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ON FOURIER SERIES

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1. Let us introduce the following notations. Let $C_{2\pi}$ denote the Banach space of continuous real-valued functions of period 2π provided with the usual uniform norm and let $L^p_{2\pi}$ $(1 \leqq p < \infty)$ be the Banach space of real-valued functions f of period 2π for which $|f|^p$ is Lebesgue integrable over the segment $[0, 2\pi]$. Let h be a real parameter and let Δ_h and ∇_h be operators on $C_{2\pi}(L^p_{2\pi})$ defined by the formulas

$$(\Delta_h f)(x) = \frac{f(x) - f(x+h)}{2},$$

$$(\nabla_h f)(x) = \frac{f(x) + f(x+h)}{2}.$$

Let, further, R be a natural integer and let

$$\omega_R(f;\ \delta)_{C_{2\pi}\left(L^p_{2\pi}\right)} = \sup_{|h| \leqslant \delta} \|(\Delta_h)^R f\|_{C_{2\pi}\left(L^p_{2\pi}\right)} \quad (\delta \geqq 0)$$

be the R-th $C_{2\pi}(L^p_{2\pi})$ modulus of smoothness of the function f. Finally, let

$$S_n(x,\ f) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

denote the n-th partial sum of the trigonometric Fourier series of the function f.

2. Let $f \in C_{2\pi}$ and R be a fixed natural integer. It is a well known inequality due to H. Lebesgue and D. Jackson that*

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* C( ) is a non-negative, finite constant depending on the parameters lying in the brackets.

$$\|f - S_n(f)\|_{C_{2\pi}} \leqq C(R)\log n\,\omega_R\left(f;\frac{\pi}{n}\right)_{C_{2\pi}} + O\left[\omega_R\left(f;\frac{1}{n}\right)_{C_{2\pi}}\right]$$

$(n=1,2,\ldots)$. In 1941 S. M. NIKOLSKII [8] showed that $C(1)=\frac{4}{\pi^2}$ is the optimal value of $C(1)$. How does $C(R)$ behave for an arbitrary R? This question was open for a long time and even the estimation $C(R) = O(1)$ was not known until 1971 when the author [6] and independently of him V. V. ŽUK [11] solved this problem. Actually, the best value of $C(R)$ can be defined easily by the following way. Denoting by I the identity operator let us remark that

$$\Delta_h + \nabla_h \equiv I$$

and therefore

$$(\Delta_h + \nabla_h)^R \equiv I$$

that is

$$\Delta_h^R + \nabla_h \sum_{k=0}^{R-1}\binom{R}{k}\Delta_h^k\nabla_h^{R-k-1} \equiv I.$$

Let us choose now a trigonometric polynomial H_n of degree at most n such that

$$\|f - H_n\|_{C_{2\pi}} = E_n(f)$$

where $E_n(f)$ denotes the measure of best approximation in $C_{2\pi}$ of f by trigonometric polynomials of degree at most n. Then we obviously have

$$S_n(f) - f = S_n(f - H_n) - (f - H_n) =$$

$$= S_n(\Delta_h^R f) - S_n(\Delta_h^R H_n) +$$

$$+ \nabla_h S_n\left(\sum_{k=0}^{R-1}\binom{R}{k}\Delta_h^k\nabla_h^{R-k-1}(f - H_n)\right) - (f - H_n).$$

But

$$S_n(\Delta_h^R H_n) = \Delta_h^R H_n = \Delta_h^R(H_n - f) + \Delta_h^R f$$

and thus

$$\|S_n(\Delta_h^R H_n)\|_{C_{2\pi}} \leqq E_n(f) + \omega_R(f;\ h)_{C_{2\pi}}$$

Let now $h = \frac{\pi}{n}$. By virtue of some result due to S. BERNSTEIN [1] and W. ROGOSINSKI [9] we have for every $g \in C_{2\pi}$

$$\|\nabla_{\frac{\pi}{n}} S_n(g)\|_{C_{2\pi}} \leqq C\|g\|_{C_{2\pi}} \qquad (n = 1,\ 2,\ \ldots).$$

Therefore

$$\|\nabla_{\frac{\pi}{n}} S_n(\sum_{k=0}^{R-1}\tbinom{R}{k}\Delta_{\frac{\pi}{n}}^k\nabla_{\frac{\pi}{n}}^{R-k-1}(f - H_n))\|_{C_{2\pi}} \leqq C(2^R - 1)E_n(f) \quad (n = 1,\ 2,\ \ldots).$$

Applying now the Jackson theorem

$$E_n(f) = O\left[\omega_R\left(f;\frac{1}{n}\right)_{C_{2\pi}}\right] \qquad (n = 1,\ 2,\ \ldots)$$

we obtain

$$(1)\qquad S_n(f) - f = S_n(\Delta_{\frac{\pi}{n}}^R f) + O\left[\omega_R\left(f;\frac{1}{n}\right)_{C_{2\pi}}\right] \qquad (n = 1,\ 2,\ \ldots).$$

From this we immediately get the following

$$\|S_n(f) - f\|_{C_{2\pi}} \leqq \frac{4}{\pi^2}\log n\,\omega_R\left(f;\frac{\pi}{n}\right)_{C_{2\pi}} + O\left[\omega_R\left(f;\frac{1}{n}\right)_{C_{2\pi}}\right] \quad (n = 1,2,\ldots)$$

and therefore $C(R) \leqq \frac{4}{\pi^2}$. For every n it is easy construct a'saw-tooth' function $f \in C_{2\pi}$ such that

$$\|S_n(f) - f\|_{C_{2\pi}} \geqq \frac{4}{\pi^2}\log n\,\omega_R\left(f;\frac{\pi}{n}\right)_{C_{2\pi}} + O\left[\omega_R\left(f;\frac{1}{n}\right)_{C_{2\pi}}\right]$$

and thus $C(R) = \frac{4}{\pi^2}$.

3. By the way we have got a very important expression (1) for the deviation $S_n(f)$ from the function $f \in C_{2\pi}$. This formula has a lot of applications, some of which give entirely new results for Fourier series. Its first obvious consequence in the following

THEOREM 1. *Let $f \in C_{2\pi}$. Then*

$$\|f - S_n(f)\|_{C_{2\pi}} \underset{n\to\infty}{\longrightarrow} 0$$

if and only if for every natural integer R

$$\omega_R(S_n(f)\ ;\ \delta)_{C_{2\pi}} \underset{\substack{n\to\infty\\ \delta\to 0}}{\longrightarrow} 0.$$

The necessity follows from the inequality

$$\omega_R(S_n(f)\ ;\ \delta)_{C_{2\pi}} \leqq \|S_n(f) - f\|_{C_{2\pi}} + \omega_R(f;\ \delta)_{C_{2\pi}}$$

and the sufficiency from (1).

Let us turn to an another application of (1). G. H. HARDY and J. E. LITTLEWOOD [2] have shown that if $f \in L^p_{2\pi}$ and $\omega_1(f;\ \delta)_{L^p_{2\pi}} = O(n^{-\alpha})$ where $\alpha p > 1$ then $f = f_1$ almost everywhere with $f_1 \in \mathrm{Lip}\,\alpha - \frac{1}{p}$ and therefore

$$|f(x) - S_n(x,\ f)| = O\left(\log n \cdot n^{-\alpha+\frac{1}{p}}\right) \qquad (n = 1,\ 2,\ \ldots)$$

uniformly almost everywhere. This result can be sharpened and generalized with the aid of the relation (1) as follows:

THEOREM 2. *Let $f \in L^p_{2\pi}$ and for some fixed natural integer R let*

$$\omega_R(f;\, \delta)_{L^p_{2\pi}} = O(n^{-\alpha}) \qquad (0 < \alpha \leqq R,\ \alpha p > 1)$$

Then

$$|f(x) - S_n(x, f)| = O\left(n^{-\alpha + \frac{1}{p}}\right) \qquad (n = 1, 2, \ldots)$$

uniformly almost everywhere.

We would like to emphasize that the logarithmic factor is absent in this latter estimation.

Let now $f \in C_{2\pi}$ and $\omega_R(f;\, \delta)_{L^1_{2\pi}} = O(\delta)$ $(\delta \to 0)$ for some fixed R. It is a classical result that in this case $S_n(f)$ converges uniformly towards f for $n \to \infty$. With the aid of (1) one can find the exact value of the speed of convergence of $S_n(f)$ to f. Namely, we have the following.

THEOREM 3. *If $f \in C_{2\pi}$ and $\omega_R(f;\, \delta)_{L^1_{2\pi}} = O(\delta)$ $(\delta \to 0)$ for some fixed natural integer R then*

$$\|f - S_n(f)\|_{C_{2\pi}} = O\left[|\log \omega_R\left(f;\, \frac{1}{n}\right)_{C_{2\pi}}| \cdot \omega_R\left(f;\, \frac{1}{n}\right)_{C_{2\pi}}\right]$$

$(n = 1, 2, \ldots)$ and this estimate cannot be improved.

We shall now define the class of piecewise R monotonic functions. The function $f \in C_{2\pi}$ is said to be piecewise R $(= 1, 2, \ldots$ but fixed) monotonic if there exists a partition $\rho(f): -\pi = x_0 < x_1 < \ldots < x_N = \pi$ of the segment $[-\pi, \pi]$ such that the expression $(\Delta^R_\delta f)(x)$ does not change his sign whenever $\delta \geqq 0$, $(x, x + R\delta) \subset (x_i, x_{i+1})$ $(i = 0, 1, \ldots, N-1)$.

Applying (1) one can prove the following statement.

THEOREM 4. *Let R be a fixed natural integer and let $f \in C_{2\pi}$ be piecewise $R + 1$ monotonic. In this case the logarithmic factor in the Lebesgue-Jackson estimate can be omitted, that is*

$$\|f - S_n(f)\|_{C_{2\pi}} = O_N\left[\omega_R\left(f;\, \frac{1}{n}\right)_{C_{2\pi}}\right] \qquad (n = 1, 2, \ldots).$$

4. Now we turn to the nicest and most important application of the formula (1). We shall generalize the classical Dini-Lipschitz (more precisely Dini-Lipschitz-Lebesgue) convergence test, which says that if $f \in C_{2\pi}$ and

$$|f(x) - f(y)| = 0\left(\frac{1}{|\log |x - y||}\right) \qquad (|x - y| \to 0)$$

uniformly in x and y then the Fourier series of f converges uniformly towards f. Let us say that the function f satisfies the one-sided Dini-Lipschitz condition if

$$(2) \qquad f(x) - f(y) \geqq -\frac{\varepsilon(y - x)}{|\log (y - x)|} \qquad (x \leqq y)$$

where $\varepsilon(\delta) \geqq 0$ and $\varepsilon(\delta) \to 0$ for $\delta \to +0$.

THEOREM 5. *If $f \in C_{2\pi}$ satisfies the one-sided Dini-Lipschitz condition then the Fourier series of f converges uniformly towards f and moreover we have the following estimate:*

$$\|f - S_n(f)\|_{C_{2\pi}} = O\left[\frac{1}{n}\int_{\frac{1}{n}}^{\pi} \frac{\omega_1(f;\, t)_{C_{2\pi}}}{t^2}\, dt + \varepsilon\left(\frac{\pi}{n}\right)\right] \qquad (n = 1, 2, \ldots)$$

Of course, this statement is more general than the Dini-Lipschitz theorem, but it can be easily checked that it is more general than the Jordan-Dirichlet convergence test as well. This fact is interesting by itself since it is well known that the Dini-Lipschitz and the Jordan-Dirichlet convergence theorems are not comparable.

To prove this theorem let us apply (1) with $R = 1$. By virtue of it we have to estimate only $S_n(\Delta_{\frac{\pi}{n}} f)$ since f is continuous. We have by the Dirichlet formula

$$S_n(\Delta_{\frac{\pi}{n}} f) = (\Delta_{\frac{\pi}{n}} f) * D_n$$

where $*$ denotes the convolution and

$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\pi \sin \frac{t}{2}}$$

is the n-th Dirichlet kernel. Hence

$$S_n(\Delta_{\frac{\pi}{n}} f) = \left[\Delta_{\frac{\pi}{n}} f + \frac{\varepsilon\left(\frac{\pi}{n}\right)}{\left|\log \frac{\pi}{n}\right|}\right] * D_n - \frac{\varepsilon\left(\frac{\pi}{n}\right)}{\left|\log \frac{\pi}{n}\right|}$$

and by virtue of (2) we have

$$|S_n(\Delta_{\frac{\pi}{n}} f)| \leqq |(\Delta_{\frac{\pi}{n}} f) * |D_n|| + \frac{\varepsilon\left(\frac{\pi}{n}\right)}{\left|\log \frac{\pi}{n}\right|} [1 + 1 * |D_n|] =$$

$$= |(\Delta_{\frac{\pi}{n}} f) * |D_n|| + O\left[\varepsilon\left(\frac{\pi}{n}\right)\right]$$

Therefore

$$|S_n(x, \Delta_{\frac{\pi}{n}} f)| \leqq \frac{1}{2}\left|(f - f(x)) * (|T_{\frac{\pi}{n}} D_n| - |D_n|)\right| + O\left[\varepsilon\left(\frac{\pi}{n}\right)\right] \leqq$$

$$\leqq \frac{1}{2}|f - f(x)| * |T_{\frac{\pi}{n}} D_n + D_n| + O\left[\varepsilon\left(\frac{\pi}{n}\right)\right]$$

where $T_{\frac{\pi}{n}} D_n(t) = D_n\left(t - \frac{\pi}{n}\right)$. From this we obtain

(3) $$|S_n(x, \Delta_{\frac{\pi}{n}} f)| = O\left[\int_{-\pi}^{\pi} \omega_1(f; t)_{L_{2\pi}} \frac{n}{1 + n^2t^2} dt + \varepsilon\left(\frac{\pi}{n}\right)\right]$$

since it can be easily computed that

$$\left|D_n\left(t - \frac{\pi}{n}\right) + D_n(t)\right| = O\left(\frac{n}{1 + n^2t^2}\right).$$

Theorem 5 follows immediately from (3).

The condition (2) can be generalized by the following way. We can suppose that for some $R(= 2, 3, \ldots)$

$$\Delta_\delta^R f \geqq -\frac{\varepsilon(\delta)}{|\log \delta|} \qquad (\delta \geqq 0)$$

and we can obtain by similar arguments that

(4) $$\|f - S_n(f)\|_{C_{2\pi}} = O\left[\omega_{R-1}\left(f; \frac{1}{n}\right)_{C_{2\pi}} + \varepsilon\left(\frac{\pi}{n}\right)\right] \quad (n = 1, 2, \ldots).$$

The results mentioned in this part have also some applications to some well known results. R. SALEM and A. ZYGMUND [10] have proved the following

THEOREM 6. *Let $f \in C_{2\pi} \cap \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ and let f be of monotonic type, that is $\exists C = C(f)$ such that the function $f(x) + C(x)$ is monotonic on the real axis. In this case we have*

$$\|f - S_n(f)\|_{C_{2\pi}} = O(n^{-\alpha}) \qquad (n = 1, 2, \ldots)$$

This follows immediately from Theorem 5. Unfortunately, for $\alpha = 1$ it is not true. At the same time, however, we can obtain from (4) the following theorem.

THEOREM 7. *Let $f \in C_{2\pi} \cap \mathrm{Lip}\,1$ and let f be of convex type that is $\exists C = C(f)$ such that the function $f(x) + Cx^2$ is convex. Then we have*

$$\|f - S_n(f)\| = O\left(\frac{1}{n}\right) \qquad (n = 1, 2, \ldots)$$

Let now $f \in L^1_{2\pi}$. It is a classical result of G. H. HARDY and J. E. LITTELWOOD [3] that for a fixed point x one can find a function $\in L^1_{2\pi}$ such that

$$|f(x + \delta) - f(x)| = o\left(\frac{1}{|\log|\delta||}\right) \qquad (\delta \to 0)$$

and $S_n(x, f)$ diverges for $n \to \infty$. At the same time, however, S. IZUMI and G. SUNOUCHI [4] have proved that if $f \in L^1_{2\pi}$ and

$$|f(y) - f(z)| = o\left(\frac{1}{|\log|y - z||}\right) \qquad (y, z \to x)$$

then $S_n(x, f) \to f(x)$ for $n \to \infty$.

The Izumi-Sunouchi's theorem also has a one-sided analogue:

THEOREM 8. *Let $f \in L^1_{2\pi}$ and let x be a Lebesgue point of the function f. If*

$$f(y) - f(z) \geqq -\frac{\varepsilon(z-y)}{|\log(z-y)|} \qquad (y < z;\ y, z \to x)$$

where $\varepsilon(\delta) \geqq 0$ and $\varepsilon(\delta) \to 0$ for $\delta \to +0$, then $S_n(x, f) \to f(x)$ for $n \to \infty$.

This theorem can be proved by localization of the main relation (1) and by the arguments used in the proof of Theorem 5.

Finally we remark that some of our results mentioned here were published in our papers [5] — [7].

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