MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 5, Nº 2, 1976, pp. 165—180

CONVERGENCE THEOREMS ON NON-COMMUTATIVE CONTINUED FRACTIONS

by

N. NEGOESCU

(Iași)

Introduction

The three lemmas on real continued fractions of the first paragraph are essential in the proofs of the theorems of paragraph 5.

In second paragraph, we state the known concepts and propositions on the Banach algebras, necessary in the paragraphs 3—7.

The definitions of non-commutative continued fractions and their convergence of paragraph 3 are given according to WYMAN FAIR [1].

The new results of this note of paragraphs **1**, **4**—**7** suggest a general method which may be used to prove the theorems of convergence for non-commutative continued fractions. In paragraph **6**, we obtain a generalization of a classical WORPITZKY'S theorem on the convergence of some continued fractions of complex numbers. In paragraph **7** too, we give some consequences of the main theorem from paragraph **6**.

1. Three lemmas on real continued fractions

Lemma 1.1. *Let be*

$$\frac{a}{1-\cfrac{a}{1-\cfrac{a}{1-\cdots}}} \tag{1.1}$$

a real continued fraction, where

$$0 < a \leqslant \frac{1}{4}.$$

Then

(i) *the convergents of continued fraction* (1.1) *are positive and its sequence is strictly monoton increasing,*

(ii) *the convergents satisfy the inequalities*

$$\frac{p_n}{q_n} \leqslant \frac{1}{2}\,\frac{n}{n+1}, \tag{1.2}$$

(iii) *the sequence of convergents of* (1.1) *is convergent to the number*

$$\frac{1}{2}\,[1-(1-4a)^{1/2}], \tag{1.3}$$

which is less or equal $\frac{1}{2}$.

Proof. (i) From the evident inequality

$$0 < \frac{a}{1} < \frac{a}{1-a}, \tag{1.4}$$

it follows, for the first two convergents of real continued fraction (1.1),

$$0 < \frac{p_1}{q_1} < \frac{p_2}{q_2}.$$

From inequality (1.4), it follows also

$$\frac{p_2}{q_2} = \frac{a}{1-a} < \frac{a}{1-\dfrac{a}{1-a}} = \frac{p_3}{q_3}.$$

By induction, we now suppose that

$$\frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n}. \tag{1.5}$$

Since

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n + a p_{n-1}}{q_n + a q_{n-1}}, \tag{1.6}$$

then the function

$$x \to \frac{p_n + x p_{n-1}}{q_n + x q_{n-1}}$$

has the derivative

$$\frac{q_n p_{n-1} - p_n q_{n-1}}{(q_n + x q_{n-1})^2}$$

which is strictly negative. From equality (1.6), it follows

$$\frac{p_{n+1}}{q_{n+1}} > \frac{p_n}{q_n}.$$

This completes the induction and the proof of (i).

(ii) By the hypothesis of the lemma, we have

$$\frac{p_1}{q_1} = a \leqslant \frac{1}{4}.$$

By induction, we now suppose that $\frac{p_n}{q_n}$ satisfies (1.2). Then

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a}{1-\dfrac{p_n}{q_n}} \leqslant \frac{\dfrac{1}{4}}{1-\dfrac{1}{2}\,\dfrac{n}{n+1}} = \frac{1}{2}\,\frac{n+1}{n+2}.$$

Therefore, (ii) is proved completely.

(iii) By (i) the sequence of the convergents is strictly monoton increasing and by (ii) the same sequence is less than $\frac{1}{2}$; therefore it is convergent.

The real continued fraction, being periodic, its value is given as the root, which is less than $\frac{1}{2}$, of equation

$$x = \frac{a}{1-x}, \text{ that is}$$

$$x = \frac{1}{2}\,[1-(1-4a)^{1/2}] \leqslant \frac{1}{2}.$$

L e m m a 1.2. *If*

$$\cfrac{a_1}{1-\cfrac{a_2}{1-\cfrac{a_3}{1-}}} \tag{1.7}$$

where $0 \leqslant a_k \leqslant a \leqslant \frac{1}{4}$, *is a real continued fraction, then*

(i) *the convergents of this continued fraction are non-negative and*

(ii) *the convergents satisfy inequalities*

$$\frac{p_n}{q_n} \leqslant \frac{1}{2}\,\frac{n}{n+1}. \tag{1.8}$$

Proof. For the first convergent $\frac{p_1}{q_1} = a_1$, we have evident

$$0 \leqslant \frac{a_1}{1} \leqslant \frac{a}{1}.$$

If we suppose

$$a' = \frac{a_2}{1 -} \frac{a_3}{1 -} \cdots \frac{a_{n+1}}{1} \geqslant 0, \tag{1.9}$$

then, since $a_1 \geqslant 0$ by hypothesis, it follows that

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_1}{1 - a'} \tag{1.10}$$

is non-negative.

(ii) If the real continued fraction (1.9) is less than $\frac{n}{2(n+1)}$, then, from (1.10), we obtain

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_1}{1 - a'} \leqslant \frac{\frac{1}{4}}{1 - \frac{n}{2(n+1)}} = \frac{1}{2} \frac{n+1}{n+2}.$$

This completes the proof of lemma 2.2.

L e m m a 1.3. *For $a > \frac{1}{4}$, the real continued fraction*

$$\frac{a}{1 -} \frac{a}{1 -} \frac{a}{1 -} \cdots \tag{1.11}$$

is divergent.

Proof. If we suppose that continued fraction (1.11) is convergent, then it converges to a real number.

On the other hand, real continued fraction (1.11), being periodic, its value is one of the complex numbers

$$\frac{1}{2}[1 \pm (1 - 4a)^{\frac{1}{2}}],$$

which are the roots of the equation

$$x = \frac{a}{1 - x}.$$

This contradiction proves the lemma.

2. Banach algebras

D e f i n i t i o n 3.1. *A* Banach algebra *with elements x, y, ... is a Banach space, that is a complete normed space endowed with a multiplication, everywhere defined, associative, distributive with respect to every linear combination and such that*

$$\|xy\| \leqslant \|x\| \, \|y\| \tag{2.1}$$

for any x, y of $\mathcal{A}$.

Algebra $\mathcal{A}$ is said to be commutative *if the multiplication is commutative.*

A subspace $\mathcal{B}$ of $\mathcal{A}$ is called a subalgebra *of $\mathcal{A}$ if it is also an algebra.*

It is easy to prove the following propositions:

P r o p o s i t i o n 2.1. *If (x_n) and (y_n) are convergent sequences, with respect the topology induced by the algebra norm, then $(x_n y_n)$ converges to xy, that is the multiplication is continuous.*

P r o p o s i t i o n 2.2. *For every x of a Banach algebra and $n \in \mathrm{N}^*$, w have*

$$\|x^n\| \leqslant \|x\|^n.$$

P r o p o s i t i o n 2.3. *If there is in $\mathcal{A}$ an element, e which satisfies the relation*

$$ex = xe = x. \tag{2.2}$$

then e is unique and $\|e\| \geqslant 1$, if $\mathcal{A} \neq \{0\}$.

P r o p o s i t i o n 2.4. *If there is an element e in $\mathcal{A}$ which satisfies (2.2), then the expresion*

$$\|x\|' = \sup\{\|xu\| : \|u\| = 1\}$$

exists in $\mathcal{A}$ and it defines a norm equivalent with the initial norm such that

$$\|e\|' = 1 \tag{2.3}$$

holds.

For the norm $\|\cdot\|'$, $\mathcal{A}$ is also a Banach algebra with properties (2.2) and (2.3).

It is easy to verify that $\|\cdot\|'$ is a norm. Then

$$\|e\|' = \sup\{\|eu\| : \|u\| = 1\} = \sup\{\|u\| : \|u\| = 1\} = 1,$$

$$\|xy\|' = \sup\{\|xyu\| : \|u\| = 1\} = \sup\left\{\left\|x\frac{yu}{\|yu\|}\right\| : \|u\| = 1\right\}\|yu\| \leqslant$$

$$\leqslant \sup\left\{\left\|x\frac{yu}{\|yu\|}\right\| : \|u\| = 1\right\} \sup\{\|yu\| : \|u\| = 1\} = \|x\|'\|y\|'$$

since $yu/\|yu\|$ has the norm 1.

This norm is equivalent to the initial norm, because

$$\|x\|' \leqslant \|x\| \leqslant \sup\{\|u\| : \|u\| = 1\}\|e\| \leqslant \|x\|'\|e\|,$$

where $u = e/\|e\|$.

Propositions (2.3) and (2.4) suggest:

Definition 2.2. *A Banach algebra is said to have the unit e* [2], *if for any $x \in \mathcal{A}$ we have the relation* (2.2) *and* (2.3).

Definition 2.3. *Let $\mathcal{A}$ be a Banach algebra with the element e, wich satisfies* (2.2). *An element $x \in \mathcal{A}$ has a left inverse, if it exists an element $x_s^{-1} \in \mathcal{A}$, such that $x_s^{-1}x = e$.*

An element $x \in \mathcal{A}$ has a right inverse, if it exists an element $x_d^{-1} \in \mathcal{A}$, such that $x_d^{-1}x = e$. The element x is called invertible, if it has a left inverse and a right inverse.

Proposition 2.5. *The invertible elements in a Banach algebra $\mathcal{A}$ with an element e, wich satisfies relations* (2.2) *have following properties:*

(i) *If $yz = xz = e$, then x is invertible in $\mathcal{A}$ and $y = z$. Therefore, the inverse of an element, if it exists, is unique. It is noted x^{-1}.*

(ii) *If x is invertible in $\mathcal{A}$ and $xy = e(yx = e)$, then $y = x^{-1}$.*

(iii) *The element e is invertible and it is the own inverse. The element* 0 *is not invertible.*

(iv) *If x is invertible in $\mathcal{A}$, then cx is invertible in $\mathcal{A}$ for any scalar $c \in \mathcal{C}$, with $c \neq 0$ and*

$$(cx)^{-1} = \frac{1}{c}x^{-1},$$

(v) *If x, y are invertible in $\mathcal{A}$, then the product is invertible and $(xy)^{-1} = y^{-1}x^{-1}$.*

It is also easy to prove

Proposition 2.6. *If $\mathcal{A}$ is a Banach algebra with unit e* (definition 2.2) *and $x \in A$ verifies inequality $\|x\| < 1$, then the element $e - x$ is invertible in $\mathcal{A}$ and*

(i) $$(e - x)^{-1} = e + x + x^2 + \ldots,$$

(ii) $$\|(e - x)^{-1}\| \leqslant \frac{1}{1 - \|x\|}.$$

One also see that *if $\mathcal{A}$ is a Banach algebra with unit, e, then*

(i) $\{p(x) : \forall$ polynomial p, $x \in \mathcal{A}$ *is a commutative subalgebra of $\mathcal{A}$, with unit e.*

(ii) $\{ae : a \in K\}$, where $\mathcal{K}$ is $\mathbf{R}$ or $\mathcal{C}$, *is a commutative subalgebra of $\mathcal{A}$, isomorphic to $\mathcal{K}$, with unit e.*

3. Non-commutative continued fractions

Definition 3.1. *The formal expression*

$$\cfrac{a_1}{e + \cfrac{a_2}{e + \cfrac{a_3}{e + \ddots}}}$$

wich may be written also as

(3.1) $$\frac{a_1}{e+}\,\frac{a_2}{e+}\,\frac{a_3}{e+\ldots},$$

where a_n are elements of a non-commutative complex Banach algebra $\mathcal{A}$ with unit e, is said to be a non-commutative continued fraction.

Definition 3.2. *The element*

(3.2) $$q_n^{-1}p_n$$

(supposing q_n invertible that is $q_n^{-1}p_n \in \mathcal{A}$), *where p_n and q_n are given by the formules*

(3.3) $$p_{n+1} = p_n + a_{n+1}p_{n-1},$$
$$q_{n+1} = q_n + q_{n+1}q_{n-1},$$
$$p_1 = p_2 = a_1,\ q_1 = e,\quad q_2 = e + a_2,$$

is called the n^{th} convergent *of the continued fraction* (3.1).

Definition 3.3. *The non-commutative continued fraction* (1.1) *is said to be convergent, if q_n are invertible for a sufficient large n and the sequence of elements* (3.2) *converges.*

4. Relations for non-commutative continued fractions

Proposition 4.1. *For the non-commutative continued fractions, we have the following identities:*

(i) $q_3 = e + a_2 + a_3,$

(ii) $p_{n+1}q_n - p_nq_{n+1} = -a_{n+1}(p_nq_{n-1} - p_{n-1}q_n),$

(iii) $p_{n+1}q_n - p_nq_{n+1} = (-1)^n a_{n+1}a_na_{n-1} \ldots a_3a_1a_2,$

(iv) $q_{n+2} = (e + a_{n+1} + a_{n+2})q_n - q_{n+1}a_nq_{n-2}.$

Proof.

(i) and (ii) follow directly from (3.3).

(iii) is a corollary of (ii).

(iv) From formulas (3.3) we obtain

$$q_{n+1} = q_n + a_{n+1}q_{n-1} = q_n + a_{n-1}(q_n - a_nq_{n-2}) =$$
$$= (e + a_{n+1})q_n - a_{n+1}a_nq_{n-2},$$

and

$$q_{n+2} = q_{n+1} + a_{n+2}q_n = (e + a_{n+1})q_n - a_{n+1}a_nq_{n-2} + a_{n+2}q_n =$$
$$= (e + a_{n+1} + a_{n+2})q_n - a_{n+1}a_nq_{n+2}.$$

Proposition 4.2. If q_{n-1}, q_n, q_{n+1} are invertible elements of Banach algebra $\mathcal{A}$ with unit e, then

(4.1) $$q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n = -q_{n+1}^{-1}a_{n+1}q_{n-1}(q_n^{-1}p_n - q_{n-1}^{-1}p_{n-1}).$$

Proof. From (3.3), we obtain

$$q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n = q_{n+1}^{-1}(p_{n+1} - q_{n+1}q_n^{-1}p_n) =$$
$$= q_{n+1}^{-1}a_{n+1}(p_{n-1} - q_{n-1}q_n^{-1}p_n) = -q_{n+1}^{-1}a_{n+1}q_{n-1}(q_n^{-1}p_n - q_{n-1}^{-1}p_{n-1}).$$

Proposition 4.3. If $q_2, q_3, \ldots, q_{n+1}$ are invertible elements of Banach algebra $\mathcal{A}$ with unit e, then

(4.2) $$q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n = (-1)^n q_{n+1}^{-1}a_{n+1}q_{n-1}\cdot q_n^{-1}a_nq_{n-2}\ldots q_3^{-1}a_3q_1\cdot q_2^{-1}a_2a_1$$

or

(4.3) $$q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n = (-1)^n q_{n+1}^{-1}c_{n+1}c_n\ldots c_3a_2a_1,$$

where

(4.4) $$c_n = a_nq_{n-2}q_{n-1}^{-1}.$$

Proof. By writing (4.1) for n, $n-1$, $\ldots$, 1 and after necessary changes one obtains (4.2). Using notations (4.4), formula (4.2) becomes (4.3).

Proposition 4.4. If q_n is invertible we have

(4.5) $$q_{n+1} = (e + c_{n+1})q_n.$$

Proof. (1.3) and (4.4) imply

$$q_{n+1} = q_n + a_{n+1}q_{n-1} = eq_n + a_{n+1}q_{n-1}q_n^{-1}q_n = (e + c_{n+1})q_n.$$

5. Conditions for invertibility of elements q_n and limitations for q_n^{-1} and $q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n$.

THEOREM 5.1. *If a_n are elements of a Banach algebra with unit e, which verify the inequalities*

$$\|a_n\| \leqslant \frac{1}{4} \text{ for } n = 2, 3, 4, \ldots,$$

then the norms of c_n for the non-commutative continued fraction (1.1) *satisfy to inequalities*

$$\|c_{n+1}\| \leqslant \frac{1}{2}\,\frac{n}{n+1} \text{ for } n = 2, 3, 4, \ldots,$$

and the elements q_n are invertible.

Proof. From proposition 2.5 (iii) $q_1 = e$ *is invertible.* $q_2 = e + a_2$ *is also invertible because* $\|-a_2\| = \|a_2\| \leqslant \frac{1}{4} < 1$ (proposition 2.6).

From proposition 2.6 (ii), we have

$$\|q_2^{-1}\| = \|(e + a_2)^{-1}\| \leqslant \frac{1}{1 - \|a_2\|}.$$

Then

$$\|c_3\| = \|a_3q_1q_2^{-1}\| \leqslant \|a_3\|\,\|(e + a_2)^{-1}\| \leqslant \frac{\|a_3\|}{1 - \|a_2\|} \leqslant \frac{1}{3} = \frac{1}{2}\,\frac{2}{3}$$

and

$q_3 = (e + c_3)q_2$ is invertible (proposition 2.5 (v)) being the product of two invertible elements ($e + c_3$ is invertible because $\|-c_3\| \leqslant \frac{1}{3} < 1$, q_2 is also invertible by the preceding argument).

We suppose inductively that q_k ($k = 1, 2, \ldots, n + 1$) are invertible and

$$\|c_{k+1}\| \leqslant \cfrac{\|a_{k+1}\|}{1 - \cfrac{\|a_k\|}{1 - \ddots\, \cfrac{}{1 - \|a_2\|}}} \leqslant \frac{1}{2}\,\frac{k}{k+1} \text{ for } k = 2, 3, \ldots n + 1.,$$

From notation (.4) and roposition 4.4, we obtain

$$\|c_{n+2}\| = \|a_{n+2}q_nq_{n+1}^{-1}\| \leqslant \|a_{n+2}\|\,\|q_n(e + c_{n+1})q_n^{-1}\| \leqslant$$
$$\leqslant \|a_{n+2}\|\,\|(e + c_{n+1})^{-1}\| \leqslant \cfrac{\|a_{n+2}\|}{1 - \cfrac{\|a_{n+1}\|}{1 - \ddots\, \cfrac{}{1 - \|a_2\|}}}.$$

Then, from lemma 1.2, we hvae

$$\|c_{n+2}\| \leqslant \frac{1}{2}\,\frac{n+1}{n+2}.$$

and $q_{n+2} = (e + c_{n+2})q_{n+1}$ is invertible, because it is product of two invertible elements ($e + c_{n+2}$ is invertible since $\|-c_{n+2}\| = \|c_{n+2}\| < 1$ and q_{n+1} is invertible by the hypothesis).

THEOREM 5.2. *If a are elements of a Banach algebra $\mathfrak{A}$ with unit e, wich verify inequalities*

$$\|a_n\| \leqslant \frac{1}{4} \text{ for } n = 2, 3, 4, \ldots,$$

then

$$\|q_n^{-1}\| \leqslant \frac{2^n}{n+1} \quad \text{for } n = 1, 2, 3, \ldots \tag{5.1}$$

Proof. The invertibility of the elements q_n is a consequence of theorem 5.1.

From formula (4.5), we have

$$\|q_n^{-1}\| = \|q_{n-1}^{-1}(e + c_n)^{-1}\| \leqslant \frac{\|q_{n-1}^{-1}\|}{1 - \|c_n\|} \leqslant \frac{\|q_{n-1}^{-1}\|}{1 - \frac{n-1}{2n}},$$

that is

$$\|q_n^{-1}\| \leqslant \frac{2n}{n+1}\|q_{n-1}^{-1}\|.$$

By writing the inequalities for $n-1, n-2, \ldots, 2$ and after their multiplication, we obtain

$$\|q_n^{-1}\| \leqslant \frac{2n}{n+1} \cdot \frac{2(n-1)}{n} \cdots \frac{4}{3},$$

that is

$$\|q_n^{-1}\| \leqslant \frac{2^n}{n+1}.$$

THEOREM 5.3. *If a_n are elements of a Banach algebra $\mathfrak{A}$ with unit e, wich verify the inequalities*

$$\|a_n\| \leqslant \frac{1}{4} \text{ for } n = 2, 3, 4, \ldots,$$

then for non-commutative continued fraction (3.1), *we have*

$$\|q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n\| \leqslant \frac{2\|a_1\|}{(n+1)(n+2)} \text{ for } n = 1, 2, 3, \ldots$$

Proof. The invertibility of q_n follows from thorem 5.1. The relations (4.2) and (2.1) imply the inequality

$$\|q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n\| \leqslant \|q_{n+1}^{-1}\|\ \|c_{n+1}\|\ \|c_n\| \cdots \|c_3\|\ \|a_2\|\ \|a_1\|.$$

Using theorems 5.2 and 5.1, we obtain

$$\|q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n\| \leqslant \frac{2^{n+1}}{n+2}\ \frac{1}{2}\ \frac{n}{n+1} \cdot \frac{1}{2}\ \frac{n-1}{n} \cdots \frac{1}{2}\ \frac{2}{3}\ \frac{1}{4}\|a_1\|$$

or

$$\|q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n\| \leqslant \frac{2\|a_1\|}{(n+1)(n+2)}.$$

6. Generalized Worpitzky's theorem and some consequences

THEOREM 6.1. *If a_n are elements of a Banach algebra $\mathfrak{A}$ with unit e, wich satisfy the inequalities*

$$\|a_n\| \leqslant \frac{1}{4} \text{ for } n = 2, 3, 4, \ldots,$$

then

(i) *the non-commutative continued fraction* (3.1) *converges uniformly to an element $x \in \mathfrak{A}$,*

(ii) *the values of non-commutative continued fraction* (3.1) *and of its convergents are in the set defined by inequality*

$$\|a_1 - x\| \leqslant \frac{1}{2}\|x\|,$$

where a_1 is a fixed element from $\mathfrak{A}$,

(iii) $\frac{1}{4}$ *is the „best'' constant $c > 0$ such that the non-commutative continued fraction* (3.1) *converges for*

$$\|a_n\| \leqslant c \quad (n = 2, 3, \ldots)$$

Proof. (i) In view of the identity

$$q_{n+k}^{-1}p_{n+k} - q_n^{-1}p_n = (q_{n+k}^{-1}p_{n+k} - q_{n+k-1}^{-1}p_{n+k-1}) + $$
$$+ (q_{n+k-1}^{-1}p_{n+k-1} - q_{n+k-2}^{-1}p_{n+k-2}) + \ldots + (q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n),$$

the subadditivity of the norm and theorem 5.3, we can write

$$\|q_{n+k}^{-1}p_{n+k} - q_n^{-1}p_n\| \leqslant 2\left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \ldots + \right.$$
$$\left. + \frac{1}{(n+k)(n+k+1)}\right]\|a_1\|.$$

One also see that we have the inequality

$$\|q_{n+k}^{-1}p_{n+k} - q_n^{-1}p_n\| \leqslant 2\|a_1\|\left(\frac{1}{n+1} - \frac{1}{n+k+1}\right) = \frac{2k\|a_1\|}{(n+1)(n+k+1)} \tag{6.1}$$

Therefore $q_{n+k}^{-1}p_{n+k} - q_n^{-1}p_n \in \mathcal{A}$ tends in norm to 0, for every natural number k, when n tends to ∞, such that $\{q_n^{-1}p_n\}$ is a Cauchy sequence. Banach algebra $\mathcal{A}$ with unit e being complete with respect the metric topology induced by the algebra norm, it follows that sequence $\{q_n^{-1}p_n\}$ converges to an element $x \in \mathcal{A}$.

(ii) First we shall prove that

$$\|x\| \le 2\|a_1\| \quad \text{and} \quad \|q_n^{-1}p_n\| \le 2\|a_1\|.$$

Let us remark that

$$x = q_1^{-1}p_1 + (q_2^{-1}p_2 - q_1^{-1}p_1) + (q_3^{-1}p_3 - q_2^{-1}p_2) + \dots$$

and

$$q_n^{-1}p_n = q_1^{-1}p_1 + (q_2^{-1}p_2 - q_1^{-1}p_1) + (q_3^{-1}p_3 - q_2^{-1}p_2) + \dots + (q_n^{-1}p_n - q_{n-1}^{-1}p_{n-1}).$$

By this remark and theorem 5.3, we obtain

$$\left.\begin{array}{l}\|x\| \\ \|q_n^{-1}p_n\|\end{array}\right\} \le \|a_1\| + \frac{2\|a_1\|}{2 \cdot 3} + \frac{2\|a_1\|}{3 \cdot 4} + \dots,$$

or

$$\left.\begin{array}{l}\|x\| \\ \|q_n^{-1}p_n\|\end{array}\right\} \le \left\{1 + 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots\right]\right\}\|a_1\|.$$

Therefore, we proved

$$\|x\| \le 2\|a_1\| \quad \text{and also} \quad \|q_n^{-1}p_n\| \le 2\|a_1\|.$$

Now in order to prove (ii), we remark that we can write every convergent of the non-commutative continued fraction (3.1) and its value, denoted by x, as follows

(6.2)
$$x = a_1\left(e + \frac{1}{4}y\right)^{-1}$$

with

(6.3)
$$y = \frac{y_1}{e+}\,\frac{a_2}{e+}\dots \quad \text{and} \quad \|y_1\| \le 1.$$

Hence

$$x\left(e + \frac{1}{4}y\right) = a_1$$

or, according to (2.1),

$$\|a_1 - x\| \le \frac{1}{4}\|x\|\,\|y\|.$$

From the first part of the proof, it follows $\|y\| \le 2$ and therefore, we have

(6.4)
$$\|a_1 - x\| \le \frac{1}{2}\|x\|.$$

(iii) In order to prove that $\frac{1}{4}$ is the „best" constant $c > 0$ such that the non-commutative continued fraction (3.1) converges for $\|a_n\| \le c$ ($n = 2, 3, \dots$), it is sufficient to observe that in the Banach subalgebra $\mathcal{A}_1 \subset \mathcal{A}$ of the real numbers multiplied by $e \in \mathcal{A}$, the continued fractions with $a_n = ae$ and $a > 1/4$ diverge according to lemma 1.3.

C o r o l l a r y 6.1. (Approximation theorem) *In the conditions of theorem* 6.1, *we have*

$$\|x - q_n^{-1}p_n\| \le \frac{2\|a_1\|}{n+1}.$$

Proof. From

$$x - q_n^{-1}p_n = (q_{n+1}^{-1}p_{n+1} - q_n^{-1}p_n) + (q_{n+2}^{-1}p_{n+2} - q_{n+1}^{-1}p_{n+1}) + \dots$$

and theorem 5.3, we obtain

$$\|x - q_n^{-1}p_n\| \le 2\|a_1\|\left[\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \dots\right] = \frac{2\|a_1\|}{n+1}.$$

C o r o l l a r y 6.2. *If a_n are some elements of a Banach algebra with unit e, which satisfy the inequalities*

$$\|a_n\| \le M \quad for \ n = 2, 3, \dots$$

and x is variable element of $\mathcal{A}$, then the non-commutative continued fraction.

$$\frac{a_1}{e+}\,\frac{a_2x}{e+}\,\frac{a_3x}{e+}\cdots\frac{a_nx}{e+}$$

converges uniformly for $\|x\| \le \frac{1}{4M}$.

Proof. We are in the conditions of theorem 6.1. and we have

$$\|a_nx\| \le \|a_n\|\,\|x\| \le M \cdot \frac{1}{4M} = \frac{1}{4} \quad \text{for } n = 2, 3, 4, \dots$$

Remark 6.1. Theorem 6.1. (i) is evidently true also in the case in which $a_1, a_2, \dots, a_n$ of $\mathcal{A}$ are fixed, $a_{n+j} \in \mathcal{A}$ with $\|a_{n+j}\| \le \frac{1}{4}$ ($j =$

$= 1, 2, \ldots$). This problem, in a particular case, is solved in proposition 7.2.

Remark 6.2. If in the formulas (3.3) we permute a_{n+1} with p_{n-1} in the first relation and a_{n+1} with q_{n-1} in the second, then we have to permute q_n^{-1} with p_n in (3.2), in order that the methode be still applicable.

7. Particular cases of theorem 6.1

Theorem 6.1 is evidently true also in the case in which the Banach algebra $\mathfrak{A}$ with unit e is commutative.

(i) $\mathfrak{A} = \mathbf{R}$ with $e = 1$ and the norm $\|\cdot\| = |\cdot|$ is a commutative Banach algebra. In this case, according to theorem 6.1, the real continued fraction

$$(7.1) \qquad \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \ddots}}} \qquad (a_n \in \mathbf{R})$$

converges and the convergents and its value is in the interval

$$\left[\frac{2|a_1|}{3},\ 2|a_1|\right] \text{ if } |a_n| \leqslant \frac{1}{4} \qquad (n = 2, 3, \ldots).$$

(ii) $\mathfrak{A} = C$, with $e = 1$ and $\|\cdot\| = |\cdot|$, is a commutative algebra. Then complex continued fraction (7.1) ($a_n \in C$) converges to the value z from the disk

$$D_1 = \left\{z \in C : \left| z - \frac{4}{3}\, a_1 \right| \leqslant \frac{2}{3}\, |a_1|\right\}$$

if $|a_n| \leqslant 1/4$ ($n = 2, 3, \ldots$).

In this case, continued fraction (7.1) can be considered as a complex function f of complex variables $a_2, a_3, \ldots,$ (a_1 is fixed) and

$$(7.2) \qquad z = f(a_1, a_2, a_3, \ldots, a_n, \ldots).$$

Proposition 7.1. *The values of the function f fill the closed disk $\mathfrak{D}_1$ when every $a_n (n = 2, 3, \ldots)$* runs over closed disk $\mathfrak{D}_0 = \left\{z \in C : |z| \leqslant \frac{1}{4}\right\}$.

Proof. Let be

$$(7.2) \qquad z = \frac{a_1}{1+}\, \frac{a_2}{1-}\, \frac{\frac{1}{4}}{1-}\, \frac{\frac{1}{4}}{1-},$$

where $a_1 \in C$ is fixed and $a_2 \in C$ is a complex variable. It is easy to see that

$$z = \frac{a_1}{2a_2 + 1},$$

which is linear and this function transforms the circle $|z| = \frac{1}{4}$ in circle $\left| z - \frac{4}{3}\, a_1 \right| = \frac{2}{3}\, |a_1|$. Therefore, this function carries the disk $\mathfrak{D}_0$ on to the disk $\mathfrak{D}_1$.

Proposition 7.2. *If $\mathfrak{A} = C$, $e = 1$, $\|\cdot\| = |\cdot|$, $a_1, a_2, \ldots, a_n$ are fixed,*

$$|a_{n+j}| \leqslant \frac{1}{4} \quad (j = 1, 2, \ldots)$$

and

$$(7.3) \qquad \left| \frac{q_{n-1}}{q_{n-2}} + \frac{4}{3}\, a_n \right| > \frac{2}{3}\, |a_n|,$$

hen the complex continued fraction (7.1) *converges and the values of function fill a closed disk $\mathfrak{D}_n \subset C$.*

Proof. If.

$$a = \cfrac{a_n}{1 + \cfrac{a_{n+1}}{1 + \cfrac{a_{n+2}}{1 +}}}$$

then, a fill the closed disk

$$\mathfrak{D}_{1n} = \left\{a \in C : \left| a - \frac{4}{3}\, a_n \right| \leqslant \frac{2}{3}\, |a_n|\right\}.$$

Because z from (7.1) is given by linear function

$$z \mapsto \frac{p_{n-1} + z p_{n-2}}{q_{n-1} + z q_{n-2}},$$

it follows, as in proposition 7.1, that the values of function f fill a closed disck $\mathfrak{D}_n \subset C$. Condition (7.3) implies that $\infty \notin \mathfrak{D}_n$ (condition $\infty \notin \mathfrak{D}_1$ is automatic satisfied in proposition 7.1.).

Remark 7.1. *If $\mathfrak{A} = C$, $e = 1$, $\|\cdot\| = |\cdot|$ and $a_1 = a$, then* (i) *of the theorem* 6.1 *gives classical* WORPITZKY's *theorem* [3].

Remark 7.2. If $\mathcal{A} = C$, $e = 1$, $\|\cdot\| = |\cdot|$ and $a_1 = 1$, then the propositions (ii) and (iii) of theorem 6.1 and propositions 7.1 are Wall and PAYDON's precisions [4] of Worpitzky's theorem.

(iii) $\mathcal{A} = C$, with $e = 1$, $\|z\| = |x| + |y|$ and $\|e\| = 1$, is a commutative algebra. Theorem 6.1 shows that, if

$$\|a_n\| \leqslant \frac{1}{4} \quad (n = 2, 3, \ldots)$$

(the complex numbers a_n belong of the square centred at 0 with the vertices at the points $\pm \frac{1}{4}$ and $\pm \frac{i}{4}$), then the value of complex continued fraction (7.1) ($a_n \in C$) belongs to the set defined by the inequation

$$(7.4) \qquad |\alpha_1 - x| + |\beta_1 - y| \leqslant \frac{1}{2} (|x| + |y|),$$

where $z = x + iy$ and $a_1 = \alpha_1 + i\beta_1$.

For instance, if $a_1 = 2 + 3i$, then (7.4) shows that z belongs to convex hull of the points $\frac{1}{3} + 3i$, $2 + \frac{4}{3} i$, $7 + 3i$, $2 + 8i$.

(iv) $\mathcal{A} = C$, with $e = 1$, $\|z\| = \max \{|x|, |y|\}$ and $\|1\| = 1$, is a commutative algebra. Then, if $\|a_n\| \leqslant \frac{1}{4}$ $(n = 2, 3, \ldots)$ (a_n runs a square with the sides parallel to the coordinate axes and centred at 0), the value of continued fraction (7.1) ($a_n \in C$) are in the set defined by the inequation

$$\max \{|\alpha_1 - x|, |\beta_1 - y|\} \leqslant \frac{1}{2} \max \{|x|, |y|\},$$

where $z = x + iy$ and $a_1 = \alpha_1 + i\beta_1$.

REFERENCES

[1] Fair Wyman, *A theorem of convergence for the non-commutative fractions.* Journal of Approximation theory, **5**, 74–76 (1972).

[2] Garnir H. G., et Gobert, I., *Algebre de Banach.* Université de Liège, Séminaire d'Analyse Mathématique et d'Algèbre, pp. 1–17 (1966–1967).

[3] Worpitzky, *Untersuchungen über die Entwickelung der Monodronen und Monogenen Funktion durck Kettenbrüch-Friedrich-Gymnasium und Realschule.* Jahresbericht, Berlin pp. 3–39 (1865).

[4] Paydon, J. F. and Wall, H.S., *The continued fraction as a sequence of linear transformations.* Duke Math Journal, **9**, 360–372 (1942).

Facultatea de matematică și mecanică
Universitatea din Iași
R. S. România

Received, 20. I. 1974.