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DIRECT AND CONVERSE THEOREMS FOR SPLINE APPROXIMATION WITH FREE KNOTS IN L_p

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Introduction. In [1], [2] we have considered direct and converse theorems for spline approximation with free knots in the uniform metric (in the space $C[0,1]$). In this note we shall consider the analogue problem for spline approximation in L_p.

We shall denote the set of all spline functions in the interval $[0, 1]$ with $n+1$ knots and of k-th degree with $S(k, n)$, i.e. $s \in S(k,n)$, if $s \in C^{k-1}[0,1]$ ($C^r[a,b]$ denotes as usually the of set all functions which have r-th continuous derivative in the interval $[a, b]$) and there exist $n+1$ points x_i, $i=0,\ldots,n$, $0=x_0<x_1<\ldots<x_n=1$, such that in each interval $[x_{i-1}, x_i]$, $i=1,\ldots,n$, s is an algebraic polynomial of a degree at most k. In the case $k=0$, $S(0,n)$ coincides with the class of all stepfunctions with $n-1$ jumps. Then we suppose that s is continuous either on the right or on the left. For the approximation in L_p the restriction $s \in C^{k-1}[0,1]$ is not essential. For this reason we shall consider also approximation by means of splines with a defect, i.e. without the restriction $s \in C^{k-1}[0,1]$. We shall denote the set of all splines with $n+1$ knots of k-th degree with a defect by $\tilde{S}(k, n)$, i.e. $s \in \tilde{S}(k, n)$ if there exist $n+1$ points x_i, $i=0,\ldots,n$, $0=x_0<x_1<\ldots<x_n=1$, such that in each interval (x_{i-1}, x_i), $i=1,\ldots,n$, s is an algebraic polynomial of a degree at most k.

The best approximation $E_n^k(f)_{L_p}$ of the function $f \in L_p$ by means of elements of $S(k, n)$ is defined by

$$E_n^k(f)_{L_p} = \inf_{s \in S(k,n)} \|f - s\|_{L_p[0,1]},$$

where

$$\|f - s\|_{L_p[0,1]} = \left\{\int_0^1 |f(x) - s(x)|^p dx\right\}^{\frac{1}{p}}$$

and the best approximation $\tilde{E}_n^k(f)_{L_p}$ of $f \in L_p$ by means of elements of $\tilde{S}(k,n)$ by

$$\tilde{E}_n^k(f)_{L_p} = \inf_{s \in \tilde{S}(k,n)} \|f - s\|_{L_p[0,1]}.$$

The best uniform approximation $E_n^k(f)$ is defined by

$$E_n^k(f) = \inf_{s \in S(k,n)} \|f - s\|_{C[0,1]},$$

where

$$\|f - s\|_{C[0,1]} = \max_{x \in [0,1]} |f(x) - s(x)|;\ f \in C[0,1].$$

The following lemma is valid (see BRUDNII [3]):

Lemma 1. *Let $f \in L_p[0,1]$. There exists a constant $c(k)$, depending only on k, such that*

$$\tilde{E}_m^k(f)_{L_p} \leqq E_m^k(f)_{L_p} \leqq c(k)\tilde{E}_n^k(f)_{L_p},$$

where $m = (n-1)k + n$.

This lemma also shows us that the restriction $s \in C^r[0,1]$ is not essential for the best spline approximation (see also [1], [2]).

In [1], [2] we introduce the following moduli by means of which it is possible to obtain direct and converse theorems for spline approximation with free knots in $C[0,1]$:

Let V be the set of all monotone functions in the interval $[0,1]$ with variation $\leqslant 1$, which are continuous either on the right or on the left. Then we defined:

$$\nu_k(f;\delta) = \inf_{\varphi \in V} \sup_{|\varphi(x+kh)-\varphi(x)| \leqslant \delta} |\Delta_h^k f(x)|, \tag{1}$$

where as usual $\Delta_h^k f(x)$ denotes the k-th difference of the function f in the point x with a step h:

$$\Delta_h^k f(x) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(x + mh)$$

and the sup in (1) is taken over all x and h for which $|\varphi(x+kh) - \varphi(x)| \leqslant \delta$.

The following theorem is announced in [1] and proved in [2]:

THEOREM A. *For every $k \geqslant 1$ there exists constants $c_1(k)$ and $c_2(k)$, such that for every $f \in C[0,1]$, we have*

$$c_1(k)\nu_k\left(f;\frac{1}{n}\right) \leqslant E_n^{k-1}(f) \leqslant c_2(k)\nu_k\left(f;\frac{k+1}{n}\right).$$

For every bounded function f we have

$$E_n^0(f) = \frac{1}{2}\nu_1\left(f;\frac{1}{n}\right).$$

The aim of this note is to give an analogue of Theorem A for the best spline approximation in $L_p[0,1]$.

Remark. Direct and inverse theorems for L_p — spline approximations with equidistant knots are obtained in [4], [5]. The case with free knots is considered in [6], where are obtained some different results, which use other characteristics of the functions.

1. First we shall introduce L_p — analogue of (1). Let V^* denote the set of all monotone increasing functions φ in the interval $[0,1]$, for which $\varphi(0) = 0$, $\varphi(1) = 1$. For every $\varphi \in V^*$ we set $\varphi(x) = 0$ for $x < 1$, $\varphi(x) = 1$ for $x > 1$.

We define:

$$\nu_k(f;\delta)_{L_p} = \inf_{\varphi \in V^*} \left\{\int_0^1 \sup_{t>0} \int_{\varphi(x-\delta)}^{\varphi(x+\delta)-kt} |\Delta_t^k f(u)|^p du\, dx\right\}^{\frac{1}{p}}. \tag{2}$$

If we set

$$\omega_k(f;a,b)_{L_p} = \left\{\sup_{t>0} \int_a^{b-kt} |\Delta_t^k f(u)|^p du\right\}^{\frac{1}{p}},$$

we can wright (2) as

$$\nu(f;\delta)_{L_p} = \inf_{\varphi \in V^*} \|\omega_k(f;\varphi(x-\delta),\varphi(x+\delta))_{L_p}\|_{L_p[0,1]}.$$

Let us mention some of the properties of $\nu_k(f;\delta)_{L_p}$.

Property 1. *$\nu_k(f;\delta)_{L_p}$ is monotone increasing function of δ, i. e. $\nu_k(f;\delta_1)_{L_p} \leqslant \nu_k(f;\delta_2)_{L_p}$, if $\delta_1 \leqq \delta_2$.*

This property is evident.

Property 2. *If $k > r$, then $\nu_k(f;\delta)_{L_p} \le 2^r \nu_{k-r}(f;\delta)_{L_p}$.*

Proof. We have:

$$\nu_k(f;\delta)_{L_p} = \inf_{\varphi\in V^*}\left\{\int_0^1 \sup_{t>0}\int_{\varphi(x-\delta)}^{\varphi(x+\delta)-kt} |\Delta_t^k f(u)|^p\, du\, dx\right\}^{\frac1p} =$$

$$= \inf_{\varphi\in V^*}\left\{\int_0^1 \sup_{t>0}\int_{\varphi(x-\delta)}^{\varphi(x+\delta)-kt}\left|\sum_{m=0}^r\binom{r}{m}(-1)^{r+m}\Delta_t^{k-r}f(u+mt)\right|^p du\,dx\right\}^{\frac1p} \le$$

$$\le \inf_{\varphi\in V^*}\left\{\int_0^1\sup_{t>0}\left(\sum_{m=0}^r\binom{r}{m}\left[\int_{\varphi(x-\delta)}^{\varphi(x+\delta)-kt}|\Delta_t^{k-r}f(u+mt)|^p du\right]^{\frac1p}\right)^p dx\right\}^{\frac1p} \le$$

$$\le \inf_{\varphi\in V^*}\left\{\int_0^1 \sup_{t>0} 2^{rp}\int_{\varphi(x-\delta)}^{\varphi(x+\delta)-(k-r)t}|\Delta_t^{k-r}f(s)|^p ds\,dx\right\}^{\frac1p} = 2^r\nu_{k-r}(f;\delta)_{L_p}.$$

We shall need some lemmas, which may be considered as known but for more clarity we shall give the full proofs.

Lemma 2. *Let the function f has integrable k-th derivative $f^{(k)}$ in the interval $[a,b]$. Then there exists an algebraic polynomial p of $(k-1)$-th degree such that*

$$\|f-p\|_{L_p(a,b)} \le \frac{(b-a)^k}{(k-1)!}\|f^{(k)}\|_{L_p(a,b)}.$$

Proof. The Taylor's formula give us

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \dots + \frac{(x-a)^{k-1}}{(k-1)!}f^{k-1}(a) + \int_a^b f^{(k)}(t)\frac{(x-t)_+^{k-1}}{(k-1)!}\,dt,$$

where

$$x_+^k = \begin{cases}0 & \text{if } x<0\\ x^k & \text{if } x\ge 0\end{cases}$$

Therefore there exists an algebraic polynomial p of $k-1$ -th degree such that

$$\|f-p\|_{L_p} = \left\|\int_a^b f^{(k)}(t)\frac{(x-t)_+^{k-1}}{(k-1)!}\,dt\right\|_{L_p} = \frac{1}{(k-1)!}\left\|\int_a^b f^{(k)}(t)(x-t)_+^{k-1}\right\|_{L_p}.$$

We have:

$$\left\|\int_a^b f^{(k)}(t)(x-t)_+^{k-1}\,dt\right\|_{L_p} \le \left\|\int_a^b f^{(k)}(t)(b-a)^{k-1}\,dt\right\|_{L_p} =$$

$$= (b-a)^{k-1+\frac1p}\left|\int_a^b f^{(k)}(t)\,dt\right| \le (b-a)^k\|f^{(k)}\|_{L_p}.$$

Lemma 3. *There exists a constant $c(k)$, depending only on k, such that for every $f\in L_p(a,b)$, there exists an algebraic polynomial q of $(k-1)$-th degree, such that*

$$\|f-q\|_{L_p\left(a,\frac{a+b}{2}\right)} \le c(k)\left\{\sup_{t>0}\int_a^{b-kt}\Delta_t^k|f(x)|^p\,dx\right\}^{\frac1p}.$$

Proof. Since $C(a,b)$ is dense in $L_p(a,b)$, we can assume that $f\in C(a,b)$. Let us set $h = \frac{(b-a)}{2k}$ and let

$$f_k(x) = (-1)^{k+1}h^{-k}\int_0^h\cdots\int_0^h\left\{f(x+(t_1+\dots+t_k)) - \binom{k}{1}f\left(x+\frac{k-1}{k}(t_1+\dots+t_k)\right)+\dots+(-1)^{k-1}f\left(x+\frac{t_1+\dots+t_k}{k}\right)\right\}dt_1\dots dt_k,$$

for $x\in\left(a,\frac{a+b}{2}\right)$.

We have:

$$f_k^{(k)}(x) = \frac{(-1)^{k+1}}{h^k}\left\{\Delta_h^k f(x) - \binom{k}{1}\Delta^k_{\frac{k-1}{k}h}f(x)+\dots+(-1)^{k-1}\Delta^k_{\frac{h}{k}}f(x)\right\},$$

for $x\in\left(a,\frac{a+b}{2}\right)$.

Consequently,

(3)
$$\|f_k^{(k)}\|_{L_p\left(a,\frac{a+b}{2}\right)} = \frac{1}{h^k}\left\{\int_a^{\frac{a+b}{2}}\left|\Delta_h^k f(x) - \binom{k}{1}\Delta^k_{\frac{k-1}{k}h}f(x)+\dots+(-1)^{k-1}\Delta^k_{\frac{h}{k}}f(x)\right|^p dx\right\}^{\frac1p} \le \frac{2^k}{h^k}\left\{\sup_{t>0}\int_a^{b-kt}|\Delta_t^k f(x)|^p\,dx\right\}^{\frac1p}.$$

On the other hand $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$\|f-f_k\|_{L_p\left(a,\frac{a+b}{2}\right)} = \left\{\int_a^{\frac{a+b}{2}} |f_k(x)-f(x)|^p\,dx\right\}^{\frac{1}{p}} \leqslant \tag{4}$$

$$\leqslant \left\{h^{kp\left(\frac{1}{q}-1\right)} \int_0^h \cdots \int_0^h \int_a^{\frac{a+b}{2}} \left|\Delta^k_{\frac{t_1+\ldots+t_k}{k}} f(x)\right|^p dx\,dt_1 \ldots dt_k\right\}^{\frac{1}{p}} \leqslant$$

$$\leqslant \left\{\sup_{t>0} \int_a^{b-kt} |\Delta_t^k f(x)|^p\,dx\right\}^{\frac{1}{p}}.$$

From (3), (4) and lemma 2 we obtain that there exists an algebraic polynomial q of a degree at most $k-1$, such that

$$\|f-q\|_{L_p\left(a,\frac{a+b}{2}\right)} \leqslant \|f-f_k\|_{L_p\left(a,\frac{a+b}{2}\right)} + \|f_k-q\|_{L_p\left(a,\frac{a+b}{2}\right)} \leqslant$$

$$\leqslant \left\{\sup_{t>0} \int_a^{b-kt} |\Delta_t^k f(x)|^p\,dx\right\}^{\frac{1}{p}} + \frac{2^k\left(\frac{b-a}{2}\right)^k}{h^k(k-1)!}\left\{\sup_{t>0}\int_a^{b-kt} |\Delta_t^k f(x)|^p\,dx\right\}^{\frac{1}{p}} =$$

$$= c(k)\left\{\sup_{t>0}\int_a^{b-kt} |\Delta_t^k f(x)|^p\,dx\right\}^{\frac{1}{p}}.$$

Remark. Lemma 3 is a L_p-analogue of the well-known result of WHITNEY [7].

2. Now we shall obtain a direct and inverse theorem for L_p — approximation with free knots.

THEOREM 1. *Let* $f \in L_p(0,1)$. *Then*

$$\nu_k\left(f;\frac{1}{4(n+1)}\right)_{L_p} \leqslant 2^k(n+1)^{-\frac{1}{p}} E_n^{k-1}(f)_{L_p}, \tag{5}$$

$$E_{2n}^{k-1}(f)_{L_p} \leqslant c_3(k) n^{\frac{1}{p}} \nu_k\left(f;\frac{k+1}{n}\right), \tag{6}$$

where the constant $c_3(k)$ depends only on k.

Proof. According to lemma 1 it is sufficient to prove

$$\nu_k\left(f;\frac{1}{4(n+1)}\right)_{L_p} \leqslant 2^k(n+1)^{-\frac{1}{p}} \tilde{E}_n^{k-1}(f)_{L_p}, \tag{5'}$$

$$\tilde{E}_{2n}^{k-1}(f)_{L_p} \leqslant c_4(k) n^{-\frac{1}{p}} \nu_k\left(f;\frac{1}{n}\right), \tag{6'}$$

where the constant $c_4(k)$ depends only on k.

Let us prove (5'). Let $\varepsilon > 0$ be arbitrary and let $s \in \tilde{S}(k-1,n)$ be such that

$$\left\{\int_0^1 |f(x)-s(x)|^p\,dx\right\}^{\frac{1}{p}} \leqslant \tilde{E}_n^{k-1}(f)_{L_p} + \varepsilon. \tag{7}$$

Let the knots of s are the points x_i, $i=0,\ldots,n$, $0=x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = 1$. We have $\Delta_t^k s(x) = 0$ for x, $x+kt \in (x_{i-1}, x_i)$.

We define:

$$\varphi(x) = \begin{cases} x_i, & \text{if } \frac{i}{n+1} \leqslant x \leqslant \frac{i+1}{n+1}, \quad i=0,\ldots,n \\ 1, & \text{if } x \geqslant 1 \\ 0, & \text{if } x \leqslant 0 \end{cases}$$

We have:

$$\int_0^1 \sup_{t>0} \int_{\varphi\left(x-\frac{1}{4(n+1)}\right)}^{\varphi\left(x+\frac{1}{4(n+1)}\right)-kt} |\Delta_t^k f(u)|^p\,du\,dx =$$

$$= \sum_{i=0}^{2(n+1)} \int_{\left(\frac{i-1}{2(n+1)}+\frac{1}{4(n+1)}\right)}^{\frac{i}{2(n+1)}+\frac{1}{4(n+1)}} \sup_{t>0} \int_{\varphi\left(x-\frac{1}{4(n+1)}\right)}^{\varphi\left(x+\frac{1}{4(n+1)}\right)-kt} |\Delta_t^k f(u)|^p\,du\,dx \leqslant$$

$$\leqslant \frac{1}{2(n+1)} \sum_{i=0}^{2(n+1)} \sup_{t>0} \int_{\varphi\left(\frac{i-1}{2(n+1)}\right)}^{\varphi\left(\frac{i+1}{2(n+1)}\right)-kt} |\Delta_t^k f(u)|^p\,du \leqslant$$

$$\leqslant \frac{1}{n+1} \sum_{i=0}^{n-1} \sup_{t>0} \int_{x_i}^{x_{i+1}-kt} |\Delta_t^k (f(u)-s(u))|^p\,du \leqslant$$

$$\leqslant \frac{2^{kp}}{n+1} \sum_{i=0}^{n-1} \sup_{t>0} \int_{x_i}^{x_{i+1}} |f(u)-s(u)|^p\,du = 2^{kp}(n+1)^{-1} \int_0^1 |f(x)-s(x)|^p\,dx.$$

From here and (7) we obtain:

$$\nu_k\left(f;\frac{1}{4(n+1)}\right)_{L_p} \leqslant \left\{\int_0^1 \sup_{t>0}\int_{\varphi\left(x-\frac{1}{4(n+1)}\right)}^{\varphi\left(x+\frac{1}{4(n+1)}\right)-kt} |\Delta_t^k f(u)|^p\,du\,dx\right\}^{\frac{1}{p}} \leqslant$$

$$\leqslant 2^k(n+1)^{-\frac{1}{p}}(\tilde{E}_n^{k-1}(f)_{L_p}+\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, (5') follows.

Let us prove (6'). Let $\varphi \in V^*$ be such that

$$\left\{\int_0^1 \sup_{t>0}\int_{\varphi\left(x-\frac{1}{n}\right)}^{\varphi\left(x+\frac{1}{n}\right)-kt} |\Delta_t^k f(u)|^p\,du\,dx\right\}^{\frac{1}{p}} \leqslant \nu_k\left(f;\frac{1}{n}\right)_{L_p}+\varepsilon$$

Let $x_i = \varphi\left(\frac{i}{n}\right)$. Then

(8) $$\nu_k\left(f;\frac{1}{n}\right)_{L_p}+\varepsilon \geqslant \left\{\sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sup_{t>0}\int_{\varphi\left(x-\frac{1}{n}\right)}^{\varphi\left(x+\frac{1}{n}\right)-kt} |\Delta_t^k f(u)|^p\,du\,dx\right\}^{\frac{1}{p}} \geqslant$$

$$\geqslant \left\{\sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sup_{t>0}\int_{x_{i-1}}^{x_i-kt} |\Delta_t^k f(u)|^p\,du\,dx\right\}^{\frac{1}{p}} =$$

$$= n^{-\frac{1}{p}}\left\{\sum_{i=1}^n \sup_{t>0}\int_{x_{i-1}}^{x_i-kt} |\Delta_t^k f(u)|^p\,du\right\}^{\frac{1}{p}}$$

From lemma 3 it follows that for every i, $i = 1,\ldots, n$, there exist two algebraic polynomials of $(k-1)$-th degree $p_i^{(j)}$, $j = 1,2$, such that

$$\int_{x_{i-1}}^{\frac{x_{i-1}+x_i}{2}} |f(x)-p_i^{(1)}(x)|^p\,dx \leqslant (c(k))^p \sup_{t>0}\int_{x_{i-1}}^{x_i-kt} |\Delta_t^k f(u)|^p\,du,$$

$$\int_{\frac{x_{i-1}+x_i}{2}}^{x_i} |f(x)-p_i^{(2)}(x)|^p\,dx \leqslant (c(k))^p \sup_{t>0}\int_{x_{i-1}}^{x_i-kt} |\Delta_t^k f(u)|^p\,du,$$

where the constant $c(k)$ depends only on k.

Therefore for the function $s \in \tilde{S}(k-1, 2n)$ given by

$$s(x) = \begin{cases} p_i^{(1)}(x) & \text{if } x \in \left[x_{i-1}, \frac{x_{i-1}+x_i}{2}\right) \\ p_i^{(2)}(x) & \text{if } x \in \left[\frac{x_{i-1}+x_i}{2}, x_i\right), \quad i = 1,\ldots,n, \end{cases}$$

we have:

(9) $$\left\{\int_0^1 |f(x)-s(x)|^p\,dx\right\}^{\frac{1}{p}} \leqslant 2c(k)\left[\sum_{i=1}^n \sup_{t>0}\int_{x_{i-1}}^{x_i-kt} |\Delta_t^k f(u)|^p\,du\right]^{\frac{1}{p}}$$

Since $\varepsilon > 0$ is arbitrary, from (8) and (9) we obtain (6').

Theorem 1 is proved.

Using theorem 1 we shall give two properties of $\nu_k(f;\delta)_{L_p}$.

From the results in [9] it follows that if f has a r-th derivative $f^{(r)} \in L_p$, $p \geqslant 1$, then

(10) $$E_n^k(f)_{L_p} \leqslant c_5(k) n^{-r} E_n^{k-r}(f^{(r)})_{L_p}, \quad k \geqslant r.$$

From (10) and theorem 1 we obtain, if $f^{(r)} \in L_p$ and $k > r$:

$$\nu_k\left(f;\frac{1}{4(n+1)}\right)_{L_p} \leqslant 2^k(n+1)^{-\frac{1}{p}} E_n^{k-1}(f)_{L_p}$$

$$\leqslant c_6(k) n^{-\frac{1}{p}} n^{-r} E_n^{k-r-1}(f^{(r)})_{L_p} \leqslant c_7(k) n^{-r} \nu_{k-r}\left(f^{(r)};\frac{4(k+1)}{n}\right)_{L_p},$$

i.e.

Property 3. *There exist a constant $c_7(k)$, depending only on k and a constant $c_8(k)$, depending only on k, such that if the function f has r-th derivative $f^{(r)} \in L_p$, $k > r$, then*

$$\nu_k(f;\ \delta)_{L_p} \leqslant c_7(k)\delta^r \nu_{k-r}(f^{(r)};\ c_8(k)\delta)_{L_p}$$

In [9] (see also [5]) it is shown that

$$E_n^{k-1}(f)_{L_p} \leqslant c_9(k)\ \omega_k\left(f;\ \frac{1}{n}\right)_{L_p}.$$

Using heorem 1 we obtain

Property 4. *A constant* $c_{10}(k)$ *exists, depending only on k such that*

$$\nu_k(f;\,\delta)_{L_p} \leqslant c_{10}(k)\omega_k(f;\,\delta)_{L_p},$$

where

$$\omega_k(f;\,\delta)_{L_p} = \sup_{0<h\leqslant\delta}\left\{\int\limits_0^{1-kh} |\Delta_h^k f(x)|^p\,dx\right\}^{\frac{1}{p}}$$

is the L_p*-modulus of continuity of the function f of k-th order.*

REFERENCES

[1] Popov, V. A., *Direct and converse theorem for spline approximation with free knots.* Comptes rendus de l'Acad. bulg. des Sci., **26**, 10, 1297–1299 (1973).
[2] Popov, V. A., *Direct and inverse theorems for spline approximation.* Math. Serdica (to appear).
[3] Брудний, Ю. А., *Кусочно- полиномиальная аппроксимация и локальные приближения,* ДАН СССР, **201**, *1,* 16–18 (1971).
[4] Butter G. J. Richards, W., *An L saturation theorem for splines* (to appear),
[5] Scherer, K., *Characterization of generalized Lipschitz classes by best approximation with splines* (to appear).
[6] Брудный, Ю. А. *Сплайн-аппроксимация и функции ограниченной вариации,* ДАН СССР, **215**, *3,* 511513 (1974).
[7] Whitney, H., *On functions with bounded n-th differences,* J. de Math. pures et appl. **36**, 67–95 (1957).
[8] Фройд, Попов, Г. В. А., *Некоторые вопросы связанные с аппроксимацией сплайн-функциями и многочленами.* Studia Sci. Math. Hung., **5**, 161–171 (1970).

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