N,i} = \frac{i}{N} \quad \text{for } i = 0, 1, \ldots, N,$$

and satisfying the boudary conditions (cf. (3.4))

$$D_j^{2m+3} = 0 \quad \text{for } j = 1, \ldots, m+1, \; n-m, \ldots, n,$$

where the notation is introduced

$$D_j^{2m+3} = D^{2m+3} f(s_{N,j}) \quad \text{for } j = 1, \ldots, N.$$

It is not difficult to obtain the correspondence between the just introduced spaces $C_{N,0}^{2m+2}$ (**I**) and the spaces $C_{N,0}^{2m+2}$ investigated in Sec. 3. In particular the formula (3.1) and Lemma 2 deal if some suitable changes are made, e.g. there is a one-to-one correspondence between the elements f of $C_{N,0}^{2m+2}$ (**I**) and their values $f(s_{N,i})$, $i = 0, \ldots, N$. Moreover the equations

$$(6.1) \qquad \sum_{j=m+2}^{N-m-1} G_{j-i}^{(m)} D_j^{2m+3} = (2m+3)!\,[s_{N,i-m-2}, \ldots, s_{N,i+m+1};\, f(\cdot)],$$

are satisfied with $i = m + 2, \ldots, N - m - 1$, for $N \geqslant 2m + 3$.

Applying the above notations we can define the nth diadic point of the interval **I** as follows

$$t_n = \begin{cases} n & \text{for } n = 0, 1, \\ s_{2N, 2\nu-1} & \text{for } n = 2, 3, \ldots \end{cases}$$

where we define N and ν as the unique numbers satisfying $n = N + \nu$ $N = 2^{\mu}$, with integers $\mu \geqslant 0$, $1 \leqslant \nu \leqslant N$, for $n \geqslant 2$.

Following Schonefeld we can define now the proposed bases. Let the parameter m be a fixed non-negative integer. The functions $\varphi_n^{(m)} \in C^{2m+2}$ (**I**) are defined as follows

1° For $0 \leqslant n \leqslant 2m + 2$ $\varphi_n^{(m)}$ is the unique polynomial of degree not exceeding n such that

$$(6.2) \qquad \varphi_n^{(m)}(t_i) = \delta_{i,n} \quad \text{for } i = 0, \ldots, n, \quad 0 \leqslant n \leqslant 2m + 2.$$

2° For $n \geqslant 2m + 3$ $\varphi_n^{(m)}$ is the unique element of $C_{2N,0}^{2m+2}$ (**I**) with the values

$$(6.2') \qquad \varphi_n^{(m)}(t_i) = \delta_{i,n} \quad \text{for } i = 0, \ldots, 2N, \; n \geqslant 2m + 3.$$

Now we are ready to complete the proof of the following theorem for all values of m.

THEOREM 2. *For each $m \geqslant 0$ the sequence $(\varphi_n^{(m)}, n \geqslant 0)$ forms an interpolating simultaneous basis for the Banach space C^{2m+2} (**I**), i.e. it is a basis in each of the spaces C^k (**I**), with any $k = 0, \ldots, 2m + 2$.*

We shall repeat here only the proof originaly given by Schonefeld [13] for $m = 0, 1$, and proposed for other values of m. Therefore we shall omit some details, for which we refer to [13] and [14].

Let us denote by

$$S_n f = \sum_{i=0}^{n} a_i(f)\varphi_i^{(m)} \quad \text{for } n = 0, 1, \ldots$$

the nth partial sum of the expansion with coefficients

$$a_n(f) = \begin{cases} f(t_0) & \text{for } n = 0 \\ f(t_n) - S_{n-1} f(t_n) & \text{for } n \geqslant 1, \end{cases}$$

for any $f \in C$(**I**). It is obvious that $(a_n, n \geqslant 0)$ is the unique sequence of coefficients for which the necessary condition $\lim S_n f(t_k) = f(t_k)$ for any $k \geqslant 0$ is fulfilled, because according to (6.2) and (6.2') $S_n f(t_k) = S_k f(t_k)$ for $n \geqslant k$. Thus the nth partial sum $S_n f$ interpolates f at the first points of the diadic sequence $t_0, t_1, \ldots, t_n$, for $n \geqslant 0$.

Further it is obvious that the operators S_n defined as above form an increasing sequence of projections defined on C(**I**). The image of S_N for $N = 2^{\mu} \geqslant 2m + 3$ is just the space $C_{N,0}^{2m+2}$(**I**). Therefore we want to prove now that the sum of subspaces

$$S^{2m+2}(\mathbf{I}) = \bigcup_{N \geqslant 0} C_{N,0}^{2m+2}(\mathbf{I})$$

where $N = 2^{\mu}$ is dense in the Banach space C^{2m+2} (**I**). Indeed, the $(2m + 2)$ nd derivatives of the elements of S^{2m+2} (**I**) form a dense set in the space of continuous functions as all polygonals with a dense set of break-points.

Therefore we can approximate any $f \in C^{2m+2}(\mathbf{I})$ by the following element of $S^{2m+2}(\mathbf{I})$

$$\tilde{f}(t) = \sum_{k=0}^{2m+1} \frac{D^k f(0)}{k!} t^k + H^{2m+2} g(t),$$

where g is the polygonal which approximates the derivative $D^{2m+2} f$, and

$$Hg(t) = \int_0^t g(s)ds \text{ for } t \in \langle 0, 1\rangle.$$

As a further consequence we obtain that $S^{2m+2}(\mathbf{I})$ is dense in each of the spaces $C^k(\mathbf{I})$ with $k = 0, \ldots, 2m+2$. Now, it is sufficient to prove that the norms of the projections S_n are uniformly bounded for $n \geqslant 0$, if we consider them as operators defined on the spaces $C^k(\mathbf{I})$ with any $k = 0, \ldots, 2m+2$. In the case of $n = N = 2^\mu \geqslant 2m+3$ it is sufficient to prove that the following estimation

$$\|D^k S_N f\| \leqslant K \max_{0 \leqslant i \leqslant N-k} |[s_{N,i}, \ldots, s_{N,i+k}; f(\cdot)]| \tag{6.3}$$

holds for each $f \in C(\mathbf{I})$, $k = 0, \ldots, 2m+2$, with a constant K depending only on m. But in fact this statement concerns the spline $S_N f$, because $f(s_{N,i}) = S_N f(s_{N,i})$ for $i = 0, 1, \ldots, N$. By a standard computation presented e.g. by Schonefeld [13] it is sufficient to prove the following

L e m m a 7. *The norms* $\|M_n^{(m)}\|$ *of the inverse matrices for* $G_n^{(m)}$ (see Sec. 2) *are uniformly bounded by a constant depending only on* m.

But this lemma is an immediate consequence of Theorem 1 and formulas (4.8).

To obtain the connection between Lemma 7 and (6.3) let us rewrite the equations (6.1) in the form

$$D_j^{2m+3} = D^{2m+3} S_N f(s_{N,j} -) = \sum_{i=m+2}^{N-m-1} (2m+3)!\, M_{N;j,i}^{(m)} \Delta_i^{2m+3}$$

where $\Delta_i^{2m+3} = [s_{N,i-m-2}, \ldots, s_{N,i+m+1}; S_N f(\cdot)]$, $i = m+2, \ldots, N-m-1$. According to Lemma 7 we obtain that

$$\max_j |D_j^{2m+3}| \leqslant K \max_i |\Delta_i^{2m+3}| \text{ for } N = 2^\mu \geqslant 2m+3,$$

where the subindices i and j run over the set $m+2, \ldots, N-m-1$, with a constant K depending only on m. In the proof of (6.3) we need only the generalized Lagrange formula: If $D^k f$ is continuous in the interval $\langle s_0, s_k\rangle$ then there exist a point $s \in \langle s_0, s_k\rangle$ such that $\frac{1}{k!} D^k f(s) =$

$= [s_0, \ldots, s_k; f(\cdot)]$ whenever $s_0 < s_1 < \ldots < s_k$. This should be applied for $S_N f$. We shall demonstrate the application in the proof of Lemma 8 below.

The other cases of $n = N + \nu$, wher $1 \leqslant \nu < N$, may be reduced to the case of $n = N$, but we shall omit the consideration. For details the reader is referred to the works of Schonefeld [13] and [14].

This section we want to finish with the following

L e m m a 8. *For each* $n \geqslant 2m+3$ *the functions* $\varphi_n^{(m)}$ *defined in* Sec. 6 *may be estimated as follows*

$$|D^k \varphi_n^{(m)}(t)| \leqslant C q^{n|t-t_n|} n^k \text{ for all } t \in I,\ k = 0, \ldots, 2m+2,\ n \geqslant 2m+3,$$

with constants $C > 0$, $q \in (0, 1)$ *depending only on* m.

Proof: Let us notice that according to (6.1) we can write

$$D_j^{2m+3} \equiv D^{2m+3} \varphi_n^{(m)}(s_{2N,j} -) = \sum_{i=m+2}^{2N-m-1} (2m+3)!\, \Delta_i^{2m+3} M_{2N;j,i}^{(m)},$$

where

$$\Delta_i^{2m+3} = [s_{2N,i-m-2}, \ldots, s_{2N,i+m+1}; \varphi_n^{(m)}(\cdot)] \text{ for } i = m+2, \ldots,$$
$$2N - m - 1, \quad j = m+2, \ldots, 2N-m-1.$$

According to (6.2') $\Delta_i^{2m+3} = 0$ for $|i - (2\nu - 1)| \geqslant m+2$. Applying Theorem 1, we can estimate the numbers D_j^{2m+3} as follows

$$|D_j^{2m+3}| \leqslant \sum_{i=i_0}^{i_1} C q^{|i-j|} |[s_{2N,i-m-2}, \ldots, s_{2N,i+m+1}; \varphi_n^{(m)}(\cdot)]| \leqslant$$
$$\leqslant C' q^{|2\nu-1-j|} N^{2m+3} \text{ for } j = m+2, \ldots, 2N-m-1,$$

where the constant C' is independent of n and $q \in (0,1)$ is independent on n too. But the $(2m+3)$ rd derivative is constant in each of the intervals $(s_{2N,j-1}, s_{2N,j})$ and moreover (3.4) holds. Therefore for each $t \in \mathbf{I}$ if we choose j such that

$$s_{2N,j-1} \leqslant t < s_{2N,j}$$

and if we denote the continuous from the right version of the $(2m+3)$ rd derivative by $D^{2m+3}\varphi_n^{(m)}$ then according to the estimation

$$|2\nu - 1 - j| \leqslant \frac{1}{2} n |t - t_n| + C'',$$

with C'' independent of n the inequality

$$|D^{2m+3} \varphi_n^{(m)}(t)| \leqslant C q^{n|t-t_n|} n^{2m+3}$$

holds with $C > 0$, $q \in (0, 1)$ independent of n. The lemma follows now by induction with respect to k. Let us suppose namely that the Lemma holds for some $k = 1, 2, \ldots, 2m+3$. Because the $(k-1)$st derivative of $\varphi_n^{(m)}$ is continuous in **I** then we have

$$D^{k-1}\varphi_n^{(m)}(t) = D^{k-1}\varphi_n^{(m)}(t') + \int_{t'}^{t} D^k\varphi_n^{(m)}(s)ds,$$

where according to the Lagrange formula t' may be choosen in such a way that

$$D^{k-1}\varphi_n^{(m)}(t') = (k-1)!\,[s_{2N,i}, \ldots, s_{2N,i+k};\ \varphi_n^{(m)}(\cdot)] = 0$$

and $|t'-t| \leqslant \frac{k}{N}$. By the hypothesis we obtain then

$$|D^{k-1}\varphi_n^{(m)}(t)| \leqslant k. \max_{t' \leqslant s \leqslant t} \frac{|D^k\varphi_n^{(m)}(s)|}{N}$$

which leads after a simple reformulation to the thesis for $k-1$.

7. Final remarks. Applying Theorem 1 and Lemma 1 we are able moreover following e.g. SUBBOTIN [16] to obtain the approximative properties of the applied splines. We shall not give them here because we have find that the Schonefeld construction slightly modified leads to systems which possesses something better properties (for the order of approximation cf. Introduction).

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