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QUASICONFORMALITY AND BOUNDARY CORRESPONDENCE

by

P. CARAMAN

(Iaşi)

Introduction

Let D be a domain in the Euclidean n-space R^n, B the unit ball, $f: D \rightleftarrows B$ a K-quasiconformal mapping (qc), E the set of points of ∂D inaccessible by rectifiable arcs from D and E^* the corresponding boundary points of B. Under the additional hypotheses that $n = 3$, f is differentiable and $mD < \infty$, M. READE [6] announced that the $(n-1)$ — dimensional Hausdorff measure $H^{n-1}(E^*) = 0$. This result was proved by D. STORVICK [7] for $n = 3$, $f \in C^1$, D simply connected and $mD < \infty$. He gives also (in the same paper) an argument belonging to F. Gehring only under the hypotheses that $n = 3$ and $mD < \infty$.

We shall begin by giving a proof of this theorem for $f: D \rightleftarrows B$ qc, without any other additional restrictive condition. Then, we shall prove the E^* is of conformal capacity and even of α-capacity zero for every $\alpha > 0$. Gehring's conjecture is that E^* is even of logarithmic capacity zero : $C_0(E^*) = 0$ and M. READE [6] assertion that E^* is of newtonian capacity zero : $C_1(E^*) = 0$.

Now we shall introduce a few concepts :

Let Γ be an arc family and $F(\Gamma)$ be a family of admissible functions $\rho(x)$ satisfying the following properties :

1° $\rho(x) \geqq 0$ in R^n,

2° $\rho(x)$ is Borel measurable in R^n,

3° $\int_\gamma \rho ds \geqq 1$ for every $\gamma \in \Gamma$.

Then the modulus $M(\Gamma)$ of Γ is given as

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{R^n} \rho^n d\tau,$$

where $d\tau$ is the volume element.

A qc according to Väisälä's geometric definition is characterized by

$$\frac{M(\Gamma)}{K} \leqq M(\Gamma^*) \leqq KM(\Gamma), \tag{1}$$

which is supposed to hold for every Γ contained in D, where $\Gamma^* = f(\Gamma)$.

A function $u: D \to R^1$ is said to be *ACL* *(absolutely continuous on lines)* in a domain D if for each interval $I = \{x;\ \alpha^i < x^i < \beta^i\ (i = 1, \ldots, n)\}$, $I \subset\subset D$ (i.e. $\bar{I} \subset D$), u is AC (absolutely continuous) (in the ordinary sense) on a.e. (almost every) line segment parallel to the coordinate axes.

The *p-capacity of two closed disjoint sets* C_0, $C_1 \subset \bar{D}$ relative to D, where C_0 is bounded and C_1 is compact, is defined as

$$\text{cap}_p(D, C_0, C_1) = \inf \int_{D-(C_0 \cup C_1)} |\nabla u|^p d\tau,$$

where $\nabla u = \left(\frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^n}\right)$ is the gradient of u and the infimum is taken over all u, which are continuous on $D \cup C_0 \cup C_1$, ACL on $D - (C_0 \cup C_1)$ and assume the boundary values 0 on C_0 and 1 on C_1. Such functions are called admissible for $\text{cap}_p\,(D, C_0, C_1)$.

The *capacity of a bounded set* $E \subset R^n$ is defined to be

$$\text{cap}_n\, E = \text{cap}\, E = \inf \int_{R^n} |\nabla u|^n\, d\tau, \tag{2}$$

where the infimum is taken over all functions u, which are continuou and ACL in R^n, have a compact support contained in a fixed ball an are $= 1$ on E.

The *α-potential,* $0 \leqq \alpha \leqq n$ of a measure μ is denoted by u^μ_α, where

$$u^\mu_\alpha(x) = \int_{R^n} \frac{d\mu(y)}{|x-y|^\alpha} \quad \text{if} \quad 0 < \alpha < n,$$

and

$$u^\mu_0(x) = \int_{R^n} \log \frac{1}{|x-y|}\, d\mu(y),$$

which is called also the *logarithmic potential.* If $I_\alpha(\mu)$ denotes the *energy integral* of μ

$$I_\alpha(\mu) = \int_{R^n} u^\mu_\alpha d\mu(x),$$

then

$$C_\alpha(E) = [\inf I_\alpha(\mu)]^{-1},$$

where the infimum is taken over all positive measures μ with total mass 1 and the support of μ contained in E, is called by WALLIN [9] *α-capacity.* When $\alpha = 0$, E is supposed to have the diameter less than 1. For any arbitrary Borel set E, $C_0(E) = 0$ iff $C_0(E \cap B_r) = 0$ for every ball $B_r = B(x, r)$ centered at x and with the radius r $(0 < r < 1)$, $C_0(E)$ is said to be the ***logarithmic capacity*** of E.

We shall prove that the conformal capacity of E^* (i.e. the n-capacity) is zero, which implies that the α-capacity of E^* is zero for every $\alpha > 0$.

1. $H^{n-1}E^* = 0$.

Proposition 1. *If* Γ_0 *is the family of unrectifiable arcs of* R^n, *then* $M(\Gamma_0) = 0$ (J. VÄISÄLÄ [8]).

THEOREM 1. $H^{n-1}E^* = 0$.

Let $f: D \rightleftarrows B$ be a $K - qc$ (quasi conformal mapping), let Γ^* be the family of radial segments joining S to $S(r^*)(0 < r^* < 1)$ in A^* and which are images by f of unrectifiable arcs, where $S(r^*) = S(0, r^*)$, $S = S(1)$ and $A^*_{r^*} = \{x^*;\ r^* < |x^*| < 1\}$. and let $E_1^* \supset E^*$ be the set of endpoints of the segments of Γ^* belonging to S. The preceding proposition implies that the modulus of the arc family $\Gamma = f^{-1}(\Gamma^*)$ is zero (i.e. Γ is exceptional). But, since f is $K - qc$, it satisfies the double inequality (1) and then $M(\Gamma^*) = 0$.

Now, let γ_{ξ^*} be a radial segment with an endpoint $\xi^* \in S$. Then clearly,

$$\chi_{E^*}(\xi^*) \leqq \left(\int_{\gamma_{\xi^*}} \rho^* ds\right)^n \leqq \left(\log \frac{1}{r^*}\right)^{n-1} \int_{r^*} \rho^{*n} r^{*n-1} dr^*$$

for every $\rho^* \in F(\Gamma^*)$ and, integrating over S, on account of Fubini's theorem, we obtain

$$H^{n-1}(E^*) \leqq H^{n-1}(E_1^*) = \int_S \chi_{E^*}(\xi^*) d\sigma \leqq \left(\log \frac{1}{r^*}\right)^{n-1} \int_{A^*_{r^*}} \rho^{*n} d\tau,$$

where $d\sigma$ is the superficial element of S. Finally, taking the infimum over all $\rho^* \in F(\Gamma^*)$, yields

$$H^{n-1}(E^*) \leqq \left(\log \frac{1}{r^*}\right)^{n-1} M(\Gamma^*) = 0,$$

as desired.

2. E^* is closed.

For any two points $x, y \in D$, we shall define the relative distance $d_D(x, y)$ to be the gratest lower bound of the length of all arcs joining x to y and lying in D. It is clear that $d_D(x, y)$ is a metric and that $d_D(x, y) \geqq$

$\geq |x - y|$ with equality if x, y lie on some convex subdomain of D. For points $x \in D$, $\xi \in \partial D$, we define $d_D(x, \xi)$ to be the infimum of $\lim_{m\to\infty} d_D(x, x_m)$ on all sequences $\{x_m\}$ tending to ξ. Let us fix $x_0 \in D$ and consider for various $t > 0$ the sets $\sigma(x_0, t) = \{x \in D;\ d_D(x_0, x) = t\}$ and $\beta(x_0, t) = \{x \in D;\ d_D(x_0, t) < t\}$, which will be called *relative spheres* and *relative balls*, respectively. For values of $t < R = d(x_0, \partial D)$, $\sigma(x_0, t)$ is an ordinary sphere and $\beta(x_0, t)$ an ordinary ball. For values of $t \geq R$, $\sigma(x_0, t)$ is more complicated and may be composed of finitely or infinitely many components depending upon the nature of ∂D. Two relative spheres about x_0 with different radii have no points in common. A relative ball is a simply connected domain contained in D and if $t \geq R$, its boundary will be composed of $\sigma(x_0, t)$ and a compact subset of ∂D.

An arc $\gamma \subset D$ with endpoints $\xi \in \partial D$ is called an *endcut* of D from ξ.

The distance $d_D(E_1, E_2)$ between the sets $E_1, E_2 \subset D$ is the infimum of the length of the polygonal arcs joining E_1 to E_2 in D.

Two endcuts γ_1, γ_2 of D from the same endpoint $\xi \in \partial D$ are called *D-equivalent* if for every neighbourhood U_ξ of ξ, $d_D(\gamma_1 \cap U_\xi, \gamma_2 \cap U_\xi) = 0$.

P r o p o s i t i o n 2. *Let $f: D \rightleftarrows B$ be a K-qc mapping $(1 \leq K < \infty)$, then $d(E_1^*, E_2^*) = 0$ implies $d_D(E_1, E_2) = 0$, where $E_1^*, E_2^* \subset B$ and $E_k = f^{-1}(E_k^*)(k = 1, 2,)$.*

Now we shall introduce (according to ZORIČ [11]) the concept of boundary elements (a generalization of prime ends).

A sequence of domains $\{U_m\}$, $U_m \subset D(m = 1, 2, \ldots)$ is said to be *regular* if

a) $\bar{U}_{m+1} \subset U_m (m = 1, 2, \ldots)$,

b) $\left(\bigcap_{m=1}^{\infty} \bar{U}_m\right) \subset \partial D$,

c) $\sigma_m = \partial U_m \cap D$ (the relative boundary of U_m in D) is a connected set,

d) $d_D(\sigma_m, \sigma_{m+1}) > 0$,

e) there is at most an accessible boundary point of D, which is accessible boundary point for each of the domains of the sequence $\{U_m\}$.

Two sequences of domains $\{U_m\}$, $\{U'_m\}$ are called *equivalent* if every term of each of them contains all the terms of the other one begining by a sufficienty great index.

A boundary element of a domain D is the pair $(F, \{U_m\})$ consisting of a regular sequence $\{U_m\}$ and a continuum $F = \bigcap_{m=1}^{\infty} \bar{U}_m$. Two boundary elements $(F, \{U_m\})$, $(F', \{U'_m\})$ are considered as *identical* iff the two regular sequences $\{U_m\}$ and $\{U'_m\}$ defining them are equivalent. In this way any of the equivalent sequences determine uniquely a boundary element.

P r o p o s i t i o n 3. *For every K — qc mapping $f: D \rightleftarrows B$, it is possible to establish an one-to-one correspondence between the boundary points $(F, \{U_m\})$ of D and the points of S, so that, to each boundary point $(F, \{U_m\})$, there corresponds on S the point determined by the sequence $\{U_m^*\} = f(\{U_m\})$.*

(For the proof, see lemma 3 in ZORIČ's paper [11]).

Let us consider the class $\{\xi, \gamma\}$ of D-equivalent endcuts γ from a boundary point $\xi \in \partial D$.

Clearly, to each pair (ξ^*, γ^*) $(\xi^* \in S,\ \gamma^* \cap B)$ there corresponds a pair (ξ, γ) $(\xi \in \partial D, \gamma \subset D)$, where $\gamma = f^{-1}(\gamma^*)$. In order to see the correspondence is one-to-one, we show first that the images of two endcuts (ξ^*, γ_1^*), (ξ^*, γ_2^*) by f^{-1} are D-equivalent. Suppose $\gamma_1^* \cap \gamma_2^* = \emptyset$. Then $\gamma_1 \cap \gamma_2 = \emptyset$ and, on account of proposition 2, $d_D(\gamma_1, \gamma_2) = 0$. We can assume γ_1^* and γ_2^* have only $\xi^* \in S$ a common endpoint (the other two endpoints x_1^*, x_2^* being different). Then clearly, γ_1 and γ_2 will have $\xi \in \partial D$ as a unique endpoint. Next, let U_ξ be a neighbourhood of ξ. Since $d_D(\gamma_1, \gamma_2) = 0$, and the disjoint open arcs γ_1, γ_2 have ξ as unique common endpoint, it follows that $d_D(\gamma_1 \cap U_\xi, \gamma_2 \cap U_\xi) = 0$ and then (ξ, γ_1), (ξ, γ_2) are D-equivalent. Thus, to each ξ^*, there corresponds a unique class $\{\xi, \gamma\}$.

On the other hand, given a class $\{\xi, \gamma\}$, let us consider a pair (ξ, γ) belonging to it. The image $\gamma^* = f(\gamma)$, will have an endpoint $\xi \in S$. Let us show that any other γ' of a pair (ξ, γ')belonging to the class $\{\xi, \gamma\}$ has an image γ'^* with an endpoint at ξ^*. It is enough to show, according to proposition 3, that (ξ, γ) and (ξ, γ') correspond to the same boundary element. Indeed, let us consider a sequence of concentric balls $\{B(\xi, r_m)\}$ and let $\{U_m\}$ be the corresponding sequence of domains, which are the components of $D \cap B(\xi, r_m)$ containing a subarc of γ with an endpoint at ξ. Clearly, such a sequence is regular. Next, let us associate to γ' a regular sequence $\{U'_m\}$. It is easy to see that $U'_m = U_m$ $(m = 1, 2, \ldots)$. Indeed, let $r < \frac{1}{2} r_m$ and $\gamma_1 \subset \gamma$, $\gamma'_1 \subset \gamma'$ be such that the diameters $d(\gamma_1)$, $d(\gamma'_1) \leq r$ and ξ is the common endpoint of γ_1, γ'_1. Since γ, γ' are D-equivalent and then, *a fortiori* γ_1, γ'_1, it follows there is an arc $\alpha \subset D$ joining them and having a lenth $l < r$, so that $d(\alpha) < l < r$. But then $\alpha \subset B(\xi, r_m)$, hence $\alpha \subset D \cap B(\xi, r_m)$, whence $\alpha \subset U_m$ and $\alpha \subset U'_m$, implying $\alpha \subset U_m \cap U'_m$, and since U_m, U'_m are components of $D \cap B(\xi, r_m)$, we are allowed to conclude that $U_m = U'_m (m = 1, 2, \ldots)$ and then that (ξ, γ) and (ξ, γ') correspond to the same boundary element. Thus, on account of proposition 3, (ξ, γ) an (ξ, γ) correspond to the same point $\xi^* \in S$, as desired.

L e m m a 1. *E^* is closed.*

Let

$$F^* = \bigcup_{n=1}^{\infty} \overline{\{S - f[\beta(x_0, m)]\}}.$$

In order to prove that E^* is closed, it is enough to show that $E^* = F^*$. We shall establish first that $E^* \subset F^*$. Indeed, suppose $\xi^* \in S$, but $\xi^* \in F^*$, then there is an integer m_0 such that $\xi^* \in \overline{\{S - \overline{f[\beta(x_0, m_0)]}}$, hence $\xi^* \overline{\in} S - \overline{f[\beta(x_0, m_0)]}$ and then $\xi^* \in \overline{f[\beta(x_0, m_0)]} \cap S$, i.e. ξ^* belongs to the boundary of the domain $f[\beta(x_0, m_0)]$. Let $\gamma^* \in f[\beta(x_0, m_0)]$ and with an endpoint at ξ^*, then clearly $\gamma = f^{-1}(\gamma^*) \subset \beta(x_0, m_0)$ and will have an endpoint $\xi \in \partial D$, so that ξ is a boundary point of D accessible by rectifiable arcs from D, hence $\xi^* \overline{\in} E^*$, as desired.

Now suppose $\xi^* \in F^*$. Then $\xi^* \in \overline{S - \overline{f[\beta(x_0, m)]}}\,(m = 1, 2, \ldots)$ and even $\xi^* \in S - \overline{f[\beta(x_0, m)]}$ $(m = 1, 2, \ldots)$, on account of proposition 2, since $\sigma(x_0, m) \cap \sigma(x_0, m + 1) = \varnothing$, hence $\xi^* \overline{\in} \overline{f[\beta(x_0, m)]}\,(m = 1, 2, \ldots)$. Suppose, to prove it is false, that $\xi^* \in E^*$, i.e. that ξ corresponds to a boundary point $\xi \in \partial D$ accessible from D by rectifiable arcs. (This correspondence is supposed to be give by the preceding proposition). Let γ_0 be such an arc joining x_0 to ξ and let l be the length of γ_0. Then, clearly $\xi \in \overline{\beta(x_0, m)} \cap \partial D$ for $m \geq l$ and $\gamma_0 \subset \beta(x_0, m)$, $\gamma_0^* = f(\gamma_0) \subset \overline{f[\beta(x_0, m)]}$ and then the endpoint ξ_0^* of γ_0^* belongs to $\overline{f[\beta(x_0, m)]}$ $(m \geq l)$, hence $\xi^* \overline{\in} F^*$, contradicting so the hypothesis $\xi^* \in F^*$.

3. Cap $E^* = 0$.

Proposition 4. *Let $C_0 \subset \bar{D}$, $C_1 \subset D$ be two non-empty disjoint closed sets, Γ the family of arcs which join C_0 and C_1 in D, $\rho \in F(\Gamma)$, $\rho \in L^n$. Then, given $\varepsilon > 0$, there is a $t(\varepsilon) > 0$ such that $\frac{\rho}{1-\varepsilon} \in F[\Gamma(t)]$ for $t < t(\varepsilon)$, where $\Gamma(t)$ is the family of arcs joining $C_0(t) = \{x ; d(x, C_0) \leq t\}$ to $C_1(t) = \{x ; d(x, C_1) \leq t\}$ in D.*

(For the proof, see our paper [2], lemma 13).

Lemma 2. *Suppose that C_0 is a continuum contained in the open half space D, C_1 a closed set contained in the plane ∂D and $\tilde{C}_0$ the symmetric image of C_0 in ∂D. If Γ is the family of arcs which join C_0 and C_1 in D and Γ_1 the family of arcs which join $C_0 \cup \tilde{C}_0$ and C_1 in R^n, then*

$$M(\Gamma) = \frac{1}{2} M(\Gamma_1). \tag{3}$$

Arguing as F. GEHRING and J. VÄISÄLÄ in lemma 3.3 of [5], we obtain that

$$M(\Gamma) \leq \frac{1}{2} M(\Gamma_1).$$

Next, let $\bar{\Gamma}$ denote the family of arcs which join C_0 and C_1 in $\bar{D}$. Then, by the same argument as in Gehring and Väisälä's lemma quoted above, we conclude that

$$\frac{1}{2} M(\Gamma_1) \leq M(\bar{\Gamma}).$$

To complete the proof of (3), we must show that

$$M(\bar{\Gamma}) \leq M(\Gamma). \tag{4}$$

Now, the fact that C_0 and C_1 are disjoint implies that $M(\Gamma) < \infty$. Fix $a = \frac{1}{1-\varepsilon} > 1$ and choose $\rho \in F(\Gamma)$ so that ρ is L^n-integrable. By the preceding proposition, we can choose $t > 0$ so that $a\rho \in F[\Gamma(t)]$. We may assume, for convenience of notations, that D is the half space $x^n > 0$. Set $\rho_1(x) = a\rho(x + te_n)$ (where e_n is the versor on the axis Ox^n), let $\gamma_1 \in \bar{\Gamma}$ and let γ be the arc γ_1 translated through the vector te_n. Then $\gamma \in \Gamma(t)$ and we have

$$\int_{\gamma_1} \rho_1(x)ds = \int_{\gamma} a\rho(x)ds \geq 1.$$

Hence $\rho_1 \in F(\bar{\Gamma})$,

$$M(\bar{\Gamma}) \leq \int_{R^n} \rho_1^n d\tau = a^n \int_{R^n} \rho^n d\tau$$

and taking the infimum over all such ρ yields

$$M(\bar{\Gamma}) \leq a^n M(\Gamma).$$

Finally, if we let $a \to 1$, we obtain (4) as desired.

Corollary. 1. *Suppose $E \subset S$ is a closed proper subset of S and $A = \left\{x ; r < |x| < \frac{1}{r}\right\}$, where $0 < r < 1$. If Γ is the family of arcs which join $|x| \leqslant r$ to E in B and Γ_1 the family of arcs which join CA to E in R^n, then, formula (3) still holds for the new meaning of Γ and Γ_1.*

Let $x_0 \in S - E$ and $x' = \varphi(x)$ be an inversion with respect to a sphere with the center of inversion x_0. Let us denote by $B(r) = \{y ; |y| < r\}$, $C_0 = \varphi[B(r)]$, $C_1 = \varphi(E)$, $\Gamma' = \varphi(\Gamma)$ and $\Gamma_1' = \varphi(\Gamma_1)$. Then, $\varphi(S) = \Pi$ is a plane and $C_1 \subset \Pi$ and we are in the hypothesis of the preceding lemma, so that

$$M(\Gamma') = \frac{1}{2} M(\Gamma_1').$$

But inequality (1) implies the invariance of the modulus with respect to the conformal mappings, allowing us to conclude that

$$M(\Gamma) = M(\Gamma') = \frac{1}{2} M(\Gamma_1') = \frac{1}{2} M(\Gamma_1)$$

as desired.

Proposition 5. *If $\Gamma \subset \bigcup_m \Gamma_m$, then $M(\Gamma) \leq \sum_m M(\Gamma_m)$.* (B. FUGLEDE [3].)

Let $M(E^*)$ be the modulus of the family of arcs with an endpoint in E^*, where E^* is defined as above.

L e m m a 3. *If* $\tilde{\Gamma}^*$ *is the family of arcs with an endpoint belonging to* E^*, *then*

$$M(E^*) = M(\tilde{\Gamma}^*) = 0.$$

Let $\{r_m\}$ be an increasing sequence of numbers $r_m > 0$ such that $\lim_{n\to\infty} r_m = 1$, and $\{\Gamma_m^*\}$ be a sequence of arc families Γ_m^* joining $\overline{B(r_m)}$ to E^* in B. Then, from the definition of E^*, we deduce that the arcs of $\Gamma_m = f^{-1}(\Gamma_m^*)$ $(m = 1, 2, \ldots)$ are not rectifiable, so that, on account of proposition 1, $M(\Gamma_m) = 0$ $(m = 1, 2, \ldots)$ and (1) yields $M(\Gamma_m^*) = 0$ $(m = 1, 2, \ldots)$. Let $\tilde{\Gamma}_m^*$ be the family of arcs which join $\overline{B(r_m)}$ and $CB\left(\frac{1}{r_m}\right)$ t_0E^* in R^n. Then, the preceding corollary allows us to conclude that $M(\tilde{\Gamma}_m^*) = 0$ $(m = 1, 2, \ldots)$. Hence, taking into account proposition 5, $M(\bigcup \tilde{\Gamma}_m^*) = 0$. Next, if $\Gamma_S^* \subset S$ and Γ_0^* is the family of the arcs γ_0^* with the endpoints belonging to E^* and with $\gamma_0^* \cap CS \neq \emptyset$, then

$$M(\Gamma_S^*) = \inf_\rho \int_{Rn} \rho^n d\tau = \inf_\rho \int_S \rho^n d\tau = 0$$

and since $\Gamma_0^* = \bigcup_m \tilde{\Gamma}_m^*$, propositions 5 yields

$$M(\Gamma_0^*) = M(\bigcup_m \tilde{\Gamma}_m^*) = \sum_m M(\tilde{\Gamma}_m^*) = 0.$$

Clearly, $\tilde{\Gamma}^* \subset \Gamma_S^* \cup \Gamma_0^*$, so that, from above, and by proposition 5, we conclude that

$$M(\tilde{\Gamma}^*) \leq M(\Gamma_S^*) + M(\Gamma_0^*) = 0,$$

as desired.

P r o p o s i t i o n 7. *If* χ *is the set of all continua in* R^n *that interesect two closed, disjoint sets* C_0, C_1, *where* C_0 *contains the complement of a ball, then* $M(\chi) = \operatorname{cap}(C_0, C_1, R^n)$.

(For the proof, see W. ZIEMER [10], theorem 3.8.)

C o r o l l a r y. $M(C_0, C_1, R^n) = \operatorname{cap}(C_0, C_1, R^n)$.

It is enough to observe that

$$M(C_0, C_1, R^n) = M(\chi). \tag{5}$$

Indeed, if Γ is the family of arcs which join C_0 and C_1 and C_1 in R^n, then, clearly, $\Gamma \subset \chi$, hence, proposition 5 yields

$$M(C_0, C_1, R^n) = M(\Gamma) \leq M(\chi). \tag{6}$$

On the other hand, let $\rho \in F(\Gamma)$ and α an arbitrary continuum of χ. Then, there exists an arc $\gamma \in \Gamma$ such that $\gamma \subset \alpha$, hence $\rho \in F(\chi)$, so that

$$M(\chi) \leq \int_{Rn} \rho^n d\tau,$$

whence, taking the infimum over all $\rho \in F(T)$, we obtain $M(\chi) \leq M(\Gamma)$, which, together with (6), gives (5), as desired.

THEOREM 2. Cap $E^* = 0$.

If $E^*(r)$ is an r-neighbourhood of E^* (i.e. the set of points within a distance r from E^*), then, clearly

$$\operatorname{cap}\,[CE^*(r),\ E^*,\ R^n] \geq \operatorname{cap} E^* \tag{7}$$

since the class of admissible functions for cap $[CE^*(r), E^*, R^n]$ is contained in that of cap E^*.

Next, let Γ_r^* denote the family of arcs, which join E^* and $CE^*(r)$ in R^n and $\tilde{\Gamma}^*$ of the preceding lemma, then evident $\Gamma_r^* \subset \tilde{\Gamma}^*$ and the preceding lemma implies

$$M[CE^*(r),\ E^*,\ R^n] = M(\Gamma_r^*) \leq M(\tilde{\Gamma}^*) = 0$$

for all $r > 0$, hence and by (7), taking into account also the preceding corollary, we obtain

$$\operatorname{cap} E^* \leq \operatorname{cap}\ [CE^*(r),\ E^*,\ R^n] = M[CE^*(r), E^*,\ R^n] \leq M(\tilde{\Gamma}^*) = 0,$$

as desired.

H. WALLIN [9] gives the following definition of the conformal capacity: „Let E be a bounded set in R^n, cap E is defined by (2), where the infimum is taken over all functions $u \in C^1$, which have compact support belonging to a certain fixed sphere $B(R_0)$ which is independent of E and $u|_E \geq 1$."

Arguing as in F. GEHRING 's paper ([4], lemma 1), it can easily be shown that the infimum appearing in the definition (2) of the conformal capacity of a bounded set E is not increased if it is taken over all $u \in C^1$ in R^n.

C o r o l l a r y. Cap $E^* = 0$, *where the conformal capacity is taken in Wallin's sense, i.e. with* $u|_E \geq 1$ *(not* $u|_E = 1$*).*

This is a consequence of the preceding theorem, since the conformal capacity of a bounded set E given in the introduction is not less than the preceding Wallin's conformal capacity.

Proposition 8. Let F *be a compact set in* R^n *with* cap $F = 0$ *(the conformal capacity in Wallin's sense). The following conclusions are true:*

If $n = 2$, *the logarithmic capacity* $C_0(F) = 0$.

If $n > 2$, *then* $C_\alpha(F) = 0$ *for every* $\alpha > 0$.

For the proof, see WALLIN 's paper ([9], theorem B).

C o r o l l a r y. *If* $n = 2$, *the logaritmic capacity* $C_0(E^*) = 0$. *If* $n > 2$, *then* $C_\alpha(E^*) = 0$ *for every* $\alpha > 0$.

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