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BEST APPROXIMATION BY CHEBYSHEVIAN SPLINES AND GENERALIZED LIPSCHITZ SPACES

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The purpose of this note is to extend results in SCHERER [7] on the characterization of generalized Lipschitz spaces by best approximation by means of polynomial splines to the more general case of Chebyshevian splines. Thereby results of RICHARDS-DEVORE [5], [6] in the case of approximation in the $C[a, b]$-norm are completed and further extended to approximation in the $L_p(a, b)$-norm, $1 \leq p < \infty$.

The procedure is the following: first a direct theorem is proved by extending a corresponding result of JEROME [2] in the $C[a, b]$ case to the $L_p(a,b)$ spaces; then, by the methods in [7], it is shown that a converse theorem holds with respect to moduli of continuity introduced in [6] via generalized divided differences. In the third section relations between these moduli of continuity and the ordinary ones are established. A combination of all these results then yields the desired characterization of generalized Lipschitz spaces.

1. A direct theorem

Let $C^m[a,b]$ be the space of m-times $(m = 0,1,2,\ldots)$ continuously differentiable functions on $[a, b]$, and $L_p^m(a, b)$, $1 \leq p \leq \infty$, be the space of functions on (a, b) whose $(m-1)$th derivative is absolutely continuous and whose m-th derivative belongs to $L_p(a,b)$. The splines in question are defined as the elements of the classes

(1) $\mathrm{Sp}(L, \Delta, m) = \{S \in L_\infty^m(a, b) : LS(x) = 0,\ x \in (x_{i-1}, x_i),\ 1 \leq i \leq N,$

where Δ is a partition of the interval $[a, b]$, i.e.

(2) $\Delta : a = x_0 < x_1 < \ldots < x_N = b$, $\underline{\Delta} = \min (x_i - x_{i-1})$, $\overline{\Delta} = \max (x_i - x_{i-1})$, L is a differential operator of the form $(0 \leq m \leq n-1)$

(3) $$L = D_{n-1} \ldots D_0,$$

where $D_i f(x) = d/dx[f(x)/w_i(x)]$ with strictly positive functions $w_i(x) \in C^{n-i}[a, b]$. Then there exists a basis of the null space of L of the form

(4)
$$u_0(x) = w_0(x)$$
$$u_1(x) = w_0(x) \int_a^x w_1(\xi_1) d\xi_1$$
$$\vdots$$
$$u_{n-1}(x) = w_0(x) \int_a^x w_1(\xi_1) \int_a^{\xi_1} \ldots \int_a^{\xi_{n-2}} w_{n-1}(\xi_{n-1}) d\xi_{n-1} \ldots d\xi_1.$$

The system $(u_0, \ldots, u_{n-1})$ forms an extended complete Chebyshev (*E.C.T*) system on $[a, b]$ (KARLIN [4, p. 276]), in particular the Wronskian $W(u_0, u_1, \ldots, u_{n-1})$ is strictly positive on $[a,b]$, i.e.,

(5) $$\begin{vmatrix} u_0(x) & \ldots\ldots & u_{n-1}(x) \\ \vdots & & \vdots \\ D^n u_0(x) & \ldots\ldots & D^n u_{n-1}(x) \end{vmatrix} > 0 \quad (x \in [a, b]).$$

Denoting by u_i^* the elements of the last column of $W^{-1}(u_0, u_1, \ldots, u_{n-1})$ one sets

$$\hat{\theta}(x, \xi) = \begin{cases} \theta(x, \xi), & x \geq \xi \\ 0, & x < \xi, \end{cases}$$

where $\theta(x, \xi) = \sum_{k=0}^{n-1} u_k(x) u_k^*(\xi)$. Then by definition

(6) $$D_x^j \hat{\theta}(x, \xi) \Big|_{x=\xi} = \delta_{j, n-1} \qquad (0 \leq j \leq n-1)$$

so that for $f \in C^n[a, b]$ the generalized Taylor formula

(7) $$f(x) = u(x) + \int_a^b \hat{\theta}(x, \xi) Lf(\xi) \Phi(\xi) d\xi$$

holds, where $\Phi(\xi) = w_0(\xi)\, w_1(\xi) \ldots w_{n-1}(\xi)$ and $u(x)$ satisfies $Lu(x) = 0$ on $[a,b]$ and $D^j u(a) = D^j f(a)$, $0 \leq j \leq n-1$. By taking a sequence of functions $\{f_m\}$ in $C^n[a,b]$ which converge in $L_1^n(a, b)$ to an element $f \in L_p^n(a, b)$, it is seen that (7) holds more generally for functions in $L_p^n(a, b)$, $1 \leq p < \infty$.

The linear approximation method of JEROME [2] (defined more generally for an arbitrary nonsingular linear differential operator with suitable constants) is now described as follows:
For each $\xi \in [a,b]$ let the points $t_1(\xi), \ldots, t_n(\xi)$ be consecutive points of the mesh (2) satisfying

$$\underline{\Delta} \leq |t_1 - \xi| \leq 2\overline{\Delta}; \ |t_1 - \xi| < |t_2 - \xi| < \ldots < |t_n - \xi|,$$

and let $\beta_0, \ldots, \beta_n$ be any (non-trivial) solution of the system

$$\beta_0 u_1^*(t_0) + \ldots + \beta_n u_1^*(t_n) = 0$$
$$\vdots$$
$$\beta_0 u_n^*(t_0) + \ldots + \beta_n u_n^*(t_n) = 0.$$

Setting $t_0(\xi) = \xi$, the approximating spline $S(x) = S(f; x)$ is defined by

(8) $$S(x) = u(x) + \int_a^b \gamma(x, \xi) Lf(\xi)\, d\xi$$

with $u(x)$ being as in (7) and

$$\gamma(x, \xi) = \hat{\theta}(x, \xi) - (1/\beta_0) \sum_{j=0}^{n} \beta_j \hat{\theta}(x, t_j).$$

Obviously $S(x)$ belongs to $\mathrm{Sp}(L,\Delta,n)$ by (6). JEROME [2] has shown that $\sum_{j=0}^{n} \beta_j \hat{\theta}(x, t_j)$ has compact support in $[t_0, t_n]$ and that $\beta_j/\beta_0 \leq C$ for $1 \leq j \leq n$ independent of the choice of $\{t_i\}$ provided $\overline{\Delta}/\underline{\Delta}$ remains bounded. By the Taylor expansion (see (6)) one has furthermore

$$\hat{\theta}(x,y) = (x-y)_+^{n-1}/(n-1)! + 0((x-y)_+^n),$$

so that one may estimate

(9) $$|f(x) - S(f; x)| \leq \int_a^b \chi(x, \xi) |Lf(\xi)| d\xi$$

where $\chi(x, \xi) \geq 0$ has support in $|x - \xi| \leq (n+2)\overline{\Delta}$ and satisfies $\chi(x, \xi) \leq C_0 \overline{\Delta}_{n-1}$, the constant * being independent of x and f.

From this one easily concludes (with constants C_1, C_2)

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* The constants will be denoted successively by C_0, C_1, ...

$$\|f-S(f)\|_\infty \leqslant C_1\bar{\Delta}^n\|Lf\|_\infty \qquad (f \in L^n_\infty(a, b)),$$

$$\|f-S(f)\|_1 \leqslant C_2\bar{\Delta}^n\|Lf\|_1 \qquad (f \in L^n_1(a, b)).$$

By arguments of the theory of interpolation of Banach spaces these inequalities imply

$$(10) \qquad \|f-S(f)\|_p \leqslant C_3\bar{\Delta}^n\|Lf\|_p, \qquad (f \in L^n_p(a, b),\ 1 \leqslant p \leqslant \infty).$$

Indeed, by definition of Peetre's K-functional one has for $f \in L^n_1(a, b)$

$$K(t, f-S(f)\,;\, L_1(a, b), L_\infty(a, b)) \leqslant \inf_{g \in L^n_\infty(a, b)} (\|S(f-g)-(f-g)\|_1 + t\|S(g)-g\|_\infty) \leqslant$$

$$\leqslant \max\,(C_1, C_2)\bar{\Delta}^n \inf_{g \in L^n_\infty(a, b)} (\|L(f-g)\|_1 + t\|Lg\|_\infty) \leqslant$$

$$\leqslant \max\,(C_1, C_2)\bar{\Delta}^n K(t, Lf\,;\, L_1(a, b), L_\infty(a, b)).$$

But since $\int_0^\infty [t^{-1+1/p} K(t, f; L_1, L_\infty)]\,dt/t$ is a norm equivalent to $\|f\|_p$ (see [1, p. 185]), there follows (10).

From this one obtains in a standard manner for the best approximation $E^{(L, m)}_\Delta(f; p) = \inf_{s \in Sp(L, \Delta, m)} \|f-s\|_p$ the Jackson-type inequality

$$E^{(L, m)}_\Delta(f; p) \leqslant C_4 K(t, f; L_p(a, b), L^n_p(a, b)).$$

Using an estimate in [3] therefore establishes the direct theorem

THEOREM 1. *Under the condition that the ratio $\bar{\Delta}/\Delta$ remains bounded there holds for each $f \in L_p(a,b)$, $1 \leqslant\ < \infty$ (or $f \in C[a,b]$ if $p = \infty$)*

$$(12) \qquad E^{(L, m)}_\Delta(f; p) \leqslant C_\sigma[\bar{\Delta}^n\|f\|_p + \omega_n(f; \bar{\Delta})_p].$$

Here $\omega_n(f; \bar{\Delta})_p$ is the n th modulus of continuity defined by

$$(13) \qquad \omega_n(f; t)_p = \begin{cases} \sup\limits_{0<|h|\leqslant t} \left\{ \int\limits_{x,\, x+nh \in [a, b]} |\Delta^n_h f(x)|^p\, dx^{1/p} \right\}, & 1 \leqslant p < \infty \\ \sup\limits_{0<|h|\leqslant t} \operatorname{ess.\,sup}\limits_{x,\, x+nh \in [a, b]} |\Delta^n_h f(x)|, & p = \infty. \end{cases}$$

Here $\Delta^n_h f(x)$ is the n th (forward) difference of $f(x)$ with increment h.

2. An inverse theorem

Following KARLIN [4, p. 523] the generalized divided difference of f at $x_i, \ldots, x_{i+n}$ is defined by

$$(14) \qquad f(x_i, \ldots, x_{i+n}) = \frac{\begin{vmatrix} u_0(x_i) & \ldots & u_{n-1}(x) & f(x_i) \\ \vdots & & & \\ u_0(x_{i+n}) & \ldots & u_{n-1}(x_{i+n}) & f(x_{i+n}) \end{vmatrix}}{\begin{vmatrix} u_0(x_i) & \ldots & u_n(x_i) \\ \vdots & & \\ u_0(x_{i+n}) & \ldots & u_n(x_{i+n}) \end{vmatrix}}$$

where $u_n(x) = w_0(x)\int_a^x w_1(\xi_1) \ldots \int_a^{\xi_{n-1}} w_n(\xi_n)\,d\xi_n \ldots d\xi_1$ with $w_n(x) \equiv 1$. The generalized divided difference has the property

$$(15) \qquad Lf = 0 \Rightarrow f(x_i, \ldots, x_{i+n}) = 0.$$

In generalization of (13) one defines (see [6])

$$(16) \qquad \omega_L(t; f)_p = \begin{cases} \sup\limits_{0<|h|\leqslant t} \{ \int\limits_{x,x+nh \in (a,b)} |\delta^n_h f(x)|^p \}^{1/p}, & 1 \leqslant p < \infty \\ \sup\limits_{0<|h|\leqslant t} \operatorname{ess.\,sup}\limits_{x,x+nh \in (a,b)} |\delta^n_h f(x)|, & p = \infty \end{cases}$$

with

$$(17) \qquad \delta^n_h f(x) = h^n f(x, x+h, \ldots, x+nh).$$

Since $f(x_i, \ldots, x_{i+n})$ is linear in f, the generalized modulus of continuity $\omega_L(t; f)_p$ is sublinear in f, i.e.,

$$(18) \qquad \omega_L(t; f_1+f_2)_p \leqslant \omega_L(t; f_1)_p + \omega_L(t; f_2)_p.$$

For the following the relation

$$(19) \qquad \delta^n_h f(x) = \frac{1}{B(x, h)} \sum_{r=0}^{n} h^r \Delta^{n-r}_h f(x+rh) B_r(x, h)$$

together with (see RICHARDS-DEVORE [6])

$$(20) \qquad \|B_r(x, t)/B(x, t)\|_\infty \leqslant C_7 \qquad (0 \leqslant r \leqslant n;\ 0 < t \leqslant 1),$$

$$(21) \qquad \|B_0(x, t)/B(x, t)\|_\infty \geqslant C_8 \qquad (0 < t \leqslant 1),$$

is essential.

A consequence of (19), (20) is

$$(22) \qquad \omega_L(t; f)_p \leqslant C_9\|f\|_p.$$

With the help of the properties (15), (18) and (22) the following converse theorem can be proved

THEOREM 2. *Let $E_N^{(L,m)}(f;p)$ denote the best approximation $E_{\Delta_N}^{(L,m)}(f;p)$, where Δ_N is the equidistant partition of $[a, b]$ into segments of lengths $(b-a)/N$. Then one has for $f \in L_p(a, b)$, $1 \leqslant p < \infty$ or $f \in C[a, b]$ if $p = \infty$, N being odd and $N \geqslant 3$,*

(23) $$\omega_L(N^{-1}; f)_p \leqslant C_{10}[E_N^{(L,0)}(f;p) + E_{N+1}^{(L,0)}(f;p)].$$

Proof. Since it is very similar to that one of the corresponding theorem in [7] for polynomial splines only a brief sketch is given.

Let S denote an element of best approximation of $\mathrm{Sp}(L, \Delta_N, 0)$ to $f \in L_p(a, b)$. By (18) one has for $0 < t \leqslant 1$

$$\omega_L(t; f)_p \leqslant C_{11}\|f - S\|_p + \omega_L(t; S)_p.$$

By (15) and (22) one may estimate further ($0 < t \leqslant C_{12}N^{-1}$ with suitable small constant C_{11}, the points $x_{i,N}$ being the knots of Δ_N)

$$\omega_L(t; S)_p \leqslant \sup_{0<h\leqslant t}\left\{\sum_{i=1}^{N-1}\int_{x_{i,N-rh}}^{x_{i,N}}|\delta S_h^n(x)|^p\,dx\right\}^{1/p} \leqslant$$

$$\leqslant C_{13}\left\{\|f - S\| + \sup_{0<h\leqslant t}\left\{\sum_{i=1}^{N-1}\int_{x_{i,N-rh}}^{x_{i,N}}|\delta_h^n f(x)|\,dx\right\}^{1/p}\right\}.$$

But the last term can be estimated by $E_{N+1}^{(L,0)}(f;p)$ in exactly the same manner as in [7] using property (15).

3. Connections between moduli of continuity

The following theorem is proved:

THEOREM 3. *There exist constances C_{14}, C_{15} and a δ, $0 < \delta \leqslant 1$, such that for every $f \in L_p(a, b)$ and $0 < t \leqslant \delta$ there hold the inequalities**

(24) $$\omega_L(t; f_p) \leqslant C_{14}[t^n\|f\|_p + \omega_n(t; f)_p],$$

(25) $$\omega_n(t; f)_p \leqslant C_{15}[t^n\|f\|_p + \omega_L(t; f)_p].$$

Proof. By relation (19) and the estimates (20), (21) there follows immediately

(26) $$\omega_n(t; f)_p \leqslant C_7\left[t^n\|f\|_p + \omega_n(t; f)_p + \sum_{r=1}^{n-1} t^r\omega_{n-r}(t; f)_p\right]$$

* In general the term $t^n\|f\|_p$ cannot be dropped since this would imply that the nul-space of D^n is contained in that of L, or conversely.

and

(27) $$\omega_n(t; f)_p \leqslant C_8^{-1}\max(1, C_7)\left[t^n\|f\|_p + \omega_L(t; f)_p + \sum_{r=1}^{n-1} t^r\omega_{n-r}(t; f)_p\right].$$

It remains to estimate the sum in (26) or (27) in a suitable manner. By Marchaud's inequality (see e.g. [3]) one has

(28) $$\sum_{r=1}^{n-1} t^r\omega_{n-r}(t; f)_p \leqslant C_{17}\sum_{r=1}^{n-1} t^n\left[\|f\|_p + \int_t^1 u^{-n+r-1}\omega_n(u; f)_p\,du\right] \leqslant$$

$$\leqslant C_{18}\left[t^n\|f\|_p + t^n\int_t^1 u^{-n}\omega_n(u; f)_p du\right].$$

Splitting up the integration from t to δ and from δ to 1, and observing

$$t_1^{-n}\omega_n(t_1; f)_p \leqslant 2^n t_2^{-n}\omega_n(t_2; f)_p \qquad (0 < t_2 \leqslant t_1 \leqslant 1),$$

one obtains

(29) $$t^n\int_t^1 u^{-n}\omega_n(u; f)_p\,du \leqslant t^n\left[\int_t^\delta u^{-n}\omega_n(u; f)_p\,du + \int_\delta^1 u^{-n-1}\omega_n(f; u)_p\,du\right] \leqslant$$

$$\leqslant 2^n[\omega_n(t; f)_p(\delta - t) + t^n\|f\|_p n(\delta^{-n} - 1)] \leqslant$$

$$\leqslant 2^n[\omega_n(t; f)_p\delta + nt^n\delta^{-n}\|f\|_p].$$

Now one takes $\delta = 2^{-n-1}[C_{18}C_8^{-1}\max(1, C_7)]^{-1}$. Then (26) and the estimates (28), (29) establish the first inequality (24) of the theorem. Furthermore, inserting (28), (29) into (27) yields with this choice of δ

$$\omega_n(t; f)_p \leqslant \frac{1}{2}\omega_n(t; f)_p + C_{19}[t^n\|f\|_p + \omega_L(t; f)_p],$$

whence (25) immediately follows.

4. Characterization of generalized Lipschitz spaces

The generalized Lipschitz spaces in question are defined by

$$\mathrm{Lip}(\Phi, q, n; p) = \left\{f \in L_p(a, b):\ \begin{array}{ll}\left\{\int_0^1[\Phi(t)^{-1}\omega_n(t; f)_p]\,dt/t\right\}^{1/q}, & 1 \leqslant q < \infty \\ \sup_{0<t\leqslant 1}\Phi(t)^{-1}\omega_n(t; f)_p, & q = \infty\end{array}\right.$$

for $1 \leqslant p \leqslant \infty$ (in case $p = \infty$ one assumes $f \in C[a, b]$ instead of $L_\infty(a,b)$), where $\Phi(t)$ is a positive non-decreasing function on $(0,1]$ with $\lim_{t \to 0+} \Phi(t) = 0$. Combining the previous results one arrives at the following characterization theorem for these Lipschitz spaces (in case $\Phi(t) = t^\alpha$ they are the familiar Besov spaces on the interval $[a, b]$) in terms of the best approximation $E_N^{(L,m)}(f; p)$:

THEOREM 4. *Let Φ be as above and satisfy $\int_0^1 \Phi(t)^{-1} t^{n-1}\, dt < \infty$. Then under the above notations the following assertions are equivalent for $1 \leqslant q < \infty$:*

i) $f \in \mathrm{Lip}\,(\Phi, q, n; p)$,

ii) $\left\{\int_0^1 [\Phi(t)^{-1}\omega_L(t; f)_p]^q\, dt/t\right\}^{1/q} < \infty$,

iii) $\left\{\sum_{N=1}^{\infty} [\Phi(N^{-1})^{-1} E_N^{(L,m)}(f; p)]^q\, N^{-1}\right\}^{1/q} < \infty$.

In case $q = \infty$ and $t^n = 0(\Phi(t))$ the following are equivalent:

i)′ $f \in \mathrm{Lip}\,(\Phi, \infty, n; p)$,

ii)′ $\omega_L(t; f)_p = 0(\Phi(t))$,

iii)′ $E_N^{(L,m)}(f; p) = 0(\Phi(N^{-1}))$.

If $\lim_{t \to 0+} \inf \Phi(t)^{-1} t^n = \infty$, assertion i)′ implies that f is a polynomial of degree $n - 1$, whereas assertion ii)′ or iii)′ imply that $Lf = 0$, $1 < p \leqslant \infty$.

Note that condition $\int_0^1 \Phi(t)^{-1} t^{n-1}\, dt < \infty$ implies a smaller decrease of $\Phi(t)$ for $t \to 0+$ than condition $t^n = 0(\Phi(t))$, since from the boundedness of the integral it follows that $t^n = 0(\Phi(t))$. On the other hand, $\Phi(t)^{-1}t = 0(1)$ means a smaller decrease than $\lim_{t \to 0+} \inf \Phi(t)^{-1} t^n = \infty$ which is just the saturation case since f being a polynomial of degree $n - 1$ or $Lf = 0$ imply that the quantities in assertion i)′ — iii)′ vanish identically with respect to t and N, respectively.

Proof of Theorem 4. The preceding theorems yield the following chain of inequalitites $(0 \leqslant m \leqslant n - 1)$

$$\begin{aligned} \omega_L(N^{-1}; f)_p &\leqslant C_{10}[E_N^{(L,m)}(f; p) + E_{N+1}^{(L,m)}(f; p)] \leqslant \\ &\leqslant C_{20}[N^{-n}\|f\|_p + \omega_n(N^{-1}; f)_p] \leqslant \\ &\leqslant C_{21}[N^{-n}\|f\|_p + \omega_L(N^{-1}; f)_p], \end{aligned}$$

from which the equivalences stated above immediately follow. The saturation case is handled by the same arguments as in RICHARDS-DEVORE [6].

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