MATHEMATICA - REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Anal. Numér. Théor. Approx., vol. 5 (1976) no. 1, pp. 31–57
ictp.acad.ro/jnaat

ON THE NUMERICAL SOLUTIONS OF SOME VOLTERRA EQUATIONS ON INFINITE INTERVALS

OLAVI NEVANLINNA

(Helsinki)

1. INTRODUCTION

In this paper we consider the numerical solution of the following Volterra equations

$$x'(t) + f(t, x(t)) + \int_0^t K(t,s)\, x(s)\,\mathrm{d}s = 0, \qquad t \geq 0;\; x(0) = \tilde{x}, \tag{1.1}$$

and

$$x(t) + \int_0^t K(t,s)\, f(s, x(s))\,\mathrm{d}s = a(t), \qquad t \geq 0. \tag{1.2}$$

On the kernel $K(t,s)$ we assume that

$$K(t,s) \quad \text{is a symmetric} \quad d \times d\text{-matrix}, \tag{1.3}$$

such that the operator

$$\mathcal{K}(\varphi)(t) = \int_0^t K(t,s)\,\varphi(s)\,\mathrm{d}s \tag{1.4}$$

is a positive operator in L^2 (see Definition 2.1). On the function f we assume that it satisfies

$$f(\cdot, u) \;\text{ is continuous on } \mathbb{R}_+ \text{ for all } u \in \mathbb{R}^d, \tag{1.5a}$$

$$\big| f(t,u) - f(t,v) \big| \leq M\,|u-v|, \qquad \text{for all } u, v \subset \mathbb{R}^d \text{ and } t \geq 0 \tag{1.5b}$$

$$\big\langle u - v, f(t,u) - f(t,v) \big\rangle \geq \mu\,|-v|^2, \qquad \text{for all } u, v \in \mathbb{R}^d \text{ and } t \geq 0, \tag{1.5c}$$

with some $\mu \geq 0$.

It is not necessary to assume that (1.5) holds globally in $\mathbb{R}^d$ but it simplifies the presentation. In (1.5) $\langle \cdot, \cdot \rangle$ denotes a fixed positive definite inner product in $\mathbb{R}^d$ and $\|\cdot\|$ the corresponding norm.

It is well known that in compact intervals $[0, T]$ we can discretize the equations (1.1) and (1.2) in such a way that, if the discretization parameter, the step size h tends to zero, then the approximative solutions converge to the solutions of the original problems. However, qualitative estimates obtained for the global error (=computed approximation - exact solution) typically contain factors of the form $\exp(MT)$ so that they become meaningless as $T \to \infty$, and can in fact, be "very pessimistic" also for, small T. The essential reason for this lies in the fact that for each fixed $h > 0$ the perturbation sensitivity of the solutions of the discretized equations can be different from that of the original equation.

In this paper we shall derive bounds for the global errors, which remain small on the whole half axis $t \geq 0$. The bounds depend on local errors in an efficient way, and since local errors can usually be easily estimated, they give us good estimates for the upper bounds of global errors. Furthermore, no restriction on the step size h is posed in the whole paper, it can be an arbitrary but fixed positive constant.

For ordinary differential equations ((1.1) with $K(t, s) \equiv 0$) some error bounds were given by Dahlquist G. [3], and further sharpened in [10]. For (proper) Volterra equations no earlier results are known to us.

In chapter 2 discretizations to (1.1) are considered. The main subject is to show that the error bounds derived in [10] for the special case $K(t, s) \equiv 0$ still hold with one modification, if $\mathcal{K}$ in (1.4) is suitably discretized. As in [10] we assume that the differential part of (1.1) is discretized using a G-stable method [3]. On the discretization of $\mathcal{K}$ we need that the resulting discrete operator is still positive, now in e^2. This happens for al $\mathcal{K}$ if and only if the quadrature itself is a positive operator, then called a positive quadrature. In chapter 2 we assume that $K(t, s)$ is continuous on $0 \leq s \leq t < \infty$, while the weakly singular case is discussed in chapter 5. All the bounds we give for (1.1) are independent on the (possibly very large) Lipschitz constant M.

In chapter 3 we investigate equation (1.2) under discretization with a positive quadrature.

In chapter 4 we consider the relationship between A-stability of a multistep method [2] and the positiveness of an associated "convolution quadrature". As a corollary we get an error bound for the ordinary differential equation $x' + f(t, x) = 0$ assuming only A-stability on the method, which is the weakest possible assumption we have to make, if nothing additional is known on f. (It is known that G-stability is properly stronger than A-stability, [3]). As an other application we consider a second order differential equation discretized using a G-stable method.

In chapter 5 we show how we can discretize $\mathcal{K}$ without destroying the positivity, if $\mathcal{K}$ is weakly singular in the form $K(t,s) = a(t-s)B(t,s)$. Here $B(t,s)$ is a continuous matrix on $0 \le s \le t < \infty$, and $a \in L^1_{loc}$ is a scalar function satisfying some monotonicity conditions.

In chapter 6 we give a result concerning discretizations to (1.1), which guarantees that the approximative solutions tend to zero at infinity. Generally, if (1.5) holds with $\mu = 0$, $\{f(nh,0)\} \in \mathrm{e}^1$ and $\mathcal{K}$ is a positive operator, then the use of a G-stable method and a positive quadrature guarantees that the solutions are bounded, but they need not tend to zero at infinity. However, if $\mathcal{K}$ satisfies a stronger condition so that the solutions of (1.1) tend to zero then the same holds for the discretized equation.

2. VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

Here we shall derive some error bounds for the numerical solutions to the following Volterra integro-differential equation

$$x'(t) + f(t, x(t)) + \int_0^t K(t,s)\, x(s)\, \mathrm{d}s = 0, \quad t \ge 0;\ x(0) = \tilde{x} \in \mathbb{R}^d. \tag{2.1}$$

When doing this we use the following concepts and notations. Let H be a real Hilbert space and $\mathcal{K}$ a densely defined operator in H.

Definition 2.1. *$\mathcal{K}$ is a positive operator in H if*

$$(\varphi, \mathcal{K}\varphi) \ge 0 \quad \text{for all } \varphi \in D\mathcal{K}. \tag{2.2}$$

We denote by L^2 and l^2 the Hilbert space defined by the inner products

$$(\varphi, \psi) = \int_0^\infty \langle \varphi(t), \psi(t) \rangle \mathrm{d}t$$

and

$$(\xi, \eta) = \sum_{k=0}^\infty \langle \xi_k, \eta_k \rangle,$$

respectively. Throughout this chapter, let $K(t,s)$ be a symmetric $d \times d$-matrix, which as a function of t and s is continuous for $0 \le s \le t < \infty$. We define an operator in L^2 by

$$(\mathcal{K}\varphi)(t) = \int_0^t K(t,s)\, \varphi(s)\, \mathrm{d}s. \tag{2.3}$$

Given a quadrature formula

$$(\mathcal{W}\xi)_n = h\sum_{k=0}^{n} w_{nk}\xi_k, \tag{2.4}$$

which we consider as an operator in l^2, and an operator $\mathcal{K}$ in L^2 we define an operator $\mathcal{WK}$ in l^2 setting

$$(\mathcal{WK}\xi)_n = h\sum_{k=0}^{n} w_{nk}K\,(nh,kn)\,\xi_k. \tag{2.5}$$

In (2.4) h is a positive real and the weights w_{nk} are reals (which can depend on h).

Definition 2.2. *A quadrature formula $\mathcal{W}$ is called positive if $\mathcal{W}$ is a positive operator in l^2.*

Note that the positivity on $\mathcal{W}$ does not depend on the dimension d. The following theorem is basic to our considerations.

Theorem 2.1. *$\mathcal{WK}$ is a positive operator in l^2 whenever $\mathcal{K}$ is a positive operator in L^2 if and only if $\mathcal{W}$ is a positive quadrature.*

A proof of this theorem in the scalar case can be found in [11]. Since $K\,(t,s)$ is a symmetric matrix for every (t,s), the proof generalizes easily. In fact, assume that $\mathcal{K}$ is a positive operator and $\mathcal{W}$ a positive quadrature. Using the identities

$$\int_0^T \left\langle \varphi\,(t)\,, \int_0^t A\,(t,s)\,\mathrm{d}s \right\rangle \mathrm{d}t = \tfrac{1}{2}\int_0^T \left\langle \varphi\,(t)\,, \int_0^T \tilde{A}\,(t,s)\,\varphi\,(s)\,\mathrm{d}s \right\rangle \mathrm{d}t, \tag{2.6}$$

$$\sum_{n=0}^{N} \left\langle \xi_n, \sum_{j=0}^{n} A_{nj}\xi_j \right\rangle = \tfrac{1}{2}\sum_{n=0}^{N} \left\langle \xi_n, \sum_{j=0}^{N} \tilde{A}_{nj}\xi_j \right\rangle, \tag{2.7}$$

where

$$\tilde{A}\,(t,s) = \begin{cases} A\,(t,s)\,, & t \geq s \\ A\,(s,t)^{\tau}\,, & t < s \end{cases}$$

and

$$\tilde{A}_{nj} = \begin{cases} A_{nj}, & n > j \\ A_{nn} + A_{nn}^T, & n = j \\ A_{jn}^T, & n < j, \end{cases}$$

and a suitable sequence of functions $\{\varphi_m\}$ approximating $\sum_{k=0}^{N} \delta_{kh}\xi_k + \delta_0\xi_0 + \delta_{Nk}\xi_N$, where δ denotes the Dirac measure, we first obtain the positivity of the operator

$$(\mathcal{L}\xi)_n = h\sum_{k=0}^{n} L_{nk}\xi_k,$$

where

$$L_{nk} = \begin{cases} \frac{1}{2} K\left(nh,\ nh\right), & n = k, \\ L\left(nh,\ kh\right), & n > k. \end{cases}$$

The operator $\mathcal{WK}$ is then a "Schur product" of $\widetilde{\mathcal{W}}$ and $\mathcal{L}$, where $\widetilde{\mathcal{W}}$ is defined by (2.7) with A_{nj} replaced by w_{nj}. The positivity of $\widetilde{\mathcal{W}}\mathcal{L}$ then follows using the symmetricity of $\widetilde{\mathcal{W}}$. That $\mathcal{W}$ itself must be positive is obvious since $\mathcal{K}$ with $K\left(t,s\right) \equiv I$ is a positive operator.

It was shown in [11] that there does not exist any explicit positive quadrature and that the quadrature $\mathcal{W}_\theta$ are positive for $\theta \in \left[\frac{1}{2}, 1\right]$, where $\mathcal{W}_\theta$ is defined by

$$\begin{aligned} &w_{00} = \tfrac{1-\theta}{4}, \\ &w_{n0} = 1-\theta, \quad n > 0, \\ &w_{nk} = 1, \quad n > k > 0, \\ &w_{nn} = \theta. \end{aligned}$$

In the following we shall use a positive quadrature when discretizing the integral operator $\mathcal{K}$ occurring in the equation (2.1). In the discretization of the differential part of the equation we shall use the so called one-leg methods [3].

Consider the Volterra integro-differential equation

$$x'\left(t\right) = g\left(t, x\left(t\right), y\left(t\right)\right), \quad y\left(t\right) = \int_0^t k\left(t, s, x\left(s\right)\right) \mathrm{d}s. \tag{2.8}$$

This can be discretized using a *linear k-step* method $\left(\rho, \sigma\right)$

$$\sum_{j=0}^{k} \alpha_j \hat{x}_{n+j} = h \sum_{j=0}^{k} \beta_j g\left(\hat{t}_{n+j}, \hat{x}_{n+j}, \hat{y}_{n+j}\right), \tag{2.9}$$

and a quadrature $\mathcal{W}$

$$\hat{y}_n = h \sum_{u=0}^{n} w_{n\mu} k\left(\hat{t}_n, \hat{t}_\mu, \hat{x}_\mu\right). \tag{2.10}$$

In (2.9) the generating polynomials

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j, \quad \sigma\left(\zeta\right) = \sum_{j=0}^{k} \beta_j \zeta^j$$

are assumed to have real coefficients and no common divisor. We also assume that the method $\left(\rho, \sigma\right)$ is consistent $\left(\rho\left(1\right) = 0,\ \rho'\left(1\right) = \sigma\left(1\right)\right)$, stable (no root of $\rho(\zeta)$ lie outside the unit circle, and those on the circle are simple), and normalized so that $\sigma(1) = 1$. The corresponding *one-leg method* for (2.8) is then defined by

$$\sum_{j=0}^{k} \alpha_j x_{n+j} = hg\Big(\sum_{j=0}^{k} \beta_j t_{n+j}, \sum_{j=0}^{k} \beta_j x_{n+j}, y_n \Big), \tag{2.11}$$

where

$$y_n = h \sum_{\mu=0}^{n} w_{n\mu} k\Big(\sum_{j=0}^{k} \beta_j t_{n+j}, \sum_{j=0}^{k} \beta_j t_{\mu+j}, \sum_{j=0}^{k} \beta_j x_{\mu+j} \Big). \tag{2.12}$$

Using the shifting operator $E : E\xi_n = \xi_{n+1}$ the approximations $\{\hat{x}_n\}$ and $\{x_n\}$ satisfy the equations

$$\rho(E)\,\hat{x}_n = h\sigma(E)\,\hat{g}_n + \hat{p}_n, \quad \hat{y}_n = h \sum_{\mu=0}^{n} w_{n\mu} k\left(\hat{t}_n, \hat{t}_\mu, \hat{x}_\mu\right) \tag{2.13}$$

and

$$\begin{cases} \rho(E)\, x_n = hg\big(\sigma(E)\, t_n, \sigma(E)\, x_n, y_n\big) + p_n \\ y_n = h \displaystyle\sum_{\mu=0}^{n} w_{n\mu} k\big(\sigma(E)\, t_n,\ \sigma(E)\, t_\mu, \sigma(E)\, x_\mu\big), \end{cases} \tag{2.14}$$

respectively, where we put $\hat{g}_n = g\left(\hat{t}_n, \hat{x}_n, \hat{y}_n\right)$ and $\hat{p}_n,\ p_n$ denote local perturbations which exist in real computations. It turns out that the error bounds are simpler to derive for the one-leg methods than for the linear multistep methods. However, due to the following theorem results obtained for one-leg methods can be transformed to results for the corresponding linear multistep methods.

Theorem 2.2. *Let $x = \{x_n\}$ satisfy* (2.14) *and put $\hat{t}_n = \sigma(E)\, t_n,\ \hat{x}_n = \sigma(E)\, x_n,\ \hat{p}_n = \sigma(E)\, p_n$. Then $\{\hat{x}_n\}$ satisfies* (2.13).

Conversely, let $\hat{x} = \{\hat{x}_n\}$ satisfy (2.13) *and define t_n, p_n by solving $\sigma(E)\, t_n = \hat{t}_n,\ \sigma(E)\, p_n = \hat{p}_n$. Let P, Q be two polynomials of degree not exceeding $k-1$, such that for some integer $m,\ 0 \le m \le k$,*

$$P(\zeta)\,\sigma(\zeta) - Q(\zeta)\,\rho(\zeta) \equiv \zeta^m,$$

and, for $n \ge k$ put $x_n = E^{-m}\{P(E)\,\hat{x}_n - hQ(E)\,\hat{g}_n - Q(E)\, p_n\}$.

Then $\sigma(E)\, x_n = \hat{x}_n$ and $\{x_n\}$ satisfies (2.14) *for $n \ge k$.*

For ordinary differential equations this theorem is due to G. Dahlquist [3]. Here we have chosen y_n and $\hat{y}_n$ in such a way that $y_n = \hat{y}_n$, which reduce the theorem into that original form.

Because of this result we shall here consider one-leg methods only. Let $\{u_n\}$ be a sequence in $\mathbb{R}^d$, then we define another sequence $\{U_n\}$ by

$$U_n = (u_{n+k-1}, \dots, u_n)\,.$$

For a $k \times k$-matrix $F = (f_{ij})$ we denote

$$F\,(U_n) = \sum_{i=1}^{k}\sum_{j=1}^{k} f_{ij}\langle u_{n+k-i}, u_{n+k-j}\rangle.$$

DEFINITION 2.3. *Let $G = (g_{ij})$ be a real symmetric positive definite matrix. The method (ρ, σ) is said to be G-stable, if for arbitrary $\{u_n\}$ we have*

$$G\,(U_1) - G\,(U_0) \le 2\big\langle \sigma\,(E)\,u_0, \rho\,(E)\,u_0\big\rangle. \tag{2.15}$$

Using the fact that for symmetric matrices

$$F\,(U_n) \ge 0 \tag{2.16}$$

if the matrix F is nonnegative definite (e.g., [3, Lemma 2.2]) we observe that the G-stability does not depend on the dimension d. All G-stable methods are A-stable (Definition 3.1) but the converse does not hold. However, every A-stable k -step method of order k (or more) is G-stable for exactly one matrix G, see [3].

Specifying the discretization (2.14) to (2.1) yields

$$\rho\,(E)\,x_n + hf\big(nh, \sigma\,(E)\,x_n\big) + h^2 \sum_{\mu=0}^{n} w_{n\mu} K\,(nh, \mu h)\,\sigma\,(E)\,x_\mu = r_n, \quad n \ge 0, \tag{2.17}$$

when $x_0, \dots, x_{k-1}$ are given.

(Observe that we put in (2.17) for notational simplicity $\sigma\,(E)\,t_n = nh$). We shall first give a result on the existence and uniqueness of the solutions of (2.17).

THEOREM 2.3. *Assume that f satisfies* (1.5) *and $\mathcal{WK}$ is a positive operator in l^2. Then, for all $\{x_0, \dots, x_{k-1}\}$* (2.17) *has a unique solution $\{x_n\}$.*

Proof. Put $\sigma\,(E)\,x_n = \xi$ and assume that we know the existence of $x_0, \dots, x_{n+k-1}$. Then (2.17) can be written as

$$A\xi \overset{def}{=} \tfrac{\alpha_k}{\beta_k}\xi + hf\,(nh, \xi) + h^2 w_{nn} K\,(nh,\ nh)\,\xi = \eta$$

where ξ is the unknown vector in $\mathbb{R}^d$. We shall use the following fact from the theory of monotone operators: If A is *continuous, monotone* (i.e $\langle u - v, Au - Av\rangle \ge 0$ for all u, v) and *coercive ($\langle u, Au\rangle\,\|u\|^{-1} \to \infty$ as $\|u\| \to \infty$, uniformly) then it is onto.* Since $\mathcal{WK}$ is a positive operator in l^2, $h^2 w_{nn} K\,(nh, nh)$ is a nonnegative definite matrix. Hence, using (1.5) $\langle u - v, Au - Av\rangle \ge \big(\frac{\alpha_k}{\beta_k} + \mu h\big)\,\|u - v\|^2$, where $\frac{\alpha_k}{\beta} + \mu h > 0$, which gives both monotonicity and coerciveness. Since f is continuous so is A and the existence of ξ follows. Uniqueness follows form

$$0 = \left\langle \xi_1 - \xi_2,\ A\xi_1 - A\xi_2 \right\rangle \geq \left(\tfrac{\alpha_k}{\beta_k} + \mu h \right) \|\xi_1 - \xi_2\|^2 .$$

In practical calculations, the local perturbation r_n consists of roundoff error and truncation error in the iterative solution of the algebraic equation determining $\sigma(E)\,x_n$. Let then $x(t)$ be a solution of (2.1) and define another sequence of local perturbations $\{\tau_n\}$ by

(2.18)

$$\rho(E)\,x(nh) + hf\big(nh, \sigma(E)\,x(nh)\big) + h^2 \sum_{\mu=0}^{n} w_{n\mu} K(nh, \mu h)\,\sigma(E)\,x(\mu h) = \tau_n,$$

$n \geq 0.$

Put $z_n = x_n - x(nh)$, $q_n = r_n - \tau_n$, then (2.17) and (2.18) imply

$$\rho(E)\,z_n + h\big\{f(nh, \sigma(E)\,x_n) - f(nh, \sigma(E)\,x(nh))\big\} + \\ + h^2 \sum_{\mu=0}^{n} w_{n\mu} K(nh, \mu h)\,\sigma(E)\,z_\mu = q_n. \tag{2.19}$$

Assume the operator $\mathcal{K}$ in (2.1) is positive and that f satisfies (1.5) with a fixed $\mu \geq 0$. Let a and b be reals such that

$$\|\sigma(E)\,u_0\|^2 \leq aG(U_1) + bG(U_0). \tag{2.20}$$

□

THEOREM 2.4. *Assume that (ρ, σ) is a G-stable method and $\mathcal{W}$ is a positive quadrature. Then, for $\mu = 0$ we have*

$$G(Z_{n+1}) \leq e\left\{G(Z_0) + (2a+b)\left[\sum_{j=0}^{n} \|q_j\|\right]^2\right\}, \tag{2.21}$$

and for $\nu > 0$

$$G(Z_{n+1}) \leq G(Z_0) + \tfrac{1}{2\mu h} \sum_{j=0}^{n} \|q_j\|^2 . \tag{2.22}$$

If we assume in addition that $\alpha = \max|\zeta_\nu| < 1$, where ζ_ν is a zero of $\sigma(\zeta)$, then the bound in (2.22) can be essentially improved. This is based on the following.

LEMMA 2.1. (see [10, Theorem 1]). *If $\alpha < 1$ then for every $\varepsilon > 0$ such that $\alpha^2 + \varepsilon < 1$, there exists a positive definite matrix H, independent on the dimension d, satisfying*

$$\|\sigma(E)\,u_0\|^2 - \left\{H(U_1) - \left(\alpha^2 + \varepsilon\right)H(U_0)\right\} \geq 0 \tag{2.23}$$

for all sequences $\{u_n\}$.

Let λ be such that $G(U_0) \leq \lambda H(U_0)$, so that we can state

THEOREM 2.5. *Assume that (ρ, σ) is a G-stable method with $\alpha < 1$ and that $\mathcal{W}$ is a positive quadrature. Then, for $\mu > 0$,*

(2.24)

$$(G+\mu hH)(Z_{n+1})+(1-\theta)\sum_{j=0}^{n}(G+\mu hH)(Z_j) \leq (G+\mu hH)(Z_0)+(\mu h)^{-1}\sum_{j=0}^{n}\|q_j\|^2,$$

where $\theta = \frac{(\alpha^2+\varepsilon)\mu h+\lambda}{\mu h+\lambda}$, $\alpha^2 + \varepsilon < 1$ *and* H *satisfies* (2.23).

Proof of Theorem 2.4. Multiply (2.19) by $\sigma(E) z_n$ and sum from 0 to N

$$\begin{aligned}(2.25)\qquad &\sum_{n=0}^{N}\langle\sigma(E) z_n, \rho(E) z_n\rangle+\\ &+h\sum_{n=0}^{N}\Big\langle\sigma(E) z_n, f\big(nh, \sigma(E) x_n\big)-f\big(nh, \sigma(E) x(nh)\big)\Big\rangle\\ &+h^2\sum_{n=0}^{N}\Big\langle\sigma(E) z_n, \sum_{\mu=0}^{n} w_{n\mu}K(nh,\ \mu h)\,\sigma(E) z_\mu\Big\rangle\\ &=\sum_{n=0}^{N}\langle\sigma(E) z_n, q_n\rangle.\end{aligned}$$

Using G-stability, (1.4), and the positivity of $\mathcal{WK}$, (2.25) gives

$$(2.26)\quad G(Z_{n+1}) - G(Z_0) + 2h\mu\sum_{m=0}^{n}\|\sigma(E) z_m\|^2 \leq 2\sum_{m=0}^{n}\|\sigma(E) z_m\|\,\|q_m\|.$$

Consider the case $\mu = 0$ first. If $q_m = 0$ for $m = 0, \ldots, n$ then (2.21) clearly holds. Assume therefore that for some $m, 0 \leq m \leq n,\ q_m \neq 0$. First observe that by (2.20) we have for every $\eta > 0$

$$(2.27)\quad 2\|\sigma(E) z_n\| \leq \eta + \tfrac{1}{\eta}\|\sigma(E) z_n\|^2 \leq \eta + \tfrac{1}{\eta}\|aG(Z_{n+1}) + bG(EZ_n)\|.$$

Choose η to satisfy $\max\limits_{0\leq m\leq n} \frac{a}{\eta}\|q_m\| < 1$. Then (2.26) and (2.27) yield

$$\begin{aligned}(2.28)\quad G(Z_{n+1}) \leq \big(1-\tfrac{a}{\eta}\|q_n\|\big)^{-1}\Big\{&G(Z_0)+\eta\sum_{m=0}^{n}\|q_m\|+bG(Z_n)\|q_n\|+\\ &+\sum_{m=0}^{n-1}\big[aG(Z_{n+1})+bG(Z_m)\big]\|q_m\|\Big\}.\end{aligned}$$

Let us define a sequence of reals $\{\zeta_m\}_0^{n+1}$ by setting $\zeta_0 = G(Z_0)$ and

(2.29)

$$\zeta_{m+1} =$$

$$= \left(1 - \tfrac{a}{\eta}\left\|q_m\right\|\right)^{-1}\Big\{G\left(Z_0\right) + \eta \sum_{k=0}^{m} \left\|q_k\right\| + b\zeta_m + \sum_{k=0}^{m-1} \left(a\zeta_{k+1} + b\zeta_k\right)\left\|q_k\right\|\Big\}.$$

Then by a simple induction $\zeta_{m+1} \geq G\left(Z_{m+1}\right)$. But ζ_{m+1} satisfies

$$\zeta_{m+1} = \Big\{1 - \tfrac{a}{\eta}\left\|q_m\right\|\Big\}^{-1}\Big\{\left(1 + \tfrac{b}{\eta}\left\|q_m\right\|\right)\zeta_m + \eta q_m\Big\}$$

and hence

$$\begin{aligned}\zeta_{n+1} &= \Big\{\prod_{m=0}^{n} \tfrac{1+\frac{b}{\eta}\|q_m\|}{1-\frac{a}{\eta}\|q_m\|}\Big\}G\left(Z_0\right) + \eta \sum_{j=0}^{n}\Big(\prod_{j=0}^{n} \tfrac{1+\frac{b}{\eta}\|q_m\|}{1-\frac{a}{\eta}\|q_m\|}\Big)\tfrac{\|q_j\|}{1+\frac{b}{\eta}\|q_j\|} \\ &\leq \Big\{G\left(Z_0\right) + \eta \sum_{j=0}^{n}\left\|q_j\right\|\Big\}\exp\Big[\sum_{m=0}^{n} \tfrac{\frac{a+b}{\eta}\|q_m\|}{1-\frac{a}{\eta}\|q_m\|}\Big].\end{aligned}$$

The choice $\eta = (2a+b)\sum_{j=0}^{n}\left\|q_j\right\|$ now yields (2.21).

Assume then $\mu > 0$. In (2.26) we estimate

$$2\eta^{\frac{1}{2}}\left\|\sigma\left(E\right)z_m\right\|\left\|q_m\right\|\eta^{-\frac{1}{2}} \leq \eta\left\|\sigma\left(E\right)z_m\right\|^2 + \eta^{-1}\left\|q_m\right\|^2$$

and get

$$G\left(Z_{n+1}\right) \leq G\left(Z_0\right) + \left(-2h\mu + \eta\right)\sum_{m=0}^{n}\left\|\sigma\left(E\right)z_m\right\|^2 + \eta^{-1}\sum_{m=0}^{n}\left\|q_m\right\|^2. \tag{2.30}$$

The choice $\eta = 2h\mu$ gives (2.22), and completes the proof. □

Proof of Theorem 2.5. In (2.30) choose $\eta = \mu h$ and use (2.23) to obtain

(2.31)

$$G\left(Z_{n+1}\right) \leq G\left(Z_0\right) - \mu h \sum_{m=0}^{n}\left\{H\left(Z_{m+1}\right) - \left(\alpha^2 + \varepsilon\right)H\left(Z_m\right)\right\} + \tfrac{1}{\mu h}\sum_{m=0}^{n}\left\|q_m\right\|^2.$$

By the given choice of θ

$$G\left(U_0\right) + \mu h\left(\alpha^2 + \varepsilon\right)H\left(U_0\right) \leq \theta\big\{G\left(U_0\right) + \mu h H\left(U_0\right)\big\},$$

and we can write

$$\sum_{m=0}^{n} (G+\mu hH)(Z_{m+1}) \le \theta \sum_{m=0}^{n} (G+\mu hH)(Z_m) + \tfrac{1}{\mu h} \sum_{m=0}^{n} \|q_m\|^2, \tag{2.32}$$

which yields (2.24). □

REMARK. Observe that when deriving error bounds we need the positivity of a quadrature only in the form

$$\sum_{n=K}^{n} \xi_n \sum_{j=K}^{n} w_{nj}\xi_j \ge 0$$

if we may assume that $x_j - x(jh)$ vanishes for $j = 0,\ 1, \ldots, K-1$. The ordinary trapezoidal rule clearly satisfies this condition but in order to be positive in the sense of Definition 2.2 a modification is needed (see, e.g., the quadrature $\mathcal{W}_{\frac{1}{2}}$, given in this chapter or the quadratures given by (4.7) and (5.9)).

If $K \equiv 0$, (2.1) reduces to an ordinary differential equation. The sharpness of these bounds in that special case is demonstrated in [10]. Note also that these results can be applied to perturbed equations

$$\begin{aligned} &x'(t) + f(t, x(t)) + \int_0^t K(t,s)\,x(s)\,\mathrm{d}s = u(t, x(t), y(t)), \\ &y(t) = \int_0^t v(t,s,x(s))\mathrm{d}s, \end{aligned} \tag{2.33}$$

assuming that u is small in some suitable sense. This only introduces a new term into the local perturbation q_n in equation (2.19).

In [1], [9] the following test equation

$$x'(t) + \alpha x(t) + \beta \int_0^t x(s)\,\mathrm{d}s = 0 \tag{2.34}$$

is considered. The problem studied is whether the asymptotic stability of the solutions of (2.34) is carried over to the solutions of the discretized equation. In [1] α and β are assumed to be reals while [9] considers the complex valued case. One notes immediately that the technique used in this chapter can be applied to (2.34) only if β is real. Correspondingly, the class of quadratures studied in [9] are of a very special type, in fact essentially our convolution quadratures (4.7). □

3. VOLTERRA INTEGRAL EQUATIONS

Here we consider Volterra integral equations of the form

$$x(t) + \int_0^t K(t,s)\, f(s, x(s))\, ds = a(t), \quad t \geq 0 \tag{3.1}$$

where K satisfies (1.3), is continuous for $0 \leq s \leq 1 \leq \infty$ and such that $\mathcal{K}$ is a positive operator in L^2, and f satisfies (1.5) with $u > 0$. We discretize (3.1) using a quadrature $\mathcal{W}$

$$x_n + h \sum_{j=0}^{n} w_{nj} K(nh, jh)\, f(jh, x_j) = a(nh) + r_n, \tag{3.2}$$

where r_n denotes a local perturbation. Let $x(t)$ be a solution of (3.1) and define the local truncation error τ_n of this discretization by

$$x(nh) + h \sum_{j=0}^{n} w_{nj} K(nh, jh)\, f(jh,\ x(jh)) = a(nh) + \tau_n. \tag{3.3}$$

Put as before, $z_n = x_n - x(nh)$, $q_n = r_n - \tau_n$, then we have the following result, where the seminorms $\|\cdot\|_N$ are defined by $\|\xi\|_N = \left\{\sum_{n=0}^{N} \|\xi_n\|^2\right\}^{\frac{1}{2}}$.

THEOREM 3.1. *If f satisfies* (1.5) *with $\mu > 0$ and $\mathcal{WK}$ is a positive operator in l^2, then* (3.2) *has a unique solution $\{x_n\}$. Furthermore, for all $N \geq 0$*

$$\|\{z_n\}\|_N \leq \tfrac{M}{\mu} \|\{q_n\}\|_N. \tag{3.4}$$

Proof. First note that (1.5) with $\mu > 0$ implies, using the argument given in the proof of Theorem 2.3, that $f(t,\cdot)$ has a (Lipschitz-continuous) inverse $f^{-1}(t,\cdot)$. Assume the existence of $x_0, \ldots, x_{n-1}$. Put $\xi = f(nh,\ x_n)$ and write (3.2) as

$$A\xi \overset{def}{=} f^{-1}(nh, \xi) + h w_{nn} K(nh,\ nh)\, \xi = \eta.$$

We show that

$$\langle u - v,\ Au - Av \rangle \geq \tfrac{\mu}{M^2} \|u - v\|^2, \tag{3.5}$$

which implies the existence and uniqueness of ξ and hence of x_n, too. To show (3.5) observe that $h w_{nn} K(nh, nh)$ is a symmetric nonnegative definite matrix and so it has a square root H. Further

$$\left\langle u - v, f^{-1}(nh, u) - f^{-1}(nh, v) \right\rangle \geq \tfrac{\mu}{M^2} \|u - v\|^2,$$

and so

$$\langle u-v, Au-Av\rangle \geq \tfrac{\mu}{M^2}\left\|u-v\right\|^2 + \big\langle H\left(u-v\right), H\left(u-v\right)\big\rangle,$$

which gives (3.5).

To obtain (3.4) first write

$$z_n + h\sum_{j=0}^{n} w_{nj}K\left(nh, jh\right)\big[f\left(jh, x_j\right) - f\left(jh, x\left(jh\right)\right)\big] = q_n.$$

Multiplying by $f\left(nh, x_n\right) - f\left(nh, x\left(nh\right)\right)$, summing from 0 to N and using the positivity of $\mathcal{WK}$ this gives

(3.6)
$$\sum_{n=0}^{N}\big\langle f\left(nh, x_n\right) - f\left(nh, x\left(nh\right)\right), z_n\big\rangle \leq \sum_{n=0}^{N}\big\langle f\left(nh, x_n\right) - f\left(nh, x\left(nh\right)\right), q_n\big\rangle.$$

But by (1.5) this further implies

$$\begin{aligned}(3.7)\qquad \mu\sum_{n=0}^{N}\left\|z_n\right\|^2 &\leq \left\{\sum_{n=0}^{N}\left\|f\left(nh, x_n\right) - f\left(nh,\ x\left(nh\right)\right)\right\|^2\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{N}\left\|q_n\right\|^2\right\}^{\frac{1}{2}}\\ &\leq M\left\{\sum_{n=0}^{N}\left\|z_n\right\|^2\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{N}\left\|q_n\right\|^2\right\}^{\frac{1}{2}},\end{aligned}$$

which finally gives (3.4).

For $f\left(t, x\right) = x$ (3.4) gives $\left\|\{z_n\}\right\|_N \leq \left\|\{q_n\}\right\|_N$.

Putting $K = 0$ we have $z_n = q_n$ and hence $\left\|\{z_n\}\right\|_N = \left\|\{q_n\}\right\|_N$.

As in the case of integro-differential equations earlier results exist only for the test equation

$$x\left(t\right) = \tilde{x} + \alpha\int_0^t x\left(s\right)\mathrm{d}s, \tag{3.8}$$

(see, e.g., [5], [6], [9]). □

Definition 3.1. *A discretization method for Volterra integral equation is said to be A-stable if it yields an approximation* $\{x_n\}$ *such that* $x_n \to 0$ *as* $n \to \infty$, *whenever applied with a fixed* h *to* (3.8) *with* $\operatorname{Re}\alpha < 0$.

In the definition of A-stability for ordinary differential equations we replace the test equation (3.8) *by*

$$x' = \alpha x. \tag{3.9}$$

Note first that $\mathcal{W} \equiv 0$ is formally a positive quadrature but certainly not A-stable since $x_n = \tilde{x}$ for all n. On the other hand, by redefining one element in such a way that it breaks the positivity one can find A-stable quadratures which are not positive. There is an important case where these concepts equal and this is the main subject of the next chapter.

4. ORDINARY DIFFERENTIAL EQUATIONS

In this chapter we consider numerical solutions to the ordinary differential equation

$$x'(t) + f(t, x(t)) = 0, \quad t \geq 0; \; x(0) = \tilde{x}, \tag{4.1}$$

where f satisfies (1.5). We discretize (4.1) using an A-stable linear multistep method (ρ, σ), write the solution of the discretized equation using an associated quadrature and show that this quadrature is positive. Then the error bound given in Theorem 3.1 can be used.

An application of a multistep method (ρ, σ) to (4.1) yields.

$$\rho(E)\, x_n + h\rho(E)\, f_n = r_n, \tag{4.2}$$

where r_n is a local perturbation and $f_n = f(nh, x_n)$.

The solution of (4.2) can be written as

$$x_{n+k} = \varphi_{n+k} + \sum_{\mu=0}^{n} a_{n-\mu}\{r_\mu - h\rho(E)\, f_\mu\}, \tag{4.3}$$

where $\{\varphi_n\}$ is a sequence which depends only on ρ and the initial values $x_0, \ldots, x_{k-1}$, and $\{a_n\}$ is the solution of

$$\rho(E)\, a_n = 0, \; \alpha_k a_0 = 1, \; \sum_{j=k-r}^{k} \alpha_j a_{r-k+j} = 0, \text{ for } r = 1, \ldots, k-1. \tag{4.4}$$

Since we assume that (ρ, σ) is stable the sequences $\{\varphi\}$ and $\{a_n\}$ are bounded. We can rewrite (4.4) as

$$x_{n+k} = \varphi_{n+k} + \sum_{\mu=0}^{n} a_{n-\mu} r_\mu - h \sum_{\mu=0}^{n+k} w_{n+k,\mu} f_\mu, \tag{4.5}$$

where

$$w_{n+k,\mu} = \sum_{j=\max\{0,\mu-n\}}^{\min\{k,\mu\}} \beta_j a_{n-\mu+j}.$$

Put

$$w_m = \sum_{j=\max\{0,k-m\}}^{k} \beta_j a_{m-k+j} \text{ and write (4.5) as}$$

$$x_n = \psi_n + \sum_{\mu=0}^{n-k} a_{n-k-\mu} r_\mu - h \sum_{\mu=0}^{n} w_{n-\mu} f_\mu, \tag{4.6}$$

where $\{\psi_n\}$ is a bounded sequence depending on $\rho, \sigma, x_n, \dots, x_{k-1},\ f_0, \dots, f_{k-1}$. In this way we have associated with a multistep method (ρ, σ) the convolution quadrature Ω

$$(\Omega\xi)_n = h \sum_{\mu=0}^{n} \omega_{n-\mu} \xi_\mu. \tag{4.7}$$

Note that Ω is well defined even if the method (ρ, σ) is not stable.

THEOREM 4.1. *A multistep method (ρ, σ) is A-stable if and only if the corresponding convolution quadrature Ω is positive.*

In the proof we need the following results.

LEMMA 4.1. *Ω is a positive quadrature if and only if $\{\omega_n\}$ is bounded and*

$$\operatorname{Re} \sum_{n=0}^{\infty} \omega_n r^n \mathrm{e}^{in\tau} \geq 0, \qquad \text{for } \tau \in [-\pi, \pi],\ r \in (0,1). \tag{4.8}$$

LEMMA 4.2. [2] *A multistep method (ρ, σ) is A-stable if and only if $\frac{\rho(\zeta)}{\sigma(\zeta)}$ is regular and has a nonnegative real part for $|\zeta| > 1$.*

Proof of Theorem 4.1. A straightforward calculation shows that

$$\tfrac{\sigma(\zeta)}{\rho(\zeta)} = \omega_0 + \omega_1 \zeta^{-1} + \omega_2 \zeta^{-2} + \cdots \tag{4.9}$$

If the method (ρ, σ) is A-stable, then it is stable [2].

Hence $\{a_n\}$ and $\{\omega_n\}$ are bounded. But by (4.9) and Lemma 4.2 also (4.8) holds and so Ω is positive.

Assume then that Ω is positive. By (4.8) and (4.9) we note that $\operatorname{Re} \frac{\sigma(\zeta)}{\rho(\zeta)} \geq 0$ for $|\zeta| > 1$. But $\operatorname{Re} \frac{\sigma(\zeta)}{\rho(\zeta)}$ is harmonic on $\{|\zeta| > 1\}$, nonnegative and not identically zero, hence $\operatorname{Re} \frac{\sigma(\zeta)}{\rho(\zeta)} > 0$ for $|\zeta| > 0$, and the method (ρ, σ) is A-stable. □

Proof of Lemma 4.1. Since the positivity of Ω implies $2h\omega_0 \geq h\,|\omega_n|$, $\{\omega_n\}$ is bounded. The proof of (4.8) is carried out in two steps. We first note that $\{\omega_n\}$ defines a positive convolution quadrature if and only if $\{\omega_n r^n\}$ does for every $r \in (0,1)$. But $\{\omega_n r^n\} \in l^1$ and the second step is to show that $\{\nu_n\} \in l^1$ defines a positive convolution quadrature if and only if $\operatorname{Re} \hat{\nu}(\tau) \geq 0$ for all $\tau \in [-\pi, \pi]$.

Assume $\{\omega_n\}$ defines a positive quadrature. Let P_r denote the Poisson kernel, then we can write

$$r^n = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\tau)\, \mathrm{e}^{-in\tau} \mathrm{d}\tau = \tfrac{1}{\pi} \int_0^{\pi} P_r(\tau) \cos(n\tau)\, \mathrm{d}\tau.$$

Since $P_r(\tau) \geq 0$ we have

$$\begin{aligned}\sum_{n=0}^{N} \xi_n \sum_{j=0}^{n} \omega_{n-j} r^{n-j} \xi_j &= \tfrac{1}{\pi} \int_0^\pi P_r(\tau) \left\{ \sum_{n=0}^{N} \xi_n \sum_{j=0}^{n} \omega_{n-j} \cos(n-j)\tau \xi_j \right\} \mathrm{d}\tau \\ &= \tfrac{1}{\pi} \int_0^\pi P_r(\tau) \left\{ \sum_{n=0}^{N} \cos n\tau \xi_n \sum_{j=0}^{n} \omega_{n-j} \cos j\tau \xi_j + \right. \\ &\qquad \left. + \sum_{n=0}^{N} \sin n\tau \xi_n \sum_{j=0}^{n} \omega_{n-j} \sin j\tau \xi_j \right\} \mathrm{d}\tau \geq 0.\end{aligned}$$

Hence $\{\omega_n r^n\}$ defines a positive quadrature. The converse follows easily since

$$\sum_{n=0}^{N} \xi_n \sum_{j=0}^{n} \omega_{n-j} \xi_j = \lim_{r \to 1} \sum_{n=0}^{N} \xi_n \sum_{j=0}^{n} \omega_{n-j} r^{n-j} \xi_j .$$

Assume then that $\{\nu_n\} \in l^1$. Define $\tilde{\nu} = \{\tilde{\nu}_n\}_{-\infty}^{\infty}$ by

$$\begin{aligned}\tilde{\nu}_n &= \nu_{|n|}, \;\; n \neq 0 \\ \tilde{\nu}_n &= 2\nu_0, \;\; n = 0.\end{aligned}$$

By (2.7) ν defines a positive quadrature if and only if

$$\sum_{n=0}^{\infty} \xi_n \sum_{j=0}^{\infty} \tilde{\nu}_{n-j} \xi_j \geq 0 \tag{4.10}$$

for all real sequences $\{\xi_n\}_0^\infty$ with compact supports.

Using Parseval's identity we notice that $\sum\limits_{n=0}^{\infty} \bar{\zeta}_n \sum\limits_{j=0}^{\infty} \tilde{\nu}_{n-j} \zeta$ is real even if $\{\zeta_n\}$ is a complex sequence.

Hence (4.10) is equivalent to the condition

$$\sum_{n=0}^{\infty} \bar{\zeta}_n \sum_{j=0}^{\infty} \tilde{\nu}_{n-j} \zeta_j \geq 0 \tag{4.11}$$

for all complex sequences $\{\zeta_n\}_0^\infty$ with compact supports.

But

$$\sum_{n=0}^{\infty} \bar{\zeta}_n \sum_{j=0}^{\infty} \tilde{\nu}_{n-j} \zeta_j = \sum_{n=M}^{\infty} \zeta_{n-M} \sum_{j=M}^{\infty} \tilde{\nu}_{n-j} \zeta_{j-M}, \quad \text{for all } M \in \mathbb{Z}$$

which shows that ν defines a positive quadrature if and only if $\tilde{\nu}$ is a positive definite function on $\mathbb{Z}$. Since $\tilde{\nu} \in l^1(\mathbb{Z})$, this happens if and only if $\hat{\tilde{\nu}}(\tau) \geq 0$

for all $\tau \in [-\pi, \pi]$ (see e.g [4, 12.13.3]). Hence the lemma is proved because $\operatorname{Re} \tilde{\nu}(\tau) = \frac{1}{2}\tilde{\nu}(\tau)$.

We now apply Theorems 3.1 and 4.1 to obtain an error bound for the numerical solution of (4.1). It has been an open problem whether A-stability (which is connected to the test equation (3.9), only) implies some stability properties for the difference equation (4.2) in more general cases. Let, again $x(t)$ be a solution of (4.1) and define τ_n by

$$\rho(E)\, x(nh) + h\sigma(E)\, \tilde{f}_n = \tau_n, \tag{4.12}$$

where $f_n = f(nh,\ x(nh))$, and put, as before $z_n = x_n - x(nh)$, $q_n = r_n - \tau_n$.

Corollary 4.1. *Assume that f satisfies (1.5) with $\mu > 0$. If the multistep method (ρ, σ) is A-stable then there exists a constant $C = C(\rho, \sigma)$ such that for all $N > k$*

(4.13)
$$\left\{\sum_{n=0}^{N} \|z_n\|^2\right\}^{\frac{1}{2}} \le C\frac{M}{\mu}\left\{(N+1)\max_{0\le j\le k-1}\left[\|z_j\|+\left\|f_j-\tilde{f}_j\right\|\right]+\left[\sum_{n=k}^{N}\left(\sum_{\mu=0}^{n-h}\|q_n\|\right)^2\right]^{\frac{1}{2}}\right\}$$

Proof. From (4.6) we get, for $n \ge k$,

$$z_n + h\sum_{\mu=0}^{n}\omega_{n-\mu}\left(f_\mu - \tilde{f}_\mu\right) = \psi_n - \tilde{\psi}_n + \sum_{\mu=0}^{n-h} a_{n-k-\mu}q_\mu \overset{def}{=} p_n,$$

where $\left\|\psi_n - \tilde{\psi}_n\right\| \le C \max_{0\le j<k\le l}\left\{\|z_j\| + \left\|f_j - \tilde{f}_j\right\|\right\}$ and $|a_n| \le C$.

By Theorem 3.1 $\|\{z_n\}\|_N \le \frac{M}{\mu}\|\{p_n\}\|_N$ which gives (4.13).

Note that if the method (ρ, σ) is also G-stable one can obtain a better bound using (2.22) and Theorem 2.2

As another application of Theorem 4.1 we consider the system

$$\begin{cases} x' + A(t)\, x + y = b(t) \\ y' - B(t)\, x = 0 \end{cases} \tag{4.14}$$

under the assumptions

$$\mu\|u\|^2 \le \langle u, A(t)\, u\rangle \le M\|u\|^2, \text{ for all } t \ge 0,\ u \in \mathbb{R}^d, \tag{4.15}$$

$$\begin{cases} B(t) \text{ is symmetric, positive definite an} \\ B'(t) \text{ is nonnegative definite, for all } t \ge 0. \end{cases} \tag{4.16}$$

Assume (4.14) is discretized using a G-stable one-leg method (ρ, σ)

$$\text{(4.17)} \qquad \begin{cases} \rho(E)\, x_n + hA(nh)\, \sigma(E)\, x_n + h\sigma(E)\, y_n = hb(nh) + r_n \\ \rho(E)\, y_n - hB(nh)\, \sigma(E)\, x_n = 0. \end{cases}$$

Let $(x(t), y(t))$ satisfy (4.14) and define τ_n, η_n by

$$\text{(4.18)} \qquad \begin{aligned} \rho(E)\, x(nh) + hA(nh)\, \sigma(E)\, x(nh) + h\sigma(E)\, \eta_n &= hb(nh) + \tau_n \\ \rho(E)\, n_n - hB(nh)\, \sigma(E)\, x(nh) &= 0. \end{aligned}$$

Put $x_n - x(nh) = z_n,\ y_n - \eta_n = w_n,\ t_n - \tau_n = q_n$, then we first have

$\rho(E)\, w_n = hB(nh)\, \sigma(E)\, z_n$ and $\rho(E)\, \sigma(E)\, w_n = h\sigma(E) \big\{ B(nh)\, \sigma(E)\, z_n \big\}$.

This can be written as

$$\sigma(E)\, w_n = \psi_n + h \sum_{\mu=0}^{n} \omega_{n-\mu} B(\mu h)\, \sigma(E)\, z_\mu,$$

where ψ_n satisfies $\|\psi_n\| \le C \max_{0 \le j \le k-1} \big\{ \|\sigma(E)\, w_j\| + \|B(jh)\, \sigma(E)\, z_j\| \big\}$.

Hence we can write

$$\text{(4.19)} \quad \rho(E)\, z_n = hA(nh)\, \sigma(E)\, z_n + h^2 \sum_{\mu=0}^{n} \omega_{n-\mu} B(\mu h)\, \sigma(E)\, z_\mu = q_n - h\psi_n,$$

which is of the form (2.19). Since (ρ, σ) is G-stable it is A-stable and the quadrature Ω in (4.19) is positive.

Therefore, if the operator

$$\text{(4.20)} \qquad (\mathcal{B}\varphi)(t) = \int_0^t B(s)\, \varphi(s)\, \mathrm{d}s$$

is positive in L^2 then all the bounds given in chapter 2 for (2.19) apply to (4.19). But, by (4.16),

$$\int_0^T \big\langle \varphi(t), (\mathcal{B}\varphi)(t) \big\rangle \mathrm{d}t =$$

$$= \tfrac{1}{2} \big\langle B^{-1}(T)\, (\mathcal{B}\varphi)(T), (\mathcal{B}\varphi)(T) \big\rangle - \tfrac{1}{2} \int_0^T \Big\langle \big[\tfrac{\mathrm{d}}{\mathrm{d}t} B^{-1} \big] (\mathcal{B}\varphi)(t), (\mathcal{B}\varphi)(t) \Big\rangle \mathrm{d}t,$$

which gives the positivity, since when $B(t)$ is positive definite so is $B^{-1}(t)$ and $-\frac{\mathrm{d}}{\mathrm{d}t} B^{-1}(t) = B^{-1}(t)\, B'(t)\, B^{-1}(t)$. $\square$

5. WEAKLY SINGULAR KERNELS

In applications one often meets equations where the positive operator $\mathcal{K}$ has a kernel which is continuous on $0 \le s < t < \infty$ but weakly singular on $s = t$. For example, $K(t,s) = (t-s)^{-\frac{1}{2}} B$, where B is symmetric and positive definite. We shall here give discretizations for some weakly singular kernels and study the positivity of the resulting quadratures.

We consider operators $\mathcal{K}$ which are of the form

$$(\mathcal{K}\varphi)(t) = \int_0^t a(t-s) B(t,s) \varphi(s) \, \mathrm{d}s, \tag{5.1}$$

where

$$a \in L^1(0,1) \cap C(0,\infty), \tag{5.2}$$

and $B(t,s)$ is symmetric and continuous on $0 \le s \le t < \infty$.

We associate with $a(t)$ a product integration quadrature $\mathcal{P} = \{w_{nk}\}$ and approximate $\mathcal{K}$ by $\mathcal{PB}$ where $\mathcal{B}$ is defined by

$$(\mathcal{B}\varphi)(r) = \int_0^t B(t,s) \varphi(s) \, \mathrm{d}s \tag{5.3}$$

and $\mathcal{PB}$ by

$$(\mathcal{PB}\xi)_n = h \sum_{j=0}^{n} w_{nj} B(nh, jh) \xi_j. \tag{5.4}$$

The problem then is to find results of the form:
if the operator

$$(\mathcal{A}\varphi)(t) = \int_0^t a(t-s) \varphi(s) \, \mathrm{d}s \tag{5.5}$$

is positive in L^2 then $\mathcal{P}$ is a positive quadrature, since when this is the case, all the error bounds in chapters 2 and 3 are applicable.

Given $a(t)$ we define sequences $\{\alpha_k\}$, $\{\beta_k\}$ by

$$\alpha_k = h^{-1} \int_0^h a\big((k+1)h - s\big) \mathrm{d}s, \quad k = 0, 1 \ldots \tag{5.6}$$

and

(5.7)
$$\beta_0 = h^{-2}\int_0^h sa\,(h-s)\,\mathrm{d}s,$$
$$\beta_k = h^{-2}\int_0^h \Big\{sa\big(\,(k+1)\,h-s\big)+(h-s)\,a\,(kh-s)\,\Big\}\mathrm{d}s,\quad k=1,2,\dots,$$

and the corresponding convolution quadratures

$$(\mathcal{P}_1\xi)_n = h\sum_{j=0}^{n}\alpha_{n-k}\xi_k \tag{5.8}$$

and

$$(\mathcal{P}_2\xi)_n = h\sum_{j=0}^{n}\beta_{n-k}\xi_k. \tag{5.9}$$

Let us at first observe that the positivity of $\mathcal{A}$ does not generally imply the positiveness of these quadratures. When $\mathcal{P}_1$ is positive, $\alpha_0 \geq 0$, but for $a\,(t)=\cos t$ we have $\alpha_0 = h^{-1}\sin h$. On $\mathcal{P}_1$ we shall assume that

(5.10) $a\,(t)$ is nonnegative, nonincreasing and convex for $t>0$.

THEOREM 5.1. *If a satisfies* (5.2) *and* (5.10), *then the quadrature $\mathcal{P}_1$ is positive.*

Proof. By (5.10) $\{\alpha_k\}$ is bounded and nonnegative and satisfies

(5.11)
$$\alpha_{k+1}-2\alpha_k+\alpha_{k-1} = h\int_0^h s\Big\{a\,((k+2)\,h-s)-2a\,((k+1)\,h-s)+a\,(kh-s)\,\Big\}\mathrm{d}s \geq 0$$

since the integrand is nonnegative by convexity. Forming a new sequence $\{\alpha_k r^k\}$ where $r\in(0,1)$, we still have

$$\alpha_{k+1}r^{k+1}-2\alpha_k r^k+\alpha_{k-1}r^{k-1}\geq 0,\quad \text{but now also } \{\alpha_k r^k\}\in l^1.$$

By [7, Theorem 4.1] there exists a nonnegative function $f\in L^1(-\pi,\pi)$ such that $f\,(n)=\alpha_{|n|}r^{|n|}$, for all $n\in\mathbb{Z}$. Thus

$$\mathrm{Re}\sum_{n=0}^{\infty}\alpha_n r^n \mathrm{e}^{in\tau} = \tfrac12\alpha_0 + \sum_{n=-\infty}^{\infty}\alpha_{|n|}r^{|n|}\mathrm{e}^{in\tau}\geq \tfrac12\alpha_0, \tag{5.12}$$

and the positivity follows from Lemma 4.1. □

From the proof of Theorem 5.1 we note that also $\mathcal{P}_2$ is positive if we have

$$\tilde{\beta}_{k-1} - 2\tilde{\beta}_k + \tilde{\beta}_{k-1} \geq 0 \text{ for } k > 0, \tag{5.13}$$

where $\tilde{\beta}_0 = 2\beta_0,\ \tilde{\beta}_k = \beta_k,\ k > 0$. However, (5.10) does not imply (5.13), consider, e.g., $a\,(t) = \begin{cases} 1-t, & t \leq 1 \\ 0, & t > 1 \end{cases}$ with $h = \frac{1}{2}$.

We shall therefore impose a stronger condition on $a(t)$:

$$a \in C^{\infty}\,(0,\infty) \text{ and } (-1)^i\, a^{(i)}\,(t) \geq 0 \quad \text{for } i = 0, 1, \ldots;\ t > 0. \tag{5.14}$$

Theorem 5.2. *If a satisfies* (5.2) *and* (5.14)*, then the quadrature $\mathcal{P}_2$ is positive.*

Proof. Since $a\,(t)$ is completely monotonic it can be represented in the form

$$a\,(t) = \int_0^{\infty} \mathrm{e}^{-pt}\mathrm{d}\lambda\,(p)\,, \tag{5.15}$$

where $\lambda\,(p)$ is nondecreasing and the integral converges for $t > 0$ [16, Theorem 12b]. Hence, for $r \in (0,1)\,,$

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n r^n \mathrm{e}^{in\tau} &= \sum_{n=0}^{\infty} \left(h^{-2} \int_0^h s a\,((n+1)\,h - s)\,\mathrm{d}s \right) r^n \mathrm{e}^{in\tau} \\
&\quad + \sum_{n=1}^{\infty} \left(h^{-2} \int_0^h (h-s)\, a\,(nh - s)\,\mathrm{d}s \right) r^n \mathrm{e}^{in\tau} \\
&= h^{-2} \int_0^{\infty} \left\{ \mathrm{e}^{-ph} \int_0^h \mathrm{e}^{sh} s \mathrm{d}s \sum_{n=0}^{\infty} r^n \mathrm{e}^{-pnh} \mathrm{e}^{in\tau} \right\} \mathrm{d}\lambda\,(p) \\
&\quad + h^{-2} \int_0^{\infty} \left\{ \int_0^h \mathrm{e}^{sh}\,(h-s)\,\mathrm{d}s \sum_{n=1}^{\infty} r^n \mathrm{e}^{-pnh} \mathrm{e}^{in\tau} \right\} \mathrm{d}\lambda\,(p) \\
&= h^{-2} \int_0^h \left\{ \mathrm{e}^{-ph} \int_0^h \mathrm{e}^{sh} s \mathrm{d}s + r\mathrm{e}^{-ph}\mathrm{e}^{i\tau} \int_0^h \mathrm{e}^{sh}\,(h-s)\,\mathrm{d}s \right\} \cdot \\
&\qquad \cdot \left\{ 1 - r\mathrm{e}^{-ph}\mathrm{e}^{i\tau} \right\}^{-1} \mathrm{d}\lambda\,(p)\,.
\end{aligned}$$

Put $\int_0^h \mathrm{e}^{sh} s\mathrm{d}s = A,\ \int_0^h \mathrm{e}^{sh}(h-s)\,\mathrm{d}s = B$, then

(5.16)

$$\operatorname{Re}\sum_{n=0}^{\infty}\beta_n r^n \mathrm{e}^{in\tau} = h^{-2}\int_0^{\infty}\left[\mathrm{e}^{-ph}A - r^2\mathrm{e}^{-2ph}B + r\left(\mathrm{e}^{-ph}B - \mathrm{e}^{-2ph}A\right)\cos\tau\right]\cdot$$
$$\cdot\left[1 - 2r\mathrm{e}^{-ph}\cos\tau + r^2\mathrm{e}^{-2ph}\right]^{-1}\mathrm{d}\lambda(p).$$

The integrand in (5.16) is nonnegative since $\lambda(p)$ is nondecreasing, $1 - 2re^{-ph}\cos\tau + r^2e^{-2ph} \geq 0$ and the first term attains its extreme values at $\tau = n\pi$. For $\cos\tau = 1$ we have

$$\left(\mathrm{e}^{-ph} - r\mathrm{e}^{-2ph}\right)(A - rB) \geq 0 \quad \text{since } A > B$$

and for $\cos\tau = -1$

$$\left(\mathrm{e}^{-ph} + r\mathrm{e}^{-2ph}\right)(A - rB) \geq 0.$$

Hence

$$\operatorname{Re}\sum_{n=0}^{\infty}\beta_n r^n \mathrm{e}^{in\tau} \geq 0$$

and the theorem follows from Lemma 4.1 □

6. ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS

Besides the error bounds there naturally arises another question whether $x_n \to 0$ as $n \to \infty$. We shall consider here only discretizations to the Volterra integro-differential equation (1.1).

On Volterra integro-differential equations we do not know any results on this problem (in addition to those on the test equation (2.34)). On ordinary differential equations some earlier results are known, see e.g [14], [15].

We consider the equation

(6.1)

$$\rho(E)\,x_n + hf\left(nh, \sigma(E)\,x_n\right) + h^2\sum_{j=0}^{n} w_{nj}K(nh, jh)\,\sigma(E)\,x_j = 0, \quad n \geq 0$$

under the assumptions that f satisfies (1.5),

(6.2) $\mathcal{WK}$ is a positive operator,

$$(6.3) \qquad (\rho, \sigma) \text{ is a G-stable method}$$

with

$$(6.4) \qquad \alpha < 1.$$

This is not sufficient to guarantee that $x_n \to 0$ (consider, e.g., $x' = 0$).

Therefore we shall impose a stronger condition on $\mathcal{WK}$.

DEFINITION 6.1. *The operator $\mathcal{WK}$ is said to be strictly positive if there exists $\{\omega_n\} = l^1$ with* $\operatorname{Re}\hat{\omega}(\tau) > 0$ *for $\tau \in [-\pi, \pi]$ such that $\mathcal{WK} - \Omega$ is a positive operator.*

Here Ω stands for the convolution quadrature associated with $\{\omega_n\}$ by (4.7).

The definition is a modification of a concept due to STAFFANS O.J. [12] on positive type functions on $\mathbb{R}_+$; $a \in L^1_{loc}$ is *of strictly positive type if there exists $b \in L^1$ satisfying* $\operatorname{Re}\hat{b}(\tau)$ for $\tau \in \mathbb{R}$ *such that $a - b$ is of positive type.* (A function a is of positive type if $\mathcal{A}$ in (5.5) is a positive operator). He used the concept when studying asymptotic behaviour of the solutions of the scalar equation

$$(6.5) \qquad y'(t) + \int_0^t a(t-s)\, g(x(s))\, \mathrm{d}s = \psi(t), \ t \geq 0.$$

THEOREM 6.1. *Assume that* (1.5), (6.2) *and* (6.3) *hold. If*

$$(6.6) \qquad \{f(mh, 0)\} \in l^1,$$

then all solutions of (6.1) *are bounded. If additionally* (6.4) *holds and*

$$(6.7) \qquad \mathcal{WK} \textit{ is a strictly positive operator,}$$

then they tend to zero as $n \to \infty$.

Proof. Writing (6.1) in the form

(6.8)

$$\rho(E)\, x_n + h\big[f(nh, \sigma(E)\, x_n) - f(nh, 0)\big] + h^2 \sum_{j=0}^{n} w_{nj} K(nh, jh)\, \sigma(E)\, x_j =$$

$$= -hf(nh, 0),$$

we notice that the first part of theorem follows from (2.21).

To show the second part we first deduce from (6.8) that

$$(6.9) \quad G(X_{N+1}) - G(X_0) + h^2 \sum_{n=0}^{N} \Big\langle \sigma(E)\, x_n, \sum_{j=0}^{n} w_{nj} K(nh, jh)\, \sigma(E)\, x_j \Big\rangle \leq$$

$$\leq 2hS \sum_{n=0}^{\infty} \|f(nh, 0)\|,$$

where $S = \sup \|\sigma(E) x_n\| < \infty$. Since $\mathcal{WK} - \Omega$ is positive (6.9) yields

$$G(x_{N+1}) - G(X_0) + h^2 \sum_{n=0}^{N} \left\langle \sigma(E) x_n, \sum_{j=0}^{n} \omega_{n-j} \sigma(E) x_j \right\rangle \leq \tag{6.10}$$

$$\leq 2hS \sum_{n=0}^{\infty} \|f(nh, 0)\|.$$

Hence for some $R < \infty$

$$0 \leq \sum_{n=0}^{N} \left\langle \sigma(E) x_n, \sum_{j=0}^{n} \omega_{n-j} \sigma(E) x_j \right\rangle \leq R, \qquad \text{for all } N \geq 0. \tag{6.11}$$

Put

$$\begin{aligned} \xi = \{\xi_n\}_{-\infty}^{\infty},\ \xi_n = \sigma(E) x_n,\ 0 \leq n \leq N \\ \xi_n = 0,\ n < 0, n > N. \end{aligned}$$

Then by (2.7) and (6.11) we have

$$0 \leq \sum_{n=-\infty}^{\infty} \left\langle \xi_n, \sum_{k=-\infty}^{\infty} \tilde{\omega}_{n-k} \xi_k \right\rangle \leq 2R$$

and, using Parseval's identity

$$0 \leq \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\tilde{w}}(\tau) \left\|\hat{\xi}(\tau)\right\|^2 \mathrm{d}\tau \leq 2R. \tag{6.12}$$

Since $0 < \hat{\tilde{w}}(\tau) \leq \|\tilde{\omega}\|_1$ we have by multiplying (6.12) by $\hat{\tilde{w}}(\tau)$ and using Parseval's identity and the definition of ξ that

$$0 \leq \sum_{n=-\infty}^{\infty} \left\| \sum_{k=0}^{N} \tilde{\omega}_{n-k} \sigma(E) x_k \right\|^2 \leq 2R \|\tilde{\omega}\|_1. \tag{6.13}$$

But then

$$\left\{ \sum_{k=0}^{\infty} \tilde{\omega}_{n-k} \sigma(E) x_k \right\} \in l^2(\mathbb{Z}), \qquad \text{and so } \sum_{k=0}^{\infty} \tilde{\omega}_{n-k} \sigma(E) x_k \to 0, \text{ as } n \to \infty.$$

Hence, for all $\{\gamma_k\} \in l^1(\mathbb{Z})$, $\sum_{k=0}^{\infty} \gamma_{n-k} \sigma(E) x_k \to 0$ by Wiener's Tauberian theorem, since $\tilde{\omega} \in l^1(\mathbb{Z})$, $\hat{\tilde{w}}(\tau) = 2\mathrm{Re}\hat{w}(\tau) > 0$ for $\tau \in [-\pi, \pi]$, and $\|\sigma(E) x_k\| \leq S$ for all $k \geq 0$. Choosing $\gamma_0 = 1$, $\gamma_k = 0$ for $k \neq 0$ we get $\alpha(E) x_k \to 0$. (Note that so far we have not needed the assumption $\alpha < 1$). By (2.23) we have

$$H(X_{n+1}) \leq \left(\alpha^2 + \varepsilon\right) H(X_n) + \|\sigma(E) x_n\|^2,$$

which gives

$$H(X_{n+1}) \le \left(\alpha^2+\varepsilon\right)^{n+1} H(X_0) + \sum_{k=0}^{n} \left(\alpha^2+\varepsilon\right)^{n-k} \|\sigma(E)\, x_k\|^2 . \tag{6.14}$$

Since $\alpha^2+\varepsilon<1$ and $\|\sigma(E)\, x_k\|^2 \to 0$ we finally get $H(X_{n+1}) \to 0$.
But H is positive definite and the theorem is proved. □

At the end we give examples on strictly positive operators.

EXAMPLE 6.1. Let $K(t,s)$ be symmetric and continuous on $0 \le t < s < \infty$ and assume that there exists a bounded nonincreasing function $a \in L^1$ with $\operatorname{Re} \hat{a}(\tau) > 0$ for $\tau \in \mathbb{R}$, such that

$$\mathcal{K} - \mathcal{A} \text{ is positive in } L^2, \tag{6.15}$$

where $\mathcal{K}$ and $\mathcal{A}$ are defined by (1.4) and (5.5) respectively. Let Ω_θ be the convolution quadrature associated with the sequence $\{\theta,1,1,1,\ldots\}$. Then $\Omega_\theta\mathcal{K}$ is strictly positive for $\theta \ge \frac{1}{2}$.

We show this. By Theorem 2.1 $\Omega_\theta(\mathcal{K}-\mathcal{A})$ is positive in l^2. Hence it suffices to show that $\Omega_\theta\mathcal{A}$ is generated by $\{\nu_n\} \in l^1$ with $\operatorname{Re}\nu(\hat{\tau}) > 0$ for $\tau \in [-\pi,\pi]$. But

$$(\Omega_\theta\mathcal{A}\xi)_n = h\left\{\theta a(0)\,\xi_0 + \sum_{j=0}^{n-1} a((n-j)\,h)\,\xi_j\right\}$$

and $a(t) = \int_{\mathbb{R}} \mathrm{e}^{-its}\alpha(s)\,\mathrm{d}s$, with $\alpha(s)>0,\ \alpha \in L^1(\mathbb{R})$. Since $a \in L^1$ and is nonincreasing,
$\nu = \{\theta a(0), a(h), a(2h), \ldots\} \in e^1$. Furthermore,

$$\begin{aligned}
\hat{\hat{\nu}}(\tau) &= \lim_{r\to 1-}\left\{2\theta a(0) + \sum_{k\neq 0} a(kh)\,\mathrm{e}^{ik\tau} r^{|k|}\right\} \\
&\ge \lim_{r\to 1-}\int_{\mathbb{R}} \alpha(s)\,\tfrac{1-r^2}{1-2r\cos(\tau-hs)+r^2}\mathrm{d}s \\
&\ge \lim_{r\to 1-} h\int_{-\pi}^{\pi} \alpha\left(th^{-1}\right)\tfrac{1-r^2}{1-2r\cos(\tau-t)+r^2}\mathrm{d}t \\
&= 2\pi h\alpha\left(\tau h^{-1}\right), \quad \text{a.e } \tau \in [-\pi,\pi].
\end{aligned}$$

But $\hat{\hat{\nu}}(\tau)$ is continuous, $\alpha\left(\tau h^{-1}\right) > 0$ for all τ, and so

$$\operatorname{Re}\tilde{\nu}(\tau) = \tfrac{1}{2}\hat{\hat{\nu}}(\tau) < 0, \quad \text{for all } \tau \in [-\pi,\pi]. \qquad \square$$

Example 6.2. If (5.2) and (5.10) hold then $\mathcal{P}_1$ in (5.8) is strictly positive if and only if $a(t) \not\equiv 0$.

This follows directly from (5.12) by choosing $\{\omega_n\}$ in the Definition 6.1 to be the sequence $\left\{\frac{1}{2}\alpha_0, 0, 0, \ldots\right\}$. Thus it follows that $\mathcal{P}_1$ may be a strictly positive operator in l^2 even if $a(t)$ is not of strictly positive type: a function satisfying (5.2) and (5.10) is of strictly positive type if and only if there does not exist a constant $\eta > 0$ such that $a(t)$ is linear on all intervals $(n\eta, (n+1)\eta)$, $n \geq 0$ [13, Theorem 2.3]. □

Example 6.3. If a satisfies (5.2) and (5.14), then the quadrature $\mathcal{P}_2$ is strictly positive if and only if $a(t) \not\equiv a(0)$.

If $a(t) \equiv a(0)$ then $\mathcal{P}_2$ is generated by the sequence $\left\{\frac{1}{2}a(0), a(0), a(0), \ldots\right\}$ and it is not strictly positive since if $\{\omega_n\}$ generates a positive quadrature, one has $2\omega_0 \geq |\omega_n|$, $n \geq 1$. On the other hand, if $a(t) \not\equiv a(0)$, there exists $\alpha > 0$, $\beta < \infty$, such that $\lambda(p) \not\equiv$ constant on $[\alpha, \beta]$ ($\lambda(p)$ is given by (5.15) and λ is continuous at α, β. Put $b(t) = \int_\alpha^\beta \mathrm{e}^{-pt} \mathrm{d}\lambda(p)$, then $a - b$ is completely monotonic and $b \in L^1$. By Theorem 5.2 $\mathcal{P}_2$ associated with $a - b$ is positive. Choosing Ω in definition 6.1 to be equal to $\mathcal{P}_2$ associated with b, the strict positivity follows, since for $p > 0$ the integrand in (5.16) (with $r = 1$) is strictly positive. □

This work was done during the authors stay at Institute Mittag-Leffler, Sweden.

REFERENCES

[1] Brunner, H. and Lambert, J.D., *Stability of numerical methods for Voterra integro-differential equations.* Computing, **12**, 75 (1974).

[2] Dahlquist, G., *A special stability problem for linear mutlistep methods,* BIT, **3**, 27, (1963).

[3] Dahlquist, G., *Error analysis for a class of methods for stiff non-linear initial value problems.* To be puyblished in *The Proceedings of the conference on numerical analysis, Dundee*, 1975.

[4] Edwards, R.E., *Fourier series: A modern introduction.* Vol. II. New York: Holt Rinehart and Winston, Inc., 1967.

[5] Garey, L., *The numerical solution of Volterra integral equations with singular kernels.* BIT, **14**, 33 (1974).

[6] De Hoog, F. and Weiss, R., *Implicit Runge-Kutta methods for second kind Volterra integral equations.* Numer. Math., **23**, 199 (1975).

[7] Katznelson, Y., *An introduction to harmonic analysis.* New-York: Wiley Inc., 1968.

[8] Malina, L., *A-stable methods of high order for Volterra integral equations.* Aplikace matematiky, **20** , 336 (1975).

[9] Matthys, S., *A-stable linear multistep methods for Volterra integro-differential equations.* To appear in Numer. Math.

[10] Nevanlinna, O., *On error bounds for G-stable methods.* To appear in BIT.

[11] Nevanlinna, O., *Positive quadratures for Volterra equations.* To appear in Computing.

[12] Staffans, O. J., *Nonlinear Volterra integral equations with positive definite kernels.* Proc. Amer. Math. Soc., **51**, 103 (1975).

[13] Staffans, O.J., *Positive definite measures with applications to a Volterra equation.* To appear in Trans. Amer. Math. Soc.

[14] Stetter, H.J., *Analysis of discretization methods for ordinary differential equations.* Berlin: Springer-Verlag, 1973.

[15] Stetter, H.J., *Discretizations of differential equations on infinite intervals and applications to function minimization.* In *Topics in numerical analysis*, Proceedings of the conference on numerical analysis, 1972, ed. J. J. H. Miller, New-York: Academic Press, 1973.

[16] Widder, D.V., *The Laplace transform.* Princeton: Princeton University Press, 1946.

Received 8.III.1976

Helsinki University of Technology,
Institute of Mathematics