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THE CONVEXITY OF INTERVAL-FUNCTIONS

by

MIRCEA OCTAVIAN GURZĂU
(Cluj-Napoca)

1. In this paper we extend the notion of convexity to functions with interval-values. Denote with $\mathbb{R}$ the set of real numbers, with $I(\mathbb{R})$ the set of closed intervals of $\mathbb{R}$, with $\underline{a}, \underline{b}, \ldots$ the left extremities of intervals, with $\bar{a}, \bar{b}, \ldots$ the right ones. On $I(\mathbb{R})$ we shal consider the usual operations from interval-arithmetics (see [1], [2]), defined by the next formulas:

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]$$

$$[\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}]$$

$$[\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min \{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}, \max \{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}]$$

$$[\underline{a}, \bar{a}] : [\underline{b}, \bar{b}] = [\underline{a}, \bar{a}] \cdot \left[\frac{1}{\bar{b}}, \frac{1}{\underline{b}}\right], \qquad 0 \notin [\underline{b}, \bar{b}]$$

Also we shall organize the set $I(\mathbb{R})$ as a metric space, with Haussdorff's metric, which in this case is given by:

(1) $$\rho([\underline{a}, \bar{a}], [\underline{b}, \bar{b}]) = \max \{|\underline{a} - \underline{b}|, |\bar{a} - \bar{b}|\}.$$

A function $f : E \to I(\mathbb{R})$ $(E \subset \mathbb{R})$ is called I-function (Interval-function). We attach to this function two real functions $\underline{f}$ and $\bar{f}$ (defined on E):

(2) $$\underline{f}(x) = \min_{t \in f(x)} \{t\}, \quad x \in E$$
$$\bar{f}(x) = \max_{t \in f(x)} \{t\}, \quad x \in E$$

The connection between the continuity of the I-function f, according to the metric (1), and the continuity of the real functions $\underline{f}$ and $\bar{f}$ is given by:

THEOREM 1. *The I-function f is continuous iff $\underline{f}$ and $\bar{f}$ are continuous. (The continuity of the real functions $\underline{f}$ and $\bar{f}$ is the usual one)*

On $I(\mathbf{R})$ we consider the following ordering:

(3) $$[\underline{a}, \bar{a}] \leqslant_I [\underline{b}, \bar{b}] \text{ iff } \underline{a} \leqslant \underline{b} \text{ and } \bar{a} \leqslant \bar{b}.$$

The notation $[x, y; \underline{f}]$ and $[x, y; \bar{f}]$ stands for the usual difference quotients on the points $x, y \in E$ (see [3]) of the functions $\underline{f}$ and $\bar{f}$.

The difference quotient of an I — function is given by:

Definition 1. *The difference quotient of I-function f on the points $x, y \in E$ is an interval, denoted with $D(x, y; f)$, given by:*

(4) $$D(x, y; f) = [\min\{[x, y; \underline{f}], [x, y; \bar{f}]\}, \max\{[x, y; \underline{f}], [x, y; \bar{f}]\}]$$

2. In analogy with the classical definition of convexity for the real functions of a real variable by an inequality (see [3]) we give:

Definition 2. *The I-function $f: E \to I(\mathbf{R})$ is non-concave on E if for every points $x, y \in E$ and for every $\lambda \in [0,1]$ the folowing inequalities hold:*

(5) $$f(x + (1 - \lambda)y) \leqslant_I f(x) + (1 - \lambda)f(y).$$

Using (3) we formulate:

THEOREM 2. *The I-fonction $f: E \to I(\mathbf{R})$ ($E \subset \mathbf{R}$, E convex set) is non-concave on E iff the functions $\underline{f}$ and $\bar{f}$ are non-concave on E.*

For example, the I-fonction $f: \mathbf{R} \to I(\mathbf{R})$ defined by:

(6) $$f(x) = [1, 2](1 + x^2), \ x \in \mathbf{R}$$

is not a non-concave function.

The disadvantage of definition 2. consists in the fact that simple I-function, like $g: \mathbf{R} \to I(\mathbf{R})$, given by:

(7) $$g(x) = [1, 2](x^2 - 1), x \in \mathbf{R},$$

is not a non-concave, function on $\mathbf{R}$.

Thus we give another definition of the I-function convexity, similar to that of a real function of real variable, using difference quotients (see [3]):

Definition 3. *The I-function $f: E \to I(\mathbf{R})$ is l-non-concave on E if for every $x \in E$ there is a neighbourhood V of the point x so that for every $y_1, y_2 \in V \cap E$ with $y_1 < y_2$ the folowing relation is satisfied:*

(8) $$D(x, y_1; f) \leqslant_I D(x, y_2; f).$$

For example the I-fonction g, given by (7), is l-non-concave. A connection between the last two definitions is given by:

THEOREM 3. *If the I-function $f: E \to I(\mathbf{R})$, ($E \subset \mathbf{R}$, E convex set) is non-concave on E, then it is also l-non-concave on E.*

Proof. Let $x \in E$. As neighbourhood V of x we consider the set $\mathbf{R}$. In the set $E = E \cap V = E \cap \mathbf{R}$ we consider two subsets E_1 and E_2, defined as folows:

(9) $$y \in E_1 \text{ if } [x, y; \underline{f}] \leqslant [x, y; \bar{f}]$$

(10) $$y \in E_2 \text{ if } [x, y; \bar{f}] \leqslant [x, y; \underline{f}]$$

We notice that $E_1 \cup E_2 = E - \{x\}$. Let $y_1, y_2 \in E$, $y_1 < y_2$. If $y_1, y_2 \in E_1$ according to the theorem 2.:

$$[x, y_1; \underline{f}] \leqslant [x, y_2; \underline{f}], \ [x, y_1; \bar{f}] \leqslant [x, y_2; \bar{f}]$$

Thus, taking into account (4) $D(x, y_1; f) \leqq D(x, y_2; f)$. The case $y_1, y_2 \in E_2$ is considered in the same way. If $y_1 \in E_1$, $y_2 \in E_2$ then, according to the theorem 2, the formulas (9) and (10):

(11) $$[x, y_1; \underline{f}] \leqslant [x, y_1; \bar{f}] \leqslant [x, y_2; \bar{f}].$$

(12) $$[x, y_1; \underline{f}] \leqslant [x, y_2; \underline{f}] \leqslant [x, y_2; \bar{f}]$$

From (11) and (12), taking into account (3) and (4), results (8). The following theorem shows the connection between l-nonconcavity and continuity:

THEOREM 4. *If the I-function $f: E \to I(\mathbf{R})$ is l-non-concave on E, then it is continuous on the interior of E.*

Proof: Let $x \in \operatorname{Int} E$ (the interior of E). Then there is a neighbourhood U of x with $U \subset E$. According to the definition 3, there is a neighbourhood V of x where (8) holds. $U \cap V$ being a neighbourhood of x, there is an $\varepsilon > 0$ so that $[x - \varepsilon, x + \varepsilon] \subset U \cap V$. From (8) results that for every $y \in [x - \varepsilon, x + \varepsilon)]$ we have:

(13) $$M_1 \leqslant [x, y; \underline{f}] \leqslant M_2$$

(14) $$M_1 \leqslant [x, y; \bar{f}] \leqslant M_2$$

with $M_1 = \min\{[x, x - \varepsilon; \underline{f}], [x, x - \varepsilon; \bar{f}]\}$ $M_2 = \max\{[x, x + \varepsilon; \underline{f}], [x, x + \varepsilon; \bar{f}]\}$ thus, denoting $M = \max\{M_1, M_2\}$

(15) $$|\underline{f}(x) - \underline{f}(y)| < M|x - y| \text{ and } |\bar{f}(x) - \bar{f}(y)| < M|x - y|, \ y \in]x - \varepsilon, x + \varepsilon[.$$

Formulas (15) implies that $\underline{f}$ and $\bar{f}$ are continuous on x, so that according to the theorem 1. f is continuous.

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