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LINEAR POSITIVE OPERATORS GENERATED BY FUNCTIONS

by

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1. Introduction

BASKAKOV [1] and SCHURER [4] investigated a certain linear positive operator. They used a sequence of real functions of the real variable x, $\{\varphi_n(x)\}$ $(n = 1, 2, \ldots)$ to construct the linear positive operator. We assume that each function has the following properties on the interval $[0, b]$ $(b > 0)$:

I. $\varphi_n(0) = 1$;

II. $\varphi_n(x)$ is infinitely differentiable on $[0, b]$ and $(-1)^k \varphi_n^{(k)}(x) \geq 0$ $(k = 0, 1, \ldots)$;

III. there exists a positive integer n_1 not depending on k, such that

(1) $$-\varphi_n^{(k)}(x) = \psi(k, n, x)\varphi_{n_1}^{(k-1)}(x) \quad (k = 1, 2, \ldots),$$

where

IV. $\psi(k, n, x)$ is such that there exists a positive function $\Lambda(n)$, monotonically increasing and tending to infinity when $n \to \infty$ for $i = 0$ and 1 $(n_0 \equiv n)$ uniformly in k.

The operator L_n for $n = 1, 2, \ldots$ is defined by

(2) $$L_n\{f(t); x\} := \sum_{k=0}^{\infty} \frac{(-1)^k \varphi_n^{(k)}(x) x^k}{k!} f\left(\frac{k}{\Lambda 1(n)}\right).$$

In some cases ψ is supposed to be independent of k. For example in theorems 5, 6 and 7 of SCHURER [4] and in theorem 4 of SIKKEMA [5].

We will show that if $\psi(k, n, x)$ is independent of k it is independent of x as well. And, on the other hand, if ψ is independent of x it is independent of k. Furthermore we will show that if $n_1 = n$ all known examples of L can be generated, viz.: Bernstein's operator, the operator of Szász-Mirakyan, and the operator of Baskakov's (in litterature $n_1 = n - 1$, $n_1 = n$, $n_1 =$ $= n + 1$ respectively).

2. Dependence of variables

We suppose ψ is independent of k, $\psi = \psi(n, x)$. We take (1) with $k + 1$ instead of k

$$-\varphi_n^{(k+1)}(x) = \psi(n, x)\varphi_{n_1}^{(k)}(x) \quad (k = 0, 1, \ldots). \tag{3}$$

Now we differentiate (1)

$$-\varphi_n^{(k+1)}(x) = \psi'(n, x)\varphi_{n_1}^{(k-1)}(x) + \psi(n, x)\varphi_{n_1}^{(k)}(x) \quad (k = 1, 2, \ldots). \tag{4}$$

From (3) and (4) we get

$$\psi'(n, x) = 0 \text{ or } \varphi_{n_1}^{(k-1)}(x) = 0 \quad (k = 1, 2, \ldots)$$

and so

$$\psi(n, x) = \Phi(n) \text{ or } \varphi_n(x) \equiv 0 \quad x \in [0, b]$$

(where Φ is an arbitrary function of n). In the case where $\varphi_n(x) \equiv 0$ we can say ψ is independent of k and x. Thus we have proved that if ψ is independent of k, $\psi = \psi(n)$.

Now we suppose that ψ is independent of x, $\psi = \psi(k, n)$. In (1) we substitute $k, + 1$ for k. We obtain

$$-\varphi_n^{(k+1)}(x) = \psi(k + 1, n)\varphi_{n_1}^{(k)}(x) \quad (k = 0, 1, \ldots). \tag{5}$$

Differentiating (1) gives

$$-\varphi_n^{(k+1)}(x) = \psi(k, n)\varphi_{n_1}^{(k)}(x) \quad (k = 1, 2, \ldots). \tag{6}$$

We find that $\psi(k + 1, n) = \psi(k, n)$ $k = 1, 2, \ldots, k_0$ for some $k_0 = k_0(n)$ and $\varphi_{n_1}^{(k_0+1)}(x) \equiv 0$. It possible that k_0 is infinite. So we have shown that if ψ is independent of x, $\psi = \psi(n)$.

3. The case $n_1 = n$

We now consider the case where $n_1 = n$. We apply (1) to (1) and get

$$\varphi_n^{(k+1)}(x) = \psi(k + 1, n, x)\psi(k, n, x)\varphi_n^{(k-1)}(x); \tag{7}$$

for $k = 1, 2, \ldots; n = 1, 2, \ldots; x \in [0, b]$. Also we differentiate (1) and apply (1)

$$\varphi_n^{(k+1)}(x) = [-\psi'(k, n, x) + \psi^2(k, n, x)]\varphi_n^{(k-1)}(x) \tag{8}$$

for $k = 1, 2, \ldots; n = 1, 2, \ldots;$ $x \in [0, b]$. From (7) and (8) we obtain the nonlinear partial differential-difference equation

$$\psi(k + 1, n, x)\psi(k, n, x) = -\psi'(k, n, x) + \psi^2(k, n, x) \tag{9}$$

We are able to give two solutions of this equation, viz.:

$\psi(k, n, x)\Phi(n)$ and $\psi(k, n, x) = \dfrac{k + \alpha(n)}{x + \beta(n)}$ $(k = 1, 2, \ldots; n = 1, 2, \ldots; x \in$ $\in [0, b])$ (Φ, α and β arbitrary functions of n). From $\psi(k, n, x) = \Phi(n)$ we get the differential equation $\varphi_n^{(k)}(x) = -\Phi(n)\varphi_n^{(k-1)}(x)$. Together with the assumption that $\varphi_n(0) = 1$ this gives $\varphi_n(x) = e^{-\Phi(n)x}$ $(n = 1, 2, \ldots)$. These φ_n characterize the operator of SZÁSZ-MIRAKYAN [3]. $\psi(k, n, x) =$ $= \dfrac{k + \alpha(n)}{x + \beta(n)}$ together with assumption I yields $\varphi_n(x) = \left[\dfrac{\beta(n)}{x + \beta(n)}\right]^{1+\alpha(n)}$ $(n =$ $= 1, 2, \ldots)$. With these φ_n we may define the linear operator M_n by

$$M_n\{f(t); x\} := \left[\frac{\beta(n)}{x + \beta(n)}\right]^{1+\alpha(n)} \sum_{k=0}^{\infty} \binom{\alpha(n) + k}{k} \left(\frac{x}{x + \beta(n)}\right)^k f\left(\frac{k}{n}\right). \tag{10}$$

The images of 1, t and t^2 are easy to find.

$$M_n(1; x) = 1, \quad M_n(t; x) = \frac{\alpha(n) + 1}{n\beta(n)} x,$$

$$M_n(t^2; x) = \frac{(\alpha(n) + 1)(\alpha(n) + 2)}{n^2\beta^2(n)} x^2 + \frac{\alpha(n) + 1}{n^2\beta(n)} x.$$

If $\dfrac{\alpha(n) + 1}{\beta(n)} = n + 0(n)$ and $\dfrac{1}{\beta(n)} = 0(n)$ then $M_n(t^i; x) \to x^i$ $(i = 0, 1, 2)$ as $n \to \infty$.

Taking $\alpha(n) = n\beta(n) - 1$ we obtain

$$M_n(t; xy) = x, \quad M_n(t^2; x) = x^2 + \frac{x}{n} + \frac{x^2}{n\,\beta(n)}.$$

With $\beta(n) \equiv -1$ we have $M_n = B_n$ (the Bernstein operator [2]) and with $B(n) \equiv 1$ we have $M_n = K_n$ (the Baskakov operator [1]). It is clear that it is not necessary that β be an integer. We see that the larger we take the absolute value of $\beta(n)$ the closer the image of t^2 under the operator M_n approximates the image of t^2 under the operator S_n (Szász-Mirakyan). We also have $\left(\dfrac{\beta}{x + \beta}\right)^{n\beta} \to e^{-nx}$ as $\beta \to 0$ $(n = 1, 2, \ldots)$.

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