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A MAXIMUM PRINCIPLE FOR SECOND ORDER PARABOLIC SYSTEM

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In this note we purpose to give a maximum principle for the modulus of the solution of a parabolic system of equations whith partial derivatives and to study the uniqueness of the solution of a boundary value problem relative to these systems. We mention that for elliptic systems the problem of establishing of maximum principles is studied in the works [3], [4], [5]. For parabolic systems we also mention the works [1], [5].

1. Let be $\Omega \subset \mathbf{R}^m$ a bounded domain with boundary Γ and closure $\overline{\Omega}$. Let be $T \in R$, $T > 0$, fixed and $D = \Omega \times]0, T[$; then each point in D will be written (x, t), where $x \in \Omega$ and $0 < t < T$. We note by S the „lateral boundary'' of domain D, that is, $S = \{(x, t) \in D,\ x \in \Gamma,\ 0 < t < T\}$, and Σ the „parabolic boundary'' of domain D, that is, $\Sigma = B \cup S$, where $B = \Omega \times \{0\}$.

Let

$$Lu = \sum_{i,j=1}^{m} A_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} A_i(x, t) \frac{\partial u}{\partial x_i} + A_0(x, t)u - \frac{\partial u}{\partial t} = f \tag{1}$$

be a second order system of differential equations with partial derivatives where

$$A_{ij},\ A_i,\ A_0 \in C(\overline{D}, M_{nn}(R)),\ f \in C(\overline{D}, R^n) \tag{2}$$

and $u = (u_1, \ldots, u_n)$ is a vectorial-function, $u : \overline{D} \to R^n$.

Definition. *The system* (1) *is called parabolic on a point* $(x, t) \in D$ *if for any* $\tau \in R^n$, $\lambda \in R^m$, τ *and* $\lambda \neq 0$ *the inequality*

$$\sum_{i,j=1}^{m} (\tau^* A_{ij}(x, t)\tau)\lambda_i \lambda_j \geqslant 0 \tag{3}$$

takes place, where τ^* *indicates transposed vector* τ.

Let be $u \in C^2(D, R^n) \cap C(\bar{D}, R^n)$. We note by $|u|$ the function $|u| : \bar{D} \to R^n$ defined by $(x, t) \to |u(x, t)|$. Then $u = |u|e$, where $e = (e_1, \ldots, e_n)$ with $|e| = 1$.
There takes place:

THEOREM 1 (MAXIMUM PRINCIPLE). *If: i) system (1) is parabolic in D, ii) there exists $\alpha \in R$, $\alpha \neq 0$, so that*

$$e^* L_1 e \leqslant -\alpha^2 \tag{4}$$

for any $e \in C^2(D, R^n)$ with $|e| = 1$, where L_1 is the elliptic operator attached to L, i.e.

$$L_1 u = \sum_{i,j=1}^{m} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} A_i \frac{\partial u}{\partial x_i} + A_0 u \tag{5}$$

then any solution $u \in C^2(D, R^n) \cap C(\bar{D}, R^n)$, for which $u \cdot f \geqslant 0$, verifies the inequality:

$$|u(x, t)| \leqslant \max\{\max_{(x,t)\in\Sigma} |u(x, t)|, \frac{1}{\alpha^2} \max_{(x,t)\in D} |f(x, t)|\} \tag{6}$$

Proof. In any point $(x, t) \in D$, in which $|u(x, t)| \neq 0$, from $e^* L(|u|e) = e^* f$ we conclude

$$e^* \Big(\sum_{i,j=1}^{m} A_{ij} \frac{\partial^2 |u|}{\partial x_i \partial x_j} e + \sum_{i=1}^{m} A_i \frac{\partial |u|}{\partial x_i} e + A_0 |u| e - \frac{\partial |u|}{\partial t} e + \sum_{i,j=1}^{m} A_{ij} \frac{\partial^2 e}{\partial x_i \partial x_j} |u| +$$

$$+ \sum_{i=1}^{m} A_i \frac{\partial e}{\partial x_i} |u| - \frac{\partial e}{\partial t} |u| \Big) = e^* f, \text{ or}$$

$$\sum_{i,j=1}^{m} (e^* A_{ij} e) \frac{\partial^2 |u|}{\partial x_i \partial x_j} + \sum_{i=1}^{m} (e^* A_i e) \frac{\partial |u|}{\partial x_i} - \frac{\partial |u|}{\partial t} + (e^* L_1 e)|u| = e^* f \tag{7}$$

where L_1 is the operator given in relation (5).
If now $(x_0, t_0) \in D$ is a point of maximum for $|u(x, t)|$ then from (7) we get

$$\sum_{i,j=1}^{m} (e^*(x_0, t_0) A_{ij} e(x_0, t_0)) \frac{\partial^2 |u(x_0, t_0)|}{\partial x_i \partial x_j} + \tag{8}$$

$$+ (e^*(x_0, t_0) L_1 e(x_0, t_0)) |u(x_0, t_0)| = e^*(x_0, t_0) f(x_0, t_0).$$

As in the point (x_0, t_0) the function $u(x, t)$ has a maximum and the system (1) is parabolic it follows that

$$\sum_{i,j=1}^{m} (e^*(x_0, t_0) A_{ij}(x_0, t_0)) \frac{\partial^2 |u(x, t)|}{\partial x_i \partial x_j} \leqslant 0. \tag{9}$$

Considering relations (4), (9) and the fact that $u \cdot f \geqslant 0$ we are brought to a contradiction in relation (8). Thus the maximum of $|u(x, t)|$ cannot be reached in an interior point of domain D. Moreover we can easily find that the inequality (6) is true, and thus the theorem is proved.

2. We consier the following boundary value problem

$$\begin{cases} Lu = f \text{ in } D \\ u \in C^2(\bar{D}, R^n) \\ u = h \text{ on } \Sigma \end{cases} \tag{10}$$

where $h \in C(\Sigma, R^n)$.
There takes place

THEOREM 2 *In the conditions of Theorem 1 the boundary value problem (10) has a unique solution.*

Proof. One immediately notices that from

$$\begin{cases} Lu = 0 \text{ in } D \\ u \in C^2(\bar{D}, R^n) \\ u = 0 \text{ on } \Sigma \end{cases} \Rightarrow u = 0 \text{ in } D. \tag{11}$$

Then if u' and u'' are two solutions of the problem (10) the difference $v = u' - u''$ verifies the homogeneous boundary value problem (11), therefore, $u' = u''$, that is the problem (10) has at most a solution. But (see, for example [2]) the problem (10) has solution and therefore theorem follows.

3. We give below some cases of effective expression for the condition (4) by means of the system's coefficients.

Case m = 1. We consider the following system

$$Lu = \frac{\partial^2 u}{\partial x^2} + A \frac{\partial u}{\partial x} + Bu - \frac{\partial u}{\partial t} = C \tag{12}$$

with

$$A, B \in C([a, b] \times]0, T[, M_{nn}(R)),\ C \in C([a, b] \times]0, T[, R^n).$$

The condition (4) is

$$e^* e'' + e^* A e' + e^* B e \leqslant -\alpha^2.$$

Taking into account that $e^* e' = 0$, it becomes

$$(e^*)' e' - e^* A e' - e^* B e \geqslant \alpha^2.$$

We suppose that $e^* B e \leqslant -\beta^2$ and in addition

$$-\frac{\|A\|^2}{2} + \beta^2 > 0, \tag{13}$$

where $\|A\|$ is the spectral norm of matrix A. Then there exists $\alpha \in R$, $\alpha \neq 0$, so that

$$\alpha^2 \leqslant -\frac{\|A\|^2}{2} + \beta^2 \leqslant |e'|^2 - \|A\|\,|e'| + \beta^2 \leqslant (e^*)'e' - e^*Ae' - e^*Be$$

and therefore condition (4) is achieved.
The following theorems takes place:

THEOREM 3. *If the relation* (13) *is achieved then for each solution* $u \in C^2([a, b] \times]0, T[, R^n)$ *of system* (12) *for which* $u \cdot C \geqslant 0$, *we have*

$$|u(x, t)| \leqslant \max\{\max_{t \in [0,T]}\{|u(a, t)|, |u(b, t)|\}, \max_{x \in [a,b]} |u(x, 0)|, \frac{1}{\beta^2 - \frac{\|A\|^2}{2}} \max_{(x,t) \in [a,b] \times [0,T]} |C(x, t)|\}$$

THEOREM 4. *In the conditions of Theorem* 3 *the solution of boundary value problem*

$$(14)\quad \begin{cases} \frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu - \frac{\partial u}{\partial t} = C \text{ in } [a, b] \times]0, T[\\ u \in C^2(\bar{D}, R^n) \\ u = h \text{ on } (\{a\} \times [0, T])U(\{b\} \times [0, T])\ U([a, b] \times \{0\}) \end{cases}$$

is unique.

Case $A_{ij} = a_{ij} I$. We consider the system (1) in which $A_{ij} = a_{ij}I$, $a_{ij} \in C(D, R)$. We suppose that the following relations are achieved

$$(15)\quad \sum_{i,j=1}^{m} a_{ij}\lambda_i\lambda_j \geqslant \gamma^2|\lambda^2|, \qquad \gamma \neq 0$$

$$(16)\quad e^*A_0 e \leqslant -\beta^2, \qquad \beta \neq 0$$

and

$$(17)\quad A_i = a_i I + A_i^{(1)},$$

where $\qquad a_i \in C(\bar{D}, R),\ A_i^{(1)} \in C(\bar{D}, M_{nn}(R)).$

In this case the condition (4) is the following

$$\sum_{i,j=1}^{m} a_{ij}e^* \frac{\partial^2 e}{\partial x_i \partial x_j} + \sum_{i=1}^{m} e^*A_i^{(1)} \frac{\partial e}{\partial x_i} + e^*A_0 e \leqslant -\alpha^2$$

and since $e^* \dfrac{\partial^2 e}{\partial x_i \partial x_j} = -\dfrac{\partial e^*}{\partial x_i}\dfrac{\partial e}{\partial x_j}$, we get

$$\sum_{i,j=1}^{m} a_{ij} \frac{\partial e^*}{\partial x_i}\frac{\partial e}{\partial x_j} - \sum_{i=1}^{m} e^*A_i^{(1)} \frac{\partial e}{\partial x_i} - e^*A_0 e \geqslant \alpha^2.$$

Taking into account the relations (15), (16) and if

$$(18)\quad \xi^* \begin{bmatrix} \gamma^2 & 0 & \cdots & 0 & -\frac{1}{2}\|A_1^{(1)}\| \\ 0 & \gamma^2 & \cdots & 0 & -\frac{1}{2}\|A_2^{(1)}\| \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \gamma^2 & -\frac{1}{2}\|A_m^{(1)}\| \\ -\frac{1}{2}\|A_1^{(1)}\| & -\frac{1}{2}\|A_2^{(1)}\| & \cdots & -\frac{1}{2}\|A_m^{(1)}\| & \beta^2 \end{bmatrix} \xi > 0$$

for any $\xi \in R^{n+1}$, $\xi \neq 0$, we have

$$(19)\quad \gamma^2\sum_{i=1}^{m}\left|\frac{\partial e}{\partial x_i}\right|^2 - \sum_{i=1}^{m}\|A_i^{(1)}\|\left|\frac{\partial e}{\partial x_i}\right| + \beta^2 \geqslant \alpha^2.$$

But the relation (18) is true if

$$(20)\quad \beta^2 - \frac{1}{4\gamma^2}\sum_{i=1}^{m}\|A_i^{(1)}\|^2 > 0$$

and, therefore, we get the following theorems

THEOREM 5. *If the system* (1) *satisfied the conditions* (15), (16), (17) *and* (20) *then any solution* $u \in C^2(\bar{D}, R^n) \cap C(\bar{D}, R^n)$ *of this system for which* $u \cdot f \geqslant 0$, *verifies the inequality*

$$|u(x, t)| \leqslant \max\left\{\max_{(x,t)\in\Sigma}|u(x, t)|, \frac{1}{\beta^2 - \frac{1}{4\gamma^2}\sum_{i=1}^{m}\|A_i^{(1)}\|^2}\max_{(x,t)\in D}|f(x, t)|\right\}.$$

THEOREM 6. *In the conditions of Theorem* 5, *the solution of boundary value problem*

$$(21)\quad \begin{cases} \sum_{i,j=1}^{m} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} a_i(x, t)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{m} A_i^{(1)}\frac{\partial u}{\partial x_i} + \\ \qquad + A_0(x, t)u - \frac{\partial u}{\partial t} = f(x, t) \text{ in } D \end{cases}$$

$$u \in C^2(\bar{D}, R^n)$$
$$u = h \text{ on } \Sigma$$

where $h \in C(\Sigma, R^n)$, *is unique.*

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