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THE DENJOY INTEGRAL IN SOME APPROXIMATION PROBLEMS

by

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1. Denote by D^* the class of all 2π-periodic functions f integrable in the Denjoy-Perron sense on the interval $\langle 0,2\pi\rangle$. Let $\{\sigma_n^\alpha(f;x)\}$ be the sequence of Cesàro sums of the Fourier series of $f \in D^*$. As well-known,

$$\sigma_n^\alpha(f;x) = \frac{1}{\pi}\,(D^*)\int_0^{2\pi} f(t)\; K_n^\alpha\,(x-t)\;dt \qquad (\alpha \geqslant 0),$$

where

$$K_n^\alpha(t) = \frac{1}{2} + \frac{1}{A_n^\alpha}\sum_{k}^{\alpha} A_{n-k}^\alpha \cos kt,\quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}.$$

From Theorem 3 of [2] follows that, for all $\alpha > 1$,

(1) $$\lim_{n\to\infty} \sigma_n^\alpha(f;x) = f(x)$$

at each point x where

(2) $$(D^*)\int_{-h}^{h} (f(x+u) - f(x))\;du = 0(h) \qquad (h \to 0).$$

Since the function f is integrable in the Denjoy-Perron sense the last condition is fulfilled almost everywhere, and so is the relation (1). By the result of MARCINKIEWICZ (see [3], p. 360) we have taht (1) holds almost everywhere for $\alpha = 1$, too.

For 2π-periodic and Lebesgue-integrable functions the following estimate

$$|\sigma_n^1(f;x) - f(x)| \leqslant \frac{3}{n+1}\sum_{k=0}^{n} w\left(f,x;\frac{\pi}{k+1}\right),$$

with

$$w(f,x;h) = \sup_{0<t\leqslant h}\left\{\frac{1}{2t}\int_{-t}^{t}|f(x+u)-f(x)|du\right\},$$

is known (see [1]). A similar inequality is also true for Cesàro means of order $\alpha > 1$. Indeed, in this case,

$$\sigma_n^\alpha(f;x) - f(x) = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-2}A_k^1\,[\sigma_k^1(f;x) - f(x)]. \tag{3}$$

Using further the symbols $C_1(\alpha)$, $C_2(\alpha), \ldots$ for positive constants depending only on α, and applying the estimates

$$C_1(\alpha) \leqslant \frac{A_n^\alpha}{(n+1)^\alpha} \leqslant C_2(\alpha) \quad (\alpha \geqslant 0,\ n \geqslant 0),$$

we obtain

$$|\sigma_n^\alpha(f;x) - f(x)| \leqslant \frac{3C_2(\alpha-2)}{C_1(\alpha)(n+1)^\alpha}\sum_{k=0}^{n}(n-k+1)^{\alpha-2}\sum_{i=0}^{k} w\left(f,x;\frac{\pi}{i+1}\right)$$

$$\leqslant \frac{C_3(\alpha)}{n+1}\sum_{i=0}^{n} w\left(f,x;\frac{\pi}{i+1}\right).$$

The aim of this paper is to obtain the analogues of these results for arbitrary 2π-periodic and D^*-integrable functions.

2. Consider now the points x for which the condition (2) holds. As a measure of deviation of $\sigma_n^\alpha(f;x)$ from $f(x)$ we take the function $w(f,x;h)$ defined by

$$w(f,x;h)_D = \sup_{0<t\leqslant h}\left\{\frac{1}{2t}\left|(D^*)\int_{-t}^{t}(f(x+u)-f(x))du\right|\right\}.$$

Evidently, $w(f,x;h)_D$ is non-decreasing in h and, by (2),

$$\lim_{h\to 0} w(f,x;h)_D = 0$$

almost everywhere. Moreover, if f is a Lebesgue-integrable function we have $w(f,x;h)_D \leqslant w(f,x;h)$.

THEOREM 1. *Let $f \in D^*$, $\alpha \geqslant 2$. Then,*

$$|\sigma_n^\alpha(f;x) - f(x)| \leqslant \frac{C_4(\alpha)}{n+1}\sum_{k=0}^{n} w\left(f,x;\frac{\pi}{k+1}\right)_D \quad (n = 0,1,2,\ldots) \tag{4}$$

and the inequality cannot be improved.

Proof. Write

$$f_x(t) = f(x+t) + f(x-t) - 2f(x), \quad F_x(t) = (D^*)\int_0^t f_x(u)du.$$

By partial integration ([4], p. 244),

$$\sigma_n^\alpha(f;x) - f(x) = \frac{1}{\pi}(D^*)\int_0^\pi f_x(t)K_n^\alpha(t)\,dt$$

$$= \frac{1}{\pi}F_x(\pi)\,K_n^\alpha(\pi) - \frac{1}{\pi}\int_0^\pi F_x(t)\frac{d}{dt}K_n^\alpha(t)dt.$$

Since

$$K_n^\alpha(\pi) = \frac{1}{2A_n^\alpha}\sum_{k=0}^{n}(-1)^k A_{n-k}^{\alpha-1} \leqslant \frac{\alpha}{2(\alpha+n)},$$

we have

$$|\sigma_n^\alpha(f;x) - f(x)| \leqslant \frac{\alpha}{2\pi(\alpha+n)}|F_x(\pi)| + \frac{1}{\pi}\left(\int_0^{\pi(n+1)} + \int_{\pi(n+1)}^{\pi}\right)\left|F_x(t)\frac{d}{dt}K_n^\alpha(t)\right|dt$$

$$\leqslant \frac{\alpha}{2\pi(\alpha+n)}|F_x(\pi)| + I_n + Y_n.$$

If $0 \leqslant t \leqslant \pi$, then

$$|F_x(t)| = \left|(D^*)\int_{-t}^{t}(f(x+u)-f(x))du\right| \leqslant 2tw(f,x;t)_D,$$

and

$$\left|\frac{d}{dt}K_n^\alpha(t)\right| \leqslant \begin{cases} C_5(\alpha)n^2 & \text{when } 0 \leqslant t \leqslant \pi, \\ \dfrac{C_5(\alpha)}{n^{\alpha-1}t^{\alpha-1}} & \text{when } \dfrac{\pi}{n} \leqslant t \leqslant \pi \end{cases}$$

for $\alpha \leqslant 2$ ([7], p. 60). Applying these inequalities we obtain

$$|F_x(\pi)| \leqslant 2\pi w(f,x;\pi)_D \leqslant 2\pi \sum_{k=0}^{n} w\left(f,x;\frac{\pi}{k+1}\right)_D,$$

$$I_n \leqslant \frac{2}{\pi} C_5(\alpha) n^2 \int_0^{\pi/(n+1)} tw(f,x;t)_D dt \leqslant \pi C_5(\alpha)\, w\left(f,x;\frac{\pi}{n+1}\right)_D$$

$$= \frac{\pi C_5(\alpha)}{n+1} \sum_{k=0}^{n} w\left(f,x;\frac{\pi}{n+1}\right)_D \leqslant \frac{\pi C_5(\alpha)}{n+1} \sum_{k=0}^{n} w\left(f,x;\frac{\pi}{k+1}\right)_D,$$

$$Y_n \leqslant \frac{2}{\pi} C_5(\alpha)(n+1)^{1-\alpha} \int_{\pi/(n+1)}^{\pi} t^{-\alpha} w(f,x;t)_D\, dt$$

$$= \frac{2C_5(\alpha)}{\pi^\alpha (n+1)^{\alpha-1}} \int_1^{n+1} t^{\alpha-1}\, w\left(f,x;\frac{\pi}{t}\right)_D dt$$

$$\leqslant \frac{2C_5(\alpha)}{\pi^\alpha (n+1)^{\alpha-1}} \sum_{k=0}^{n} \frac{1}{(k+1)^{2-\alpha}}\, w\left(f,x;\frac{\pi}{k+1}\right).$$

Collecting the results we get the desired formula for $\alpha = 2$. In the case of $\alpha > 2$, we can write

$$|\sigma_n^\alpha(f;x) - f(x)| = \frac{1}{A_n^\alpha}\left|\sum_{k=0}^{n} A_{n-k}^{\alpha-3} A_k^2 (\sigma_k^2(f;x) - f(x)\right|$$

$$\leqslant \frac{C_4(2)}{A_n^\alpha} \sum_{k=0}^{n} \frac{1}{k+1} A_{n-k}^{\alpha-3} A_k^2 \sum_{i=0}^{k} w\left(f,x;\frac{\pi}{i+1}\right)_D$$

$$\leqslant \frac{C_6(\alpha)}{(n+1)^{(\alpha)}} \sum_{k=0}^{n} (2+k)(n-k+1)^{\alpha-3} \sum_{i=0}^{n} w\left(f,x;\frac{\pi}{i+1}\right)_D$$

$$\leqslant \frac{C_7(\alpha)}{n+1} \sum_{i=0}^{n} w\left(f,x;\frac{\pi}{i+1}\right)_D.$$

Thus, estimate (4) is proved for all $\alpha \geqslant 2$.

In order to show that the inequality (4) cannot be essentially improved we shall consider the continuous function $g(t) = 2\left|\sin\frac{1}{2}t\right|$. Then, for $\alpha \geqslant 1$,

$$|\sigma_n^\alpha(g;0) - g(0)| = |\sigma_n^\alpha(g;0)| = \frac{4}{\pi}\int_0^\pi \sin\frac{1}{2}t\; K_n^\alpha(t)dt$$

$$= \frac{2}{\pi A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \int_0^\pi \sin\left(k+\frac{1}{2}\right)t\; dt = \frac{2}{\pi A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1}\left(k+\frac{1}{2}\right)^{-1}$$

$$\geqslant \frac{C_8(\alpha)}{(n+1)^\alpha} \sum_{k=0}^{n} (n-k+1)^{\alpha-1}(k+1)^{-1} \geqslant \frac{C_9(\alpha)}{(n+1)^\alpha} \sum_{k=0}^{[n/2]} (n-k+1)^{\alpha-1}(k+1)^{-1}$$

$$\geqslant \frac{C_9(\alpha)}{n+1} \sum_{k=0}^{[n/2]} \frac{1}{k+1} \geqslant C_{10}(\alpha)\frac{\ln n}{n}.$$

Since for $0 < t \leqslant h$ we have

$$(D^*)\int_{-t}^{t} (g(0+u) - g(0))du = \int_{-t}^{t} 2\left|\sin\frac{1}{2}u\right| du \leqslant 2th,$$

it follows that $w(g,0;h)_D \leqslant h$. Hence, by (4),

$$|\sigma_n^\alpha(g;0) - g(0)| \leqslant \frac{C_4(\alpha)}{n+1} \sum_{k=0}^{n} w\left(g,0;\frac{\pi}{k+1}\right)_D$$

$$\leqslant \frac{\pi C_4(\alpha)}{n+1} \sum_{k=0}^{n} \frac{1}{k+1} \leqslant C_{11}(\alpha)\frac{\ln n}{n},$$

and this completes the proof.

Remark. In the case of $f \in D^*$ and $1 < \alpha < 2$, the above calculation leads to

$$|\sigma_n^\alpha(f;x) - f(x)| \leqslant \frac{C_{12}(\alpha)}{(n+1)^{\alpha-1}} \sum_{k=0}^{n} \frac{1}{(k+1)^{2-\alpha}}\, w\left(f,x;\frac{\pi}{k+1}\right)_D.$$

3. Following DŽVARŠEIŠVILI ([3]), let us introduce in the space D^* the norm

$$\|f\|_D = \sup_{\varphi\in V}\left|(D^*)\int_0^{2\pi} \varphi(x)f(x)dx\right|,$$

where V denote the class of all 2π-periodic functions of bounded variation in $\langle 0,2\pi\rangle$ and such that

$$\sup_{0\leqslant x\leqslant 2\pi} |\varphi(x)| \leqslant 1 \text{ and } \operatorname*{var}_{0\leqslant x\leqslant 2\pi} \varphi(x) \leqslant 1.$$

Denote by $\omega(f;\delta)_D$ the modulus of continuity of $f \in D^*$, i.e.

$$\omega(f;\delta)_D = \sup_{|h|\leqslant\delta} \|f(\cdot+h) - f(\cdot)\|_D.$$

We shall investigate the convergence of $\sigma_n^\alpha(f;x)$ in the norm $\|\ \ \|_D$.

THEOREM 2. *If $f \in D^*$, then for all $\alpha \geqslant 1$,*

(5) $$\|\sigma_n^\alpha(f;\cdot)-f(\cdot)\|_D \leqslant \frac{C_{13}(\alpha)}{n+1}\sum_{k=0}^{n}\omega\left(f;\frac{\pi}{k+1}\right)_D \quad (n=0,1,2,\ldots).$$

Proof. Consider first the case $\alpha = 1$. Applying Theorem 19 I of [3] we find

$$\|\sigma_n^1(f;\cdot)-f(\cdot)\|_D = \frac{1}{\pi}\sup_{\varphi\in V}\left|(D^*)\int_0^{2\pi}\left((D^*)\int_0^{\pi}\varphi(x)f_x(t)\,K_n^1(t)\,dt\right)dx\right|$$

$$= \frac{1}{\pi}\sup_{\varphi\in V}\left|\int_0^{\pi}\left((D^*)\int_0^{2\pi}\varphi(x)f_x(t)dx\right)K_n^1(t)\,dt\right| \leqslant \frac{2}{\pi}\int_0^{\pi}\omega(f;t)_D\,K_n^1(t)dt$$

$$= \frac{1}{\pi(n+1)}\left(\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi}\right)\omega(f;t)_D\left(\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{1}{2}t}\right)^2 dt \leqslant P_n+Q_n.$$

Clearly,

$$P_n \leqslant \frac{1}{\pi}(n+1)\int_0^{\pi/(n+1)}\omega(f;t)_D dt \leqslant \omega\left(f;\frac{\pi}{n+1}\right)_D \leqslant \frac{1}{n+1}\sum_{k=0}^{n}\omega\left(f;\frac{\pi}{k+1}\right)_D,$$

$$Q_n \leqslant \frac{\pi}{n+1}\int_{\pi/(n+1)}^{\pi} t^{-2}\,\omega(f;t)_D\,dt = \frac{1}{n+1}\int_1^{n+1}\omega\left(f;\frac{\pi}{t}\right)_D dt \leqslant \frac{1}{n+1}\sum_{k=0}^{n}\omega\left(f;\frac{\pi}{k+1}\right)_D.$$

Hence,

$$\|\sigma_n^1(f;\cdot)-f(\cdot)\|_D \leqslant \frac{2}{n+1}\sum_{k=0}^{n}\omega\left(f;\frac{\pi}{k+1}\right)_D.$$

Now, using the identity (3) we easily get the desired formula for all $\alpha > 1$.

Inequality (5) cannot be improved. In order to show this, let us consider the function

$$q(x) = \sum_{k=1}^{\infty}\frac{\sin kx}{k}.$$

Since q is 2π-periodic and Lebesgue-integrable function, we have

$$\|\sigma_n^\alpha(q;\cdot)-q(\cdot)\|_D = \sup_{\varphi\in V}\left|\int_0^{2\pi}\varphi(x)\,(\sigma_n^\alpha(q;x)-q(x))\,dx\right|$$

$$\geqslant \left|\int_0^{2\pi}\varphi_1(x)\,(\sigma_n^\alpha(q;x)-q(x))\,dx\right|,$$

where $\varphi_1(x)$ is of class V and such that $\varphi_1(x) = \frac{1}{2}$ for $0 \leqslant x < \pi$ and $\varphi_1(x) = 0$ for $\pi \leqslant x < 2\pi$. Reasoning as in [6], p. 428 we find

$$\|\sigma_n^\alpha(q;\cdot)-q(\cdot)\|_D \geqslant \left|\int_0^{\pi}(\sigma_n^\alpha(q;x)-q(x))dx\right|$$

$$\geqslant \frac{C_{14}(\alpha)}{n+1}\sum_{k=0}^{[n/2]}\frac{1}{k+1} \geqslant C_{15}(\alpha)\frac{\ln n}{n}.$$

On the other hand, if $|h| \leqslant \delta$, then

$$\omega(q;\delta)_D = \sup_{\varphi\in V}\left|\int_0^{2\pi}\varphi(x)\,(q(x+h)-q(x))\,dx\right|$$

$$\leqslant \int_0^{2\pi}|q(x+h)-q(x)|\,dx \leqslant 2\pi\delta.$$

Hence, by (5),

$$\|\sigma_n^\alpha(q;\cdot)-q(\cdot)\|_D \leqslant \frac{2\pi^2 C_{13}(\alpha)}{n+1}\sum_{k=0}^{n}\frac{1}{k+1} \leqslant C_{16}(\alpha)\frac{\ln n}{n},$$

and the assertion follows.

4. Similar results can be obtained for the Riesz means $S_n^r(f;x)$ given by the integral formula

$$S_n^r(f;x) = \frac{1}{\pi}(D^*)\int_0^{2\pi}f(t)\,Q_n^r(x-t)dt \quad (r>0,\ f\in D^*),$$

where

$$Q_n^r(t) = \frac{1}{2}+\sum_{k=1}^{n}\left\{1-\frac{k^2}{\left(n+\frac{1}{2}\right)^2}\right\}^r\cos kt.$$

It is known ([5]) that for $r > 0$

$$|Q_n^r(t)| \leqslant \begin{cases} 2n & \text{if } 0 \leqslant t \leqslant \pi,\\ \dfrac{C_{17}(r)}{n^r\,t^{r+1}} & \text{if } \dfrac{\pi}{n} \leqslant t \leqslant \pi,\end{cases}$$

and for $r \geqslant 1$

$$\left|\frac{d}{dt} Q_n^r(t)\right| \leqslant \begin{cases} 2\, n^2 & \text{if } 0 \leqslant t \leqslant \pi, \\ \dfrac{C_{18}(r)}{n^{r-1}\, t^{r+1}} & \text{if } \dfrac{\pi}{n} \leqslant t \leqslant \pi. \end{cases}$$

Applying these inequalities and arguing as in the preceding two sections we get the following statements.

THEOREM 3. *Suppose that* $f \in D^*$, $r \geqslant 2$. *Then,*

$$|S_n^r(f;x) - f(x)| \leqslant \frac{C_{19}(r)}{n+1} \sum_{k=0}^{n} w\left(f, x; \frac{\pi}{k+1}\right)_D$$

at each point x *where the condition* (2) *holds.*

THEOREM 4. *Given any* $f \in D^*$, *we have*

$$\|S_n^r(f;\cdot) - f(\cdot)\|_D \leqslant \begin{cases} \dfrac{C_{20}(r)}{n+1} \displaystyle\sum_{k=0}^{n} \omega\left(f; \frac{\pi}{k+1}\right)_D & \text{if } r = 1, \\ C_{21}(r)\ \omega\left(f; \dfrac{\pi}{n+1}\right)_D & \text{if } r > 1. \end{cases}$$

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