MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 5, N° 2, 1976, pp. 127–143

ANTIPROXIMINAL SETS IN BANACH SPACES OF CONTINUOUS FUNCTIONS

by

S. COBZAŞ
(Cluj-Napoca)

Let X be a normed linear space, M a non-void subset of X and $x \in X$. We use the following notations:

$d(x, M) = \inf \{||x - y|| : y \in M\}$ — the distance from x to M;

$P_M(x) = \{y \in M : ||x - y|| = d(x, M)\}$ — the metric projection;

$E(M) = \{x \in X : P_M(x) \neq \emptyset\}$.

The set M is called *proximinal* if $E(M) = X$ and *antiproximinal* if $E(M) = M$ (see [5]) — Following V. KLEE [6], let us denote by N_1 the class of all normed linear spaces which contain an antiproximinal closed convex set, and by N_2 the class of all normed linear spaces which contain an antiproximinal bounded closed convex set. Evidently, $N_2 \subseteq N_1$. As was shown by V. KLEE [6], a Banach space belongs to the class N_1 if and only if it is non-reflexive. The characterization of the Banach spaces of class N_2 is more complicated. The first example of a Banach space of class N_2 was given by M. EDELSTEIN and A. C. THOMPSON [4]: $c_0 \in N_2$. In [2] it was shown that the space c belongs also to the class N_2. We use standard notation; all undefined terms are as in [3]. We consider only real Banach spaces.

We say that two normed linear spaces X, Y are *isomorphic* (notation $X \sim Y$) if there exists a linear homeomorphic bijection $\varphi : X \to Y$. The map φ is called an *isomorphism* of X onto Y. If further, $||\varphi(x)|| = ||x||$ for all $x \in X$, then φ is called an *isometric isomorphism* and we say that the spaces X and Y are *isometrically isomorphic* (notation $X \cong Y$).

Let S be a compact Hausdorff space and $C(S)$ be the Banach space of all real — valued, continuous functions defined on S. An *ideal* I is a linear subspace of $C(S)$ such that $xy \in I$ for all $x \in I$ and $y \in C(S)$. If S_1 is a subset of S let $Z(S_1) = \{x \in C(S) : x(s) = 0$ for all $s \in S_1\}$. Then, I is a closed ideal in $C(S)$ if and only if there is a closed subset S_1 of S such that, $I = Z(S_1)$ (see [8], p. 119).

We are now in position to state the main result of this paper:

THEOREM 1. *Let S be a compact Hausdorff space such that $C(S)$ is isomorphic to c_0. If I is an infinite dimensional closed ideal in $C(S)$ then I belongs to N_2. More precisely, I contains a closed bounded symmetric (with respect to zero), antiproximinal, convex body.*

By a *convex body* we mean a convex set with non-void interior. We agree to call *convex cell* a closed bounded symmetric convex body. It follows that the Minkowski functional $||\cdot||_1$ corresponding to a convex cell M in a normed linear space $(X, ||\cdot||)$, is a norm on X equivalent to $||\ ||$, and $M = \{x \in X : ||x||_1 \leqslant 1\}$. Conversely, if $||\cdot||_1$ is an equivalent norm on X, then the set $M = \{x \in X : ||x||_1 \leqslant 1\}$ is a convex cell in X.

In the sequel we shall use some results from the theory of ordinal numbers following the treatise [9]. Concerning topological spaces of ordinal numbers, see [8]. We denote by ω the first infinite ordinal number. If α, β are ordinal numbers, then $[\alpha, \beta] = \{\gamma : \gamma$ is an ordinal number and $\alpha \leqslant \gamma \leqslant \beta\}$, $[\alpha, \beta[= [\alpha, \beta] \setminus \{\beta\}$ etc. We shall identify the sets $[1, \omega[$ and $\mathbf{N}$ (the set of positive integers). If α, β are ordinal numbers then $[\alpha, \beta]$ equipped with the interval topology is a compact Hausdorff space. If α is an ordinal number, we use the notation $C(\alpha) = C([1, \alpha])$. We shall use also the Cantor normal expansion of ordinal numbers (see [9], p. 322) with the difference that in an expansion $\omega^{\alpha_1}\cdot n_1 + \ldots + \omega^{\alpha_k}\cdot n_k$ we admit the numbers n_i to be zero. The occurrence of a number $n_i = 0$, means that the corresponding term misses, e.g. $\omega^3\cdot 0 + \omega^2\cdot 5 + \omega\cdot 0 + 6 = \omega^2\cdot 5 + 6$.

It is well known that c is isomorphic to c_0. As $c = C(\omega)$ and $c_0 = \{x \in C(\omega) : x(\omega) = 0\}$ is a closed ideal in $C(\omega)$, from Theorem 1 one gets

Corollary 2. ([4], [2]). *The spaces c_0 and c belong to the class N_2. More precisely, they contain antiproximinal convex cells.*

By a result of D. AMIR [1], the space $C(S)$ is isomorphic to c_0, if and only if there exist the natural numbers k, n, $1 \leqslant k$, $n < \omega$, such that $C(S)$ is isometrically isomorphic to $C(\omega^k \cdot n)$. It follows that it is sufficient to prove Theorem 1 for a space $C(\omega^k \cdot n)$. The following lemma shows that it is sufficient to prove it for a space $C(\omega^k)$.

Lemma 3. *Let k, $1 \leqslant k < \omega$ be a natural number. If every infinite dimensional closed ideal in $C(\omega^k)$ contains an antiproximinal convex cell, then every infinite dimensional closed ideal in $C(\omega^k \cdot n)$ contains an antiproximinal convex cell, for $1 \leqslant n < \omega$.*

Proof. Let I be an infinite dimensional closed ideal in $C(\omega^k\cdot n)$. Then there exists a closed set $\Lambda \subseteq [1, \omega^k\cdot n]$ such that $I = Z(\Lambda)$. Put

$$\Delta_i = [\omega^k\cdot(i-1) + 1,\ \omega^k\cdot i],\ \Gamma_i = [1, \omega^k\cdot n]\setminus\Delta_i$$

and

$$Z_i = Z(\Gamma_i) \text{ for } i = 1, 2, \ldots, n.$$

Is Ω is a set and $\Gamma \subseteq \Omega$ we denote by χ_Γ the characteristic function of the set Γ, i.e.

$$\chi_\Gamma(t) = \begin{cases} 1 & \text{for } t \in \Gamma \\ 0 & \text{for } t \in \Omega\setminus\Gamma. \end{cases}$$

Then

$$Z_i = \{x\cdot\chi_{\Delta_i} : x \in C(\omega^k\cdot n)\} \cong C(\Delta_i).$$

Since Δ_i is homeomorphic to $[1, \omega^k]$ it follows that $Z_i \cong C(\omega^k)$ (i.e. they are isometrically isomorphic). Put $\Lambda_i = \Lambda \cap \Delta_i$ and $X_i = \{x \in Z_i : x(\alpha) = 0$ for $\alpha \in \Lambda_i\}$, for $i = 1, 2, \ldots, n$.

Then X_i is a closed ideal in Z_i, and

$$I = X_1 \oplus \ldots \oplus X_n, \tag{1}$$

(the direct sum holds algebrically and topologically).

For all $x \in I$, $x = x_1 + \ldots + x_n$, $x_i \in X_i$, $i = 1, 2, \ldots, n$, we have

$$||x|| = \max\{||x_1||, \ldots, ||x_n||\}. \tag{2}$$

We intend to show that without loosing the generality we can suppose that all the ideals X_i, in the direct sum (1), are infinite dimensional. Indeed, let us suppose that $[1, n] = M_1 \cup M_2$, and X_i is finite dimensional for $i \in M_1$ and infinite dimensional for $i \in M_2$. The ideal X_i is finite dimensional if and only if the set $\Delta_i\setminus\Lambda_i$ is finite.

Put $\Delta' = \bigcup_{i \in M_1}(\Delta_i\setminus\Lambda_i)$, $m = \mathrm{card}\,(\Delta')$, $M_1 = \{i_1, \ldots, i_p\}$, $i_1 < \ldots < i_p$, $M_2 = \{j_1, \ldots, j_l\}$, $j_1 < \ldots < j_l$, $l + p = n$. Since the set $\Delta_i\setminus\Lambda_i$ is finite and the set Λ_i is closed, it follows that all the accumulation points of the set Δ_i belong to the set Λ_i, so that the sets Λ_i and Δ_i are homeomorphic. Therefore, there exist the homeorphisms: $\varphi_{j_1} : \Delta_{j_1} \to [m+1, \omega^k]$, ..., $\varphi_{j_l} : \Delta \to \Delta_{j_l}$, $\varphi_{i_1} : \Lambda_{i_1} \to \Delta_{l+1}$, ..., $\varphi_{i_p} : \Lambda_{i_p} \to \Delta_n$.

Let $\psi : \Delta' \to [1, m]$ be a bijection, and let us define the map

$$\varphi : [1, \omega^k\cdot n] \to [1, \omega^k\cdot n]$$

$$\varphi(\alpha) = \begin{cases} \psi(\alpha), & \alpha \in \Delta' \\ \varphi_i\ (\alpha), & \alpha \in \Delta_i\setminus\Delta',\ i = 1, 2, \ldots, n. \end{cases}$$

Then φ will be a homeorphism of $[1, \omega^k\cdot n]$ onto $[1, \omega^k\cdot n]$, and the map $T : C(\omega^k\cdot n) \to C(\omega^k\cdot n)$, defined by

$$Tx(\alpha) = x(\varphi(\alpha)),\ \alpha \in [1, \omega^k\cdot n],\ x \in C(\omega^k\cdot n),$$

will be an isometric isomorphism of $C(\omega^k\cdot n)$ onto $C(\omega^k\cdot n)$. This isomorphism maps the ideal I onto an ideal $J = T(I)$, of the form:

$$J = Y_1 \oplus \ldots \oplus Y_l \oplus Z(\Delta_{l+1}) \oplus \ldots \oplus Z(\Delta_n) =$$
$$= Y_1 \oplus \ldots \oplus Y_l \oplus \{0\} \oplus \ldots \oplus \{0\},$$

where Y are infinite dimensional closed ideals in Z_i, $i = 1, \ldots, l$. Therefore, we can suppose that J is a closed ideal in $C(\omega^k \cdot l)$ of the form $J = Y_1 \oplus \ldots \oplus Y_l$, Y_i being infinite dimensional closed ideals in Z_i, $i = 1, 2, \ldots, l$.

Let us then suppose that in the expansion (1), X_i is an infinite dimensinal closed ideal in Z_i, $i = 1, 2, \ldots, n$. Since $Z_i \cong C(\omega^k)$, by hypothesis X_i will contain an antiproximinal convex cell V_i, $i = 1, 2, \ldots, n$. We will show that the set $V = V_1 + \ldots + V_n$, is an antiproximinal convex cell in I. Since the sum (1) holds algebraically and topologically, V is a convex cell in I. To complete the proof of the lemma, we have to show that V is an antiproximinal set. Let $x \in I \setminus V$, $y \in V$, $x = x_1 + \ldots + x_n$, $y = y_1 + \ldots + y_n$, $x_i \in X_i$, $y_i \in V_i$, $i = 1, 2, \ldots, n$. By (2)

$$0 < \|x - y\| = \max\ \{\|x_i - y_i\| : i = 1, 2, \ldots, n\}.$$

Put $N_1 = \{i \in [1, n] : \|x_i - y_i\| = \|x - y\|\}$ and $N_2 = [1, n] \setminus N_1$. Then, $N_1 \neq \emptyset$, and for all $i \in N_1$, $\|x_i - y_i\| = \|x - y\| > 0$, so that if $x_i \notin V_i$, the set V_i being antiproximinal, there exists $y'_i \in V_i$, such that $\|x_i - y'_i\| < \|x_i - y_i\|$. If $i \in N_1$ and $x_i \in V_i$, put $y'_i = \frac{x_i + y_i}{2} \in V_i$. Set

$$y' = \sum_{i \in N_1} y'_i + \sum_{i \in N_2} y_i$$

$\left(\text{we agree that } \sum_{i \in \emptyset} y_i = 0\right)$. Then, putting $\max \emptyset = 0$, we have $\|x - y'\| = \max\ (\max\{\|x_i - y'_i\| : i \in N_1\},\ \max\ \{\|x_i - y_i\| : i \in N_2\}) < \|x - y\|$.

Therefore, the set V is antiproximinal, Q.E.D.

By Lemma 3, it follows that we can suppose

(3) $$C(S) = C(\omega^k).$$

If S is a compact Hausdorff space then, the conjugate of the Banach space $C(S)$ is the space $M(S)$ of all regular Borel measures on S. For $\mu \in M(S)$, we have

$$\|\mu\| = v(\mu,\ S) \ \text{(the total variation of } \mu)$$

and

$$\mu(x) = \int_S x(s) \cdot d\mu(s),\ x \in C(S),$$

(see [3], IV. 6.3). (In [3] the space $M(S)$ is denoted by rca (S)). If S is countable, then

(4) $$\|\mu\| = \Sigma_{s \in S}|\mu(s)|$$

and

(5) $$\mu(x) = \Sigma_{s \in S} x(s) \cdot \mu(s),\ x \in C(S).$$

Conversely, every function $\mu : S \to \mathbf{R}$, such that $\Sigma_{s \in S}|\mu(s)| < \infty$, defines through (5) a continuous linear functional on $C(S)$, whose norm is (4). If α is an ordinal number, we denote $M(\alpha) = M([1,\ \alpha])$, i.e. the conjugate of the space $C(\alpha)$.

The isomorphism A from the following lemma, was constructed by M. EDELSTEIN and A. C. THOMPSON [4], in the particular case of the space c_0. The proof given here is the same as in [4]. If S is a compact Hausdorff space, we denote by δ_s the evaluation functionals on S, i.e.

(6) $$\delta_s(x) = x(s),\ x \in C(S),\ s \in S.$$

L e m m a 4. *Let S be a countable and compact Hausdorff space, S_1 a closed subset of S, $I = Z(S_1)$ a closed ideal in $C(S)$, $S_2 = S \setminus S_1 \neq \emptyset$, $u_s \in M(S)$, $\|u_s\| \leqslant a < 1$, $s \in S_2$, and $g \in M(S)$ defined by*

$$g_s = \delta_s + u_s,\ s \in S_2.$$

If for all $x \in Z(S_1)$ the function Ax defined by

$$Ax(s) = \begin{cases} g_s\,(x), & s \in S_2, \\ 0, & s \in S_1, \end{cases}$$

belongs to $Z(S_1)$, then A is an isomorphism of $Z(S_1)$ onto $Z(S_1)$, and its adjoint A^* verifies the relations:

$$A^*\delta_s = g_s,\ s \in S_2.$$

Proof. By the definition of A, it is clear that A is a linear operator. Put

$$D = \{x \in Z(S_1) : |g_s(x)| \leqslant 1,\ s \in S_2\}.$$

Evidently, D is a symmetric, convex, closed subset of $Z(S_1)$. Since $\|x\| < (1 + a)^{-1}$ implies $|g_s(x)| < 1$, for all $s \in S_2$, it follows that 0 is an interior point of D. If $x \in Z(S_1)$ is such that $\|x\| > (1 - a)^{-1}$, and $x \in S_2$ is such that $|x(s)| = \|x\|$, then

$$|g_s(x)| \geqslant |x(s)| - |u_s(x)| \geqslant \|x\| - a \cdot \|x\| = (1 - a)\|x\| > 1.$$

Therefore, $x \notin D$ and D is also bounded. If B denotes the closed unit ball of $Z(S_1)$, then

$$D = \{x \in Z(S_1) : |g_s(x)| \leqslant 1,\ s \in S_2\} = \{x \in Z(S_1) : |Ax(s)| \leqq 1,\ s \in S_2\} = \{x \in Z(S_1) : \|Ax\| \leqslant 1\} = A^{-1}(B).$$

Since D is a convex cell it follows that A is an isomorphism of $Z(S_1)$ onto $Z(S_1)$.

Finally

$A^*\delta_s\,(x) = \delta_s\,(Ax) = Ax(s) = g_s\,(x),\ x \in Z(S_1),\ s \in S_2$, i.e. $A^*\delta_s = g_s,\ s \in S_2$, Q.E.D.

If X is a Banach space and M a non-void convex subset of X, we say that a functional $f \in X^*$ (X^* the conjugate of X) *supports* the set M, if there exists a point $x \in M$, such that $f(x) = \sup f(M)$ or $f(x) = \inf f(M)$. We denote by $\mathcal{S}(M)$ the set of all support functionals of the set M.

We mention the following two results from [4]:

L e m m a 5. ([4], Prop. 1. (iii)). *Let X be a Banach space, M a non-void, bounded closed convex subset of X and B the closed unit ball of X. Then, the set M is antiproximinal if and only if*

$$\mathcal{S}(M) \cap \mathcal{S}(B) = \{0\}.$$

L e m m a 6. ([4], p. 555). *If X, Y are Banach spaces, M a nonvoid, convex subset of X, and $A : X \to Y$ is an isomrphism, then $f \in \mathcal{S}(M)$ if and only if $(A^*)^{-1}f \in \mathcal{S}(A(M))$* (i.e. $\mathcal{S}(M) = A^*\mathcal{S}(A(M))$).

If X is a linear space and $M \subseteq X$, we denote by $\mathrm{sp}(M)$ the linear subspace of X spanned by M. If X is a linear topological space, we denote by $\overline{\mathrm{sp}}\ (M)$ the closed linear space spanned by M. We have $\overline{\mathrm{sp}}\ (M) = \overline{\mathrm{sp}\ (M)}$. A sequence $\{x_k : k \in \mathbf{N}\}$ in a Banach space X is called *complete* if $\overline{\mathrm{sp}}(\{x_k\}) = X$. If every $x \in X$ can be uniquely represented in the form $x = \sum_{k=1}^{\infty} a_k x_k$, where $a_k \in \mathbf{R}$, then $\{x_k\}$ is called a *basis* for X. A sequence $\{f_k : k \in \mathbf{N}\} \subseteq X^*$, is called *conjugate* to $\{x_k\}$ if $f_k(x_i) = \delta_{ki}$. A system $\{(x_k, f_k) : k \in \mathbf{N}\} \subseteq X \times X^*$ is called *biorthogonal* if $\{f_k\}$ is conjugate to $\{x_k\}$. If further, the system $\{x_k\}$ is complete in X, we say that $\{(x_k, f_k)\}$ is an *X-complete biorthogonal system*. For complete systems and bases in Banach spaces, see [10].

In the sequel we will use several times the following lemma, whose simple proof is omitted.

L e m m a. 7. *Let X, Y be Banach spaces, $A : X \to Y$ an isomorphism, and $\{(x_k, f_k) : k \in \mathbf{N}\} \subseteq X \times X^*$ a biorthogonal system. Then the system $\{(Ax_k, (A^*)^{-1}f_k) : k \in \mathbf{N}\}$ is biorhtogonal in $Y \times Y^*$. Further if $\{x_k\}$ is complete (a basis) in X then the system $\{Ax_k\}$ is complete (resp. a basis) in Y.*

Let $\{e_i : i \in \mathbf{N}\}$ be the usual basis of c_0 and $\{\delta_i : i \in \mathbf{N}\} \subseteq c_0^* = l_1$ its conjugate system.

L e m m a 8. *Let $\{(x_i, h_i) : i \in \mathbf{N}\}$ be a c_0 — complete biorthogonal system in $c_0 \times l_1$ such that the set $B_1 = \{x \in c_0 : |h_i(x)| \leqslant 1,\ i \in \mathbf{N}\}$, is a convex cell in c_0, and let $\|\cdot\|_1$ be the Minkowski functional of the set B_1. Then the map*

$$Tx_i = e_i,\ i \in \mathbf{N},$$

extends to an isometric isomorphism T of $(c_0, \|\cdot\|_1)$ onto $(c_0, \|\cdot\|)$. The adjoint operator of T, verifies

$$T^*\delta_i' = h_i, \qquad i \in \mathbf{N}.$$

Proof. Since B_1 is a convex cell in c_0, its Minkowski functional will be a norm on c_0, equivalent to the usual norm. By the definition of the Minkowski functional, we have for all $x \in c_0$:

$$\|x\|_1 = \inf\{\lambda > 0 : x \in \lambda B_1 = \inf\{\lambda < 0 : \lambda^{-1} x \in B_1\} =$$
$$= \inf\{\lambda > 0 : |h_i(\lambda^{-1} \cdot x)| \leqslant 1,\ i \in \mathbf{N}\} =$$
$$= \inf\{\lambda > 0 : |h_i(x)| \leqslant \lambda,\ i \in \mathbf{N}\} =$$
$$= \sup\{|h_i(x)| : i \in \mathbf{N}\}.$$

Put $X = \mathrm{sp}\ (\{x_i : i \in \mathbf{N}\})$, $Y = \mathrm{sp}\ (\{e_i' : i \in \mathbf{N}\})$, and define the linear operator $T : X \to Y$ by $T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i e_i'$, $n \in \mathbf{N}$.

Then, by (7) and taking into account that $h_i(y) = 0$ for $i > n$, we have for all $y = \sum_{i=1}^{n} a_i x_i \in X$,

$$\|Ty\| = \|\sum_{i=1}^{n} a_i e_i'\| = \max\{|a_1|,\ \ldots,\ |a_n|\} = \max\{|h_1(y)|,\ \ldots,\ |h_n(y)|\} =$$
$$= \sup\{|h_i(y)| : i \in \mathbf{N}\} = \|y\|_1.$$

Therefore, T is an isometric isomorphism of $(X, \|\cdot\|_1)$ onto $(Y, \|\cdot\|)$. Since the system $\{x_i\}$ is complete, $\overline{X} = c_0$ and T extends to a linear isometry of $(c_0, \|\cdot\|_1)$ into $(c_0, \|\cdot\|)$. Because the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent, there exist two real numbers $\alpha, \beta > 0$ such that, $\alpha\|x\| \leqslant \|x\|_1 \leqslant \beta\|x\|$, $x \in c_0$. Therefore, $\|Tx\| = \|x\|_1 \geqslant \alpha\|x\|$, $x \in c_0$. From this inequality, easily follows that the operator T has closed range. Then

$$T(c_0) = \overline{T(c_0)} \supseteq \overline{T(X)} = \overline{Y} = c_0.$$

This shows that T is an isometric isomorphism of $(c_0, \|\cdot\|_1)$ onto $(c_0, \|\cdot\|)$.

Finally, by the definition of T and by the biorthogonality of the systems (e_i', δ_i') and $\{(x_i, h_i)\}$ it follows,

$$T^*\delta_i'(x_j) = \delta_i'(Tx_j) = \delta_i'(e_j') = \delta_{ij} = h_i(x_j).$$

The system $\{x_i\}$ being complete in c_0, we have

$$T^*\delta_i' = h_i,\ i \in \mathbf{N}, \quad \text{Q.E.D.}$$

R e m a r k. It follows that the sequence $\{x_i\}$ from Lemma 8, is a basis for c_0, equivalent to the usual basis of c_0.

Proof of Theorem 1.

As was shown (see (3)), it is sufficient to prove Theorem 1 for a space $C(\omega^k)$. In order to avoid tedious notations, we suppose $k = 3$. The proof of the general case proceeds analogously.

Let us then suppose that Λ is a closed subset of $[1, \omega^3]$, and $I = Z(\Lambda)$ is an infinite dimensional closed ideal in $C(\omega^3)$. We have to consider several cases.

Case I. $\Lambda = \emptyset$, *i.e.* $I = C(\omega^3)$.

Firstly, we give a complete biorthogonal system $\{(e_i, f_i) : 1 \leqslant i \leqslant \omega^3\}$ in $C(\omega^3) \times M(\omega^3)$. Let us define the elements $e_i \in C(\omega^3)$ for $1 \leqslant i \leqslant \omega^2$, by

$$(8)\quad \begin{cases} e_{\omega^2\cdot(k-1)+\omega\cdot(l-1)+m}(i) = \begin{cases} 1 \text{ if } i = \omega^2\cdot(k-1)+\omega\cdot(l-1)+m \\ 0 \text{ in rest.}\end{cases} \\ \qquad \text{for } 1 \leqslant k, l, m < \omega; \\ e_{\omega^2\cdot(k-1)+\omega\cdot l}(i) = \begin{cases} 1 \text{ if } \omega^2\cdot(k-1)+\omega\cdot(l-1)+1 \leqslant i \leqslant \omega^2\cdot(k-1)+\omega\cdot l \\ 0 \text{ in rest}\end{cases} \\ \qquad \text{for } 1 \leqslant k, l < \omega; \\ e_{\omega^2\cdot k}(i) = \begin{cases} 1 \text{ if } \omega^2\cdot(k-1)+1 \leqslant i \leqslant \omega^2\cdot k \\ 0 \text{ in rest;}\end{cases} \\ \qquad \text{for } 1 \leqslant k < \omega; \\ e_{\omega^3}(i) = 1, \quad 1 \leqslant i \leqslant \omega^3. \end{cases}$$

Let define the functionals $f_i \in M(\omega^2)$, $1 \leqslant i \leqslant \omega^3$, by

$$(9)\quad \begin{cases} f_{\omega^2\cdot(k-1)+\omega\cdot(l-1)+m} = \delta_{\omega^2\cdot(k-1)+\omega\cdot(l-1)+m} - \delta_{\omega^2\cdot(k-1)+\omega\cdot l} - \\ \qquad - \delta_{\omega^2\cdot k} - \delta_{\omega^3}\ ; \text{ for } 1 \leqslant k, l, m < \omega; \\ f_{\omega^2\cdot(k-1)+\omega\cdot l} = \delta_{\omega^2\cdot(k-1)+\omega\cdot l} - \delta_{\omega^2\cdot k} - \delta_{\omega^3}, \\ \qquad \text{for } 1 \leqslant k, l < \omega; \\ f_{\omega^2\cdot k} = \delta_{\omega^2\cdot k} - \delta_{\omega^3} \\ f_{\omega^3} = \delta_{\omega^3} \end{cases}$$

where δ_i are the evaluation functionals (6).

L e m m a 9. *The system* $\{(e_i, f_i) : 1 \leqslant i \leqslant \omega^3\}$ *is a* $C(\omega^3)$*-complete biorthogonal system in* $C(\omega^3) \times M(\omega^3)$.

Proof of Lemma 9. The biorthogonality is obvious. To complete the proof of Lemma 9, we have to show that the system $\{e_i : 1 \leqslant i \leqslant \omega^3\}$ is complete in $C(\omega^3)$.

Let $x \in C(\omega^3)$ and $\varepsilon > 0$. Using the continuity of the function x on ω^3, $\omega^2 \cdot k$ and $\omega^2 \cdot (k-1) + l$, one can find successively the natural numbers k_0, l_0, m_0, $1 \leqslant k_0 \cdot l_0$, $m_0 < \omega$, such that

$$(10)\quad |x(\omega^3) - x(i)| < \varepsilon, \text{ for } i > \omega^2 \cdot k_0;$$

$$(11)\quad |x(\omega^2\cdot(k-1)+i) - x(\omega^2\cdot k)| < \varepsilon, \text{ for } \omega\cdot l_0 < i < \omega^2 \text{ and } k = 1, 2, \ldots, k_0;$$

$$(12)\quad |x(\omega^2\cdot(k-1)+\omega\cdot(l-1)+m) - x(\omega^2\cdot(k-1)+\omega\cdot l)| < \varepsilon, \text{ for } m_0 < m < \omega,\ k = 1, 2, \ldots, k_0 \text{ and } l = 1, 2, \ldots, l_0.$$

Let $N_0 = [1, k_0]$, $N_1 = [1, k_0] \times [1, l_0]$, $N_2 = [1, k_0] \times [1, l_0] \times [1, m_0]$, and let

$$y = x(\omega^3)\cdot e_{\omega^3} + \sum_{k\in N_0}[x(\omega^2\cdot k) - x(\omega^3)]e_{\omega^2\cdot k} + \\ + \sum_{(k,l)\in N_1}[x(\omega^2\cdot(k-1)+\omega\cdot l) - x(\omega^2\cdot k)]e_{\omega^2\cdot(k-1)+\omega\cdot l} + \\ + \sum_{(k,l,m)\in N_2}[x(\omega^2\cdot(k-1)+\omega\cdot(l-1)+m) - x(\omega^2\cdot(k-1)+\omega\cdot l)]\cdot \\ \cdot e_{\omega^2\cdot(k-1)+\omega\cdot(l-1)+m}.$$

Using the definition of y and the inequalities (10), (11), (12), it is easy to show that $|x(i) - y(i)| < \varepsilon$ for $1 \leqslant i \leqslant \omega^3$, that is $\|x - y\| < \varepsilon$. Therefore, $\{e_i\}$ is complete in $C(\omega^3)$, Q.E.D.

Let S be a compact Hausdorff space. Every regular Borel measure $\mu \in M(S)$ can be uniquely writen as $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are non-negative regular Borel measures on S (the Jordan decomposition of a measure, see [3], III. 4.11). The *support* $S(\mu)$ of a measure μ, is the complementary set to the largest open subset of S, on which the variation of μ is zero. Obviously, $S(\mu)$ is a closed subset of S.

Let $U = \{x \in C(S) : \|x\| \leqslant 1\}$. By a result, proved by S.I. ZUHOVICKI [11], in the case of a metric compact S, and by R.R. PHELPS [7], in general, $\mu \in \mathcal{S}(U)$ if and only if

$$(13)\quad S(\mu^+) \cap S(\mu^-) = \emptyset.$$

If the compact S is countable, then

$$(14)\quad \begin{aligned} S(\mu) &= \overline{\{s \in S : \mu(s) \neq 0\}} \\ S(\mu^+) &= \overline{\{s \in S : \mu(s) > 0\}} \\ S(\mu^-) &= \overline{\{s \in S : \mu(s) < 0\}}. \end{aligned}$$

Now, we want to define the functionals $g_i \in M(\omega^3)$, such that $\mathrm{sp}(\{g_i : 1 \leqslant i \leqslant \omega^3\}) \cap \mathcal{S}(U) = \{0\}$. Let us consider the maps $\sigma_k : [1, \omega[\to [1, \omega[$, be defined by

$$\sigma_k(i) = 2^{k-1}(2i+1), \ 1 \leqslant i < \omega; \qquad 1 \leqslant k < \omega, \tag{15}$$

(see [4], p. 554)

Define

$$\left\{\begin{aligned}
&g_{\omega^3}(x) = x(\omega^3) + 2^{-2} \sum_{1 \leqslant i < \omega} (-1)^i 2^{-i} \cdot x(\omega^2 \cdot i); \\
&g_{\omega^2 \cdot k}(x) = x(\omega^2 \cdot k) + 2^{-2} \sum_{i=1}^{3 \cdot 2^{k-1}-1} (-1)^i 2^{-i} \cdot x(\omega^2 \cdot i) + \\
&\quad + 2^{-(k+2)} \sum_{1 \leqslant i < \omega} (-1)^i 2^{-i} \cdot x(\omega^2 \cdot \sigma_k(i)), \\
&\qquad \text{for } 1 \leqslant k < \omega; \\
&g_{\omega^2 \cdot (k-1) + \omega \cdot l}(x) = x(\omega^2 \cdot (k-1) + \omega \cdot l) + g_{\omega^2 \cdot k}(x) - x(\omega^2 \cdot k) + \\
&\quad + 2^{-(k+l+2)} \sum_{1 \leqslant i < \omega} (-1)^i \cdot 2^{-i} \cdot x(\omega^2 \cdot (k-1) + \omega \cdot \sigma_l(i)), \\
&\qquad \text{for } 1 \leqslant k, \ l < \omega; \\
&g_{\omega^2 \cdot (k-1) + \omega \cdot (l-1) + m}(x) = x(\omega^2 \cdot (k-1) + \omega \cdot (l-1) + m) + \\
&\quad + g_{\omega^2 \cdot (k-1) + \omega \cdot l}(x) - x(\omega^2 \cdot (k-1) + \omega \cdot l) + \\
&\quad + 2^{-(k+l+m+2)} \sum_{1 \leqslant i < \omega} (-1)^i 2^{-i} x(\omega^2 \cdot (k-1) + \omega \cdot (l-1) + \sigma_m(i)), \\
&\qquad \text{for } 1 \leqslant k, l, m < \omega.
\end{aligned}\right. \tag{16}$$

Taking into account the formulae (14), (15), (16), an examination of all possible combinations $g = a_1 g_{i'} + a_2 g_{i''}$, of the elements g_i, shows that $S(g^+) \cap S(g^-) = \emptyset$.

Put

$$U = \{x \in C(\omega^3) : \|x\| \leqslant 1\}, \tag{17}$$

$$Y = \mathrm{sp}(\{g_i : 1 \leqslant i \leqslant \omega^3\}). \tag{18}$$

Then

$$Y \cap \mathcal{S}(U) = \{0\}. \tag{19}$$

L e m m a 10. *The operator A defined by*

$$Ax(i) = g_i(x), \ 1 \leqslant i \leqslant \omega^3, \ x \in C(\omega^3),$$

is an isomorphism of $C(\omega^3)$ onto $C(\omega^3)$. Its adjoint A^* verifies

$$A^* \delta_i = g_i, \qquad 1 \leqslant i \leqslant \omega^3.$$

Proof of Lemma 10. We intend to apply Lemma 4. Evidently, $g_i = \delta_i + u_i$ and $\|u_i\| \leqslant 2^{-1}$ (see (16). We have to show that Ax so defined belongs to $C(\omega^3)$. But, if $\{\alpha_n\}$ is a sequence in $[1, \omega^3]$ converging to α, then it is a routine verification to show that $g_{\alpha_n}(x)$ converges to $g_\alpha(x)$, that is Ax is a continuous function. Therefore, Lemma 4 applies, Q.E.D.

Put

$$V = \{x \in C(\omega^3) : |f_i(x)| \leqslant 1, \ 1 \leqslant i \leqslant \omega^3\}, \tag{20}$$

where f_i are defined by (9). It easy to see that V is a convex cell in $C(\omega^3)$. There exists an isomorphism, say

$$H : C(\omega^3) \to c_0. \tag{21}$$

Put

$$V_1 = A^{-1}(V), \tag{22}$$

and

$$B_1 = H(V_1) = HA^{-1}(V), \tag{23}$$

where A is the isomorphism of $C(\omega^3)$ onto $C(\omega^3)$, constructed in Lemma 10. The maps A and H being isomorphisms, B_1 will be a convex cell in c_0.

Let

$$y_i = A^{-1} e_i, \tag{24}$$

$$u_i = A^* f_i, \tag{25}$$

and

$$x_i = H y_i, \tag{26}$$

$$h_i = (H^*)^{-1} u_i, \tag{27}$$

for $1 \leqslant i \leqslant \omega^3$.

Applying twice Lemma 7, it follows that $\{(x_i, h_i) : 1 \leqslant i \leqslant \omega^3)$ is a c_0-complete biorthogonal system. By (23), (20), (27) and the fact that $(H^{-1})^* = (H^*)^{-1}$ (see [3]. VI. 3.7), one gets

$$\begin{aligned}
B_1 = HA^{-1}(V) &= \{x \in c_0 : (AH^{-1})x \in V\} = \{x \in c_0 : |f_i(AH^{-1}(x)| \leqslant 1, \\
1 \leqslant i \leqslant \omega^3\} &= \{x \in c_0 : |(H^{-1})^* A^* f_i(x)| \leqslant 1, \ 1 \leqslant i \leqslant \omega^3\} = \\
&= \{x \in c_0 : |h_i(x)| \leqslant 1, \ 1 \leqslant i \leqslant \omega^3\}.
\end{aligned}$$

Let $\sigma : \mathbf{N} \to [1, \omega^3]$, be a bijection and let us define

$$\tilde{x}_i = x_{\sigma(i)} \tag{28}$$

$$\tilde{h}_i = h_{\sigma(i)}$$

for all $i \in \mathbf{N}$. it follows that, the c_0-complete biorthogonal system $\{x_i, h_i) : i \in \mathbf{N}\}$, verifies the hypothesis of Lemma 8. Denoting by $\|\cdot\|_1$ the Minkowski functional of the convex cell B_1 ($\|\cdot\|_1$) will be a norm on c_0 equivalent to the usual norm), it follows that there exists an isometric isomorphism $T : (c_0, \|\cdot\|_1) \to (c_0, \|\cdot\|)$, such that

$$T\bar{x}_i = e_i' \tag{29}$$
$$T^*\delta_i' = \bar{h}_i$$

for $i \in \mathbf{N}$. Here $\{e_i'\}$ denotes the usual basis of c_0 and $\{\delta_i'\}$, its conjugate system.

Let us define a new norm $\|\cdot\|_2$ on c_0, by

$$\|x\|_2 = \|H^{-1}x\|, \ x \in c_0,$$

where H is the isomorphism (21). It follows that, $\|\cdot\|_2$ will be a norm on c_0, equivalent to the usual norm and, H will be an isometric isomorphism of $(C(\omega^3), \|\cdot\|)$ onto $(c_0, \|\cdot\|_2)$.

L e m m a 11. *The set* B_1 *is an antiproximinal convex cell in* $(c_0, \|\cdot\|_2)$.

Proof of Lemma 11. Let $B = \{x \in c_0 : \|x\| \leqslant 1\}$ and $B_2 = \{x \in c_0 : \|x\|_2 \leqslant 1\}$. By the definition of the norm $\|\cdot\|_2$,

$$B_2 = H(U), \tag{30}$$

where U denotes the closed unit ball in $(C(\omega^3), \|\cdot\|)$. Since T is an isometric isomorphism of $(c_0, \|\cdot\|_1)$ onto $(c_0, \|\cdot\|)$, it follows that

$$B = T(B_1). \tag{31}$$

Let Y be defined by (18) and let

$$Z = \mathrm{sp}(\delta_i' : i \in \mathbf{N})\}. \tag{32}$$

By (9), Lemma 10 and (25),

$$Y = \mathrm{sp}(\{u_i : i \in \mathbf{N}\}). \tag{33}$$

We intend to apply Lemma 5. It is well known and easy to see, that

$$\mathcal{S}(B) = Z. \tag{34}$$

By Lemma 6, (31), (34), (32), (29). (28), (27), and (33)

$$\mathcal{S}(B_1) = T^*\mathcal{S}(B) = T^*(Z) = \mathrm{sp}(\{\bar{h}_i : i \in \mathbf{N}\}) =$$
$$= \mathrm{sp}\ (\{h_i : 1 \leqslant i \leqslant \omega^3\}) = (H^*)^{-1}(Y).$$

On the other hand, by Lemma 6 and (30).

$$\mathcal{S}(B_2) = (H^*)^{-1}(\mathcal{S}(U)).$$

Then, by (19)

$$\mathcal{S}(B_1) \cap \mathcal{S}(B_2) = (H^*)^{-1}(Y) \cap (H^*)^{-1}(\mathcal{S}(U)) =$$
$$= (H^*)^{-1}(Y \cap \mathcal{S}(U)) = (H^*)^{-1}(0) = \{0\}.$$

By Lemma 5, the set B_1 is antiproximinal in $(c_0, \|\cdot\|_2)$, Q.E.D.

Now, since H is an isometric isomorphism of $(C(\omega^3), \|\cdot\|)$ onto $(c_0, \|\cdot\|_2)$, the set $V_1 = H^{-1}(B_1)$will be an antiproximinal convex cell in $C(\omega^3)$, which concludes the proof of Theorem 1 in Case I.

Remark. By (29) and the fact that T is an isomorphism of $C(\omega^3)$ onto c_0, it follows that $\{\bar{x}_i : i \in \mathbf{N}\}$ is, in fact, a basis for $C(\omega^3)$ equivalent to the usual basis of c_0.

Case II. $I = Z(\Lambda)$, Λ *a closed subset of* $[1, \omega^3]$ *and* $\omega^3 \in \Lambda$.

The proof is the same as in Case I, with some changes in the definitions of the elemente e_i, f_i, g_i.

Since $C(\omega^3)$ is isomorphic to c_0, and c_0 is isomorphic to c, it follows that $C(\omega^3)$ is isomorphic to c. This isomorphism carries the ideal I onto an infinite dimensional closed ideal in c. But, every infinite dimensional closed ideal in c is isomorphic to c_0. Therefore, there exists an isomorphism

$$H : I \to c_0, \tag{35}$$

(the analog of the isomorphism H from (21)).

Instead of (14), we can use the following result of R.R. PHELPS [7]: let S be a compact Hausdorff space, S_1 a closed subset of S, $J = Z(S_1)$ a closed ideal in $C(S)$ and $U = \{x \in J : \|x\| \leqslant 1\}$. Then, $f \in J^*$ supports U, if and only if for every norm-preserving extension μ of f to $C(S)$, the sets S_1, $S(\mu^+)$ and $S(\mu^-)$ are pairwise disjoint. Using this fact we shall define the elements $g_i \in I^*$, such that

$$\mathrm{sp}(\{g_i\}) \cap \mathcal{S}(U) = \{0\}.$$

(the analog of (19)). We observe that, if $f \in M(\omega^3)$ is such that $f(\alpha) = 0$ for $\alpha \in \Lambda$, then

$$\|f\| = \textstyle\sum_{\alpha\in\Delta}|f(\alpha)| = \|f|_I\|,$$

where $\Delta = [1, \omega^3] \setminus \Lambda$ and $f|_I$ denotes the restriction of f to I. Therefore, f is a norm-preserving extension of $f|_I$ to $C(\omega^3)$. We shall define $g_i \in M(\omega^3)$, such that $g_i(\alpha) = 0$ for $\alpha \in \Lambda$ and for $g \in sp(\{g_i\})$.

$$S(g^+) \cap S(g^-) \neq \emptyset.$$

Then, by the above quoted result of R. R. Phelps, the restriction of g to I does not attain its supremum on U.

Let $\Delta = [1, \omega^3] \setminus \Lambda$ and let

$$\alpha_1 < \alpha_2 < \ldots,$$

be the accumulation points of the set Δ of the form

$\alpha_k = \omega^2 \cdot \lambda_k$, $1 \leqslant \lambda_k < \omega$. If $\alpha_k \in \Delta$ then, by the closedness of the set Δ there exists a number l_k, $1 \leqslant l_k < \omega$, such that

$$[\omega^2 \cdot (\lambda_k - 1) + \omega \cdot l_k + 1,\ \omega^2 \cdot \lambda_k] \subseteq \Delta. \tag{36}$$

By the properties of ordinal numbers, it will exist a homeomorphism

$$\eta_k : [1, \omega^2] \to [\omega^2 \cdot (\lambda_k - 1) + \omega \cdot l_k + 1,\ \omega^2 \cdot \lambda_k], \tag{37}$$

(η_k can be defined e.g. by $\eta_k(i) = \omega^2 \cdot (\lambda_k - 1) + \omega \cdot l_k + i$, for $1 \leqslant i < \omega^2$, $\eta_k(\omega^2) = \omega^2 \cdot \lambda_k$)

Put

$$g_{\alpha_k}(x) = x(\alpha_k) + 2^{-(k+1)} \sum_{1 \leqslant i < \omega} (-1)^i 2^{-i} \cdot x(\eta_k(\omega \cdot i)),$$

$$e_{\alpha_k}(i) = \begin{cases} 1 & \text{for } i \in \eta_k([1, \omega^2]) \\ 0 & \text{in rest,} \end{cases}$$

and

$$f_{\alpha_k} = \delta_{\alpha_k}, \text{ for } k = 1, 2, \ldots$$

Let now

$$\alpha_{k,1} < \alpha_{k,2} < \ldots,$$

be the accumulation points of the set Δ of the form

$$\alpha_{k,j} = \omega^2 (\lambda_{k,j} - 1) + \omega \cdot \mu_{k,j},\ 1 \leqslant \lambda_{k,j},\ \mu_{k,j} < \omega,$$

belonging to the interval $[\alpha_{k-1}, \alpha_k]$. (We put $\alpha_0 = 1$ and, if there are only a finite number of α_k, we consider also the interval $[\alpha_n, \omega^3]$ where α_n is the last of α_k). In this case there exist the homeomorphisms

$$\eta_{k,j} : [1, \omega] \to]\alpha_{k,j-1}, \alpha_{k,j}]. \tag{38}$$

We have now to consider some different situations. The symbol σ_k will have the same meaning as in (15).

II. *a*. $\alpha_k \in \Delta$.

Preserving the notations from (36), we consider the sub-cases

II. *a*.1. $\alpha_{k,j} \in \Delta \cap [\alpha_{k-1}, \omega^2 \cdot (\lambda_k - 1) + \omega \cdot l_k]$.

Put

$$g_{\alpha_{k,j}}(x) = x(\alpha_{k,j}) + 2^{-(k+j+1)} \sum_{1 \leqslant i < \omega} (-1)^i 2^{-i} x(\eta_{k,j}(i)),$$

$$e_{\alpha_{k,j}}(i) = \begin{cases} 1 & \text{for } i \in \eta_{k,j}([1, \omega]) \\ 0 & \text{in rest,} \end{cases}$$

$$f_{\alpha_{k,j}} = \delta_{\alpha_{k,j}},$$

where $\eta_{k,j}$ is the homeomorphism (38).

II.*a*.2. $\alpha_{k,j} \in \Delta \cap]\omega^2 \cdot (\lambda_k - 1) + \omega \cdot l_k, \omega^2 \cdot \lambda_k]$.

By (37) there exists a number j' such that $\alpha_{k,j} = \eta_k(\omega \cdot j')$. Put

$$g_{\alpha_{k,j}}(x) = x(\alpha_{k,j}) + 2^{-(k+1)} \sum_{i=1}^{3 \cdot 2^{j'-1}-1} (-1)^i 2^{-i} x(\eta_k(\omega \cdot i)) +$$
$$+ 2^{-(k+j'+1)} \sum_{1 \leqslant i < \omega} (-1)^i 2^{-i} x(\eta_k(\omega \cdot \sigma_{j'}(i))),$$

$$e_{\alpha_{k,j}}(i) = \begin{cases} 1 & \text{for } i \in \eta_k \ ([1, \omega^2]) \\ 0 & \text{in rest,} \end{cases}$$

$$f_{\alpha_{k,j}} = \delta_{\alpha_{k,j}} - \delta_{\alpha_k}.$$

II.*b*. $\alpha_k \notin \Delta$.

Put

$$g_{\alpha_{k,j}}(x) = x(\alpha_{k,j}) + 2^{-(k+j+1)} \sum_{1 \leqslant i < \omega} (-1)^i 2^{-i} x(\eta_{k,j}(i)),$$

$$e_{\alpha_{k,j}}(i) = \begin{cases} 1 & \text{for } i = \alpha_{k,j} \\ 0 & \text{in rest} \end{cases}$$

$$f_{\alpha_{k,j}} = \delta_{\alpha_{k,j}}$$

Let now pass to the isolated points of $[1, \omega^3]$ which belong to Δ. By the homeomorphism (38), every isolated point α from $\Delta \cap]\alpha_{k,j-1}, \alpha_{k,j}[$ is of the form $\alpha = \eta_{k,j}(l)$ for a number $l \in [1, \omega[$. We consider now the following cases:

II.*a*.1.α.

$$\alpha_k \in \Delta,\ \alpha_{k,j} \in \Delta \cap [\alpha_{k-1}, \omega^2 \cdot (\lambda_k - 1) + \omega \cdot l_k] \text{ (see (36)).}$$

Put

$$g_\alpha(x) = x(\alpha) + 2^{-(k+j+1)} \sum_{i=1}^{3 \cdot 2^{l-1}-1} (-1)^i \cdot 2^{-i} \cdot x(\eta_{k,j}(i)) +$$
$$+ 2^{-(k+j+l+1)} \sum_{1 \leqslant i < \omega} (-1)^i \cdot 2^{-i} \cdot x(\eta_{k,j}(\sigma_l(i))),$$

$$e_\alpha(i) = \begin{cases} 1 & \text{for } i = \alpha \\ 0 & \text{in rest} \end{cases}$$

$$f_\alpha = \delta_\alpha$$

II.*a*.2.α.

$$\alpha_k \in \Delta,\ \alpha_{k,j} \in \Delta \cap]\omega^2 \cdot (\lambda_k - 1) + \omega \cdot l_k, \alpha_k] \text{(see (36)).}$$

In this case, by (37), there exist $l, j' \in [1, \omega[$, such that

$$\alpha = \eta_k(\omega \cdot (j' - 1) + l)$$

Put

$$g_\alpha(x) = x(\alpha) + g_{\alpha_{k,j}}(x) - x(\alpha_{k,j}) +$$

$$+ 2^{-(k+j+l+1)} \sum_{1 \leqslant i < \omega} (-1)^i \cdot 2^{-i} \cdot x(\eta_k(\omega \cdot (j' - 1) + \sigma_l(i))),$$

$$e_\alpha(i) = \begin{cases} 1 & \text{for } i = \alpha \\ 0 & \text{in rest} \end{cases}$$

$$f_\alpha = \delta_\alpha - \delta_{\alpha_{k,j}} - \delta_{\alpha_k}.$$

II.b.α.

$$\alpha_k \notin \Delta, \ \alpha_{k,j} \in \Delta.$$

By (38), there exists $l \in [1, \omega[$, such that $\alpha = \eta_{k,j}(l)$. Put

$$g_\alpha(x) = x(\alpha) + 2^{-(k+j+l)} \sum_{i=1}^{3 \cdot 2^{j-1}-1} (-1)^i \cdot 2^{-i} \cdot x(\eta_{k,j}(i)) +$$

$$+ 2^{-(k+j+l+1)} \sum_{1 \leqslant i < \omega} (-1)^i \cdot 2^{-i} \cdot x(\eta_{k,j}(\sigma_l(i))),$$

$$e_\alpha(i) = \begin{cases} 1 & \text{for } i = \alpha \\ 0 & \text{in rest} \end{cases}$$

$$f_\alpha = \delta_\alpha.$$

II.b.β.

$$\alpha_k \notin \Delta, \qquad \alpha_{k,j} \notin \Delta.$$

If $\alpha = \eta_{k,j}(l)$, put

$$g_\alpha(x) = x(\alpha) + 2^{-(k+j+1)} \sum_{1 \leqslant i < \omega} (-1)^i \cdot 2^{-i} \cdot x(\eta_{k,j} \cdot (\sigma_l(i))),$$

$$e_\alpha(i) = \begin{cases} 1 & \text{for } i = \alpha \\ 0 & \text{in rest,} \end{cases}$$

$$f_\alpha = \delta_\alpha$$

This finishes the definitions of the elements e_i, f_i, g_i in Case II.

Case III. $I = Z(\Lambda)$, $\Lambda \neq \emptyset$ *and* $\omega^3 \notin \Lambda$.

This case reduces to Case I or to Case II. Since Λ is closed, from $\omega^3 \notin \Lambda$ it follows the existence of a $k_0 \in [1, \omega[$, such that $[\omega^2 \cdot (k_0 + 1), \omega^3] \subseteq \Delta$, where $\Delta = [1, \omega^3] \setminus \Lambda$. Denoting

$$\Delta_i = [\omega^2 \cdot (i-1) + 1, \ \omega^2 \cdot i], \quad \Gamma_i = [1, \omega^3] \setminus \Delta_i,$$

$$\Lambda_i = \Delta_i \cap \Lambda, \qquad X_i = Z(\Lambda_i),$$

$$\Gamma = [1, \omega^2 \cdot k_0], \qquad X = Z(\Gamma),$$

one can write

$$Z(\Lambda) = X_1 \oplus \ldots \oplus X_{k_0} \oplus X.$$

Each X_i is a closed ideal in $Z(\Gamma_i) \cong C(\omega^2)$, and $X = Z(\Gamma) \cong C(\omega^3)$ Reasoning like in the proof of Lemma 3, we can suppose that all X are infinite dimensional. Now, if $\omega^2 \cdot i \notin \Lambda_i$ we decompose again $Z(\Lambda_i)$ as above. Continuing in such a manner, we obtain finally a decomposition of $Z(\Lambda)$.

$$Z(\Lambda) = Z_1 \oplus \ldots \oplus Z_p,$$

where each Z is isometrically isomorphic to an infinite dimensional closed ideal $Z(\Lambda_i')$ in $C(\omega^k)$, such that $\Lambda_i' = \emptyset$ or $\omega^k \in \Lambda_i$,

$$k \in \{1, 2, 3\}. \text{ For } x \in Z(\Lambda), \ x = z_1 + \ldots + z_p, \ z_i \in Z_i,$$

$i = 1, \ldots, p$ we have $\|x\| = \max\{\|z_1\|, \ldots, \|z_p\|\}$. By Case I or Case II of Theorem 1, each $Z(\Lambda_i')$ contains an antiproximinal convex cell. Reasoning again, like in the proof of Lemma 3, one can show that $Z(\Lambda)$ contains an antiproximinal convex cell.

Theorem 1 is completely proved.

REFERENCES

[1] A m i r, D., *Continuous function spaces with the bounded extension property*, Bull. Res. Counc. of Israel, **10F** 133–138, (1962).

[2] С. К о б з а ш, С. *Выпуклые антипроксиминальные множества в пространствах* c_0 и c Мат. Заметки, т. 17, № *3* 449–457 (1975).

[3] D u n f o r d, N., and S c h w a r t z, J. T., *Linear operators. I. General theory*, Interscience, New York, 1958 (Russian translation Moscow 1962).

[4] E d e l s t e i n, M., and T h o m p s o n, A. C., *Some results on nearest points and support properties of convex sets in* c_0, Pacific J. Math. **40**, 4, 553–560, (1972).

[5] H o l m e s, R.R., *A course on optimization and best approximation*, Lecture Notes in Math. v. **257**, Springer Verlag, Berlin-Heidelberg-New York, 1972.

[6] K l e e, V., *Remarks on nearest points in normed linear spaces*, Proc. Colloq. Concexity, Copenhagen 1965, 168–176, Copenhagen 1967.

[7] P h e l p s, R. R., *Some subreflexive Banach spaces*, Archiv der Math. **10**, 162–169 (1959).

[8] S e m a d e n i Z., *Banach spaces of continuous functions*, I, Warszawa 1971.

[9] S i e r p i n s k i, W., *Cardinal and ordinal numbers*, Warszawa 1965.

[10] S i n g e r, I., *Bases in Banach spaces*, I. Springer Verlag, Berlin-Heidelberg- New Yrok, 1970.

[11] Зуховицкий С.И., *О минимальных расширениях линейных функционалов в пространстве непрерывных функц* Изв. Акад. Наук СССР, Сер матем **21**; *3* (409–422),

Received 30.6.1976.

University Babeş-Bolyai, Cluj-Napoca
Institute of Mathematics.