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# QUASI-ANALITIC SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUTIONS WITH DISCONTINUOUS COEFFICIENTS

by

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In the exploitation of oil reservoirs appear hydrodinamic systems of flow which are constituted of two zones, where the flow is single phase in a zone and two phase in another. The mathematical model of such a flow has been studied in the papers [1], [2], and [3]. In this paper we give a method for determining a quasi-analytic solution.

**1.** Consider the equation

$$\frac{\partial p}{\partial t} = \chi(x) \frac{\partial^2 p}{\partial t^2}, \tag{1}$$

in a region $R: 0 < x < 1,\ 0 < t < T$ where

$$\chi(x) = \begin{cases} a^2 & \text{for } 0 < x < l \\ b^2 & \text{for } l < x < 1 \end{cases}$$

and one requires the continuous and positive solution in $R$ subject to the conditions

$$p(x, 0) = f(x), \qquad 0 \leqslant x \leqslant 1, \tag{2}$$

$$\left.\frac{\partial^{k_1} p}{\partial x^{k_1}}\right|_{x=0} = g_1(t), \qquad \left.\frac{\partial^{k_2} p}{\partial x^{k_2}}\right|_{x=1} = g_2(t), \qquad t > 0, \tag{3}$$

where $k_1$ and $k_2$ may be 0 or 1,

$$\text{(4)} \qquad \left.\frac{\partial p}{\partial x}\right|_{x=l+0} = F[t,\ p(l,\ t)]\left.\frac{\partial p}{\partial x}\right|_{x=l-0},$$

$f$, $g_1$, $g_2$ and $F$ being continuous functions such that $F(t,\ p) \geqslant 0$ for $t > 0$ and $p \geqslant 0$.

In order to find an approximate solution of the above problem, we propose a method based on on the discretization of the equation (1) and the approximation of the function $F(t,\ p)$ by a step function. We shall begin the investigation of this problem with a particular case.

**2.** We consider the condition (4) of the form

$$\text{(5)} \qquad \left.\frac{\partial p}{\partial x}\right|_{x=l+0} = F(t)\left.\frac{\partial p}{\partial x}\right|_{x=l-0},$$

and we suppose that the problems (1), (2), (3) and (5) have sufficiently smooth solution in the regions $D_1 : 0 < x < l,\ \ 0 < t < T$ and $D_2 : l < x < 1$, $0 < t < T$. In order to determine an approximate quasi-analytic solution of this problem, we consider the lines $x = x_i$, $i = 0,\ 1,\ \ldots,\ n + m$, where $x_i = ih$, $h = \frac{1}{n}$, $i = 0,\ 1,\ \ldots,\ n$ and $x_i = x_n + ik$ for $i = n + 1$, $n + 2,\ \ldots,\ n + m$ where $k = \frac{1 - l}{m}$ and $m = \left[\frac{1 - l}{h}\right]$.

The solution of the above problem on the lines $x = x_i$ will be noted by $p_i(t)$, that is $p_i(t) = p(x_i,\ t)$.

For the discretization of the equation (1) when one takes $k_1 = k_2 = 0$, in the conditions (3) we approximate the derivative $\frac{\partial^2 p}{\partial x^2}$ on the lines $x = x_i$, $i = 1,\ 2,\ \ldots,\ n - 1,\ n + 1,\ \ldots,\ n + m - 1$, by the divided difference, that is

$$\left.\frac{\partial^2 p}{\partial x^2}\right|_{x=x_i} = \frac{p_{i-1}(t) - 2p_i(t) + p_{i+1}(t)}{h_i^2} - \frac{h_i^2}{11}\left.\frac{\partial^4 p}{\partial x^4}\right|_{x=\xi_i},\quad x_{i-1} < \xi_i < x_{i+1},$$

where

$$h_i = \begin{cases} h & \text{for } i = 1,\ 2,\ \ldots,\ n - 1 \\ k & \text{for } i = n + 1,\ n + 2,\ \ldots,\ n + m - 1 \end{cases}$$

In the case $k_1 = 1$, on the line $x = x_1$ one uses the formula [4]

$$\text{(6)} \qquad \left.\frac{\partial^2 p}{\partial x^2}\right|_{x=x_1} = \frac{2}{11\,h^2}\left[-3\,h\left.\frac{\partial p}{\partial x}\right|_{x=0} - 2p_1(t) + p_2(t) + p_3(t)\right] - $$

$$- \frac{29}{132}\,h^2\left.\frac{\partial^4 p}{\partial x^4}\right|_{x=\xi_1},\quad 0 < \xi_1 < x_3,$$



and for $k_2 = 1$ at $x = x_{n+m-1}$ the analogue formula

$$\text{(7)} \qquad \left.\frac{\partial^2 p}{\partial x^2}\right|_{x=x_{n+m-1}} = \frac{2}{11\,k^2}\left[3k\left.\frac{\partial p}{\partial x}\right|_{x=1} - 2p_{n+m-1}(t) + p_{n+m-2}(t) + p_{n+m-3}(t)\right] - $$

$$- \frac{29}{132}\,k^2\left.\frac{\partial^4 p}{\partial x^4}\right|_{x=\xi_{n+m-1}},\quad x_{n+m-3} < \xi_{n+m-1} < 1.$$

In this way we derive from the equation (1) the following system of ordinary differential equations.

$$p_1' = \left(1 - \frac{9}{11}k_1\right)\frac{a^2}{h^2}(-2p_1 + p_2 + k_1 p_3) + \frac{a^2}{h^2}\left[1 - \left(1 - \frac{6h}{11}\right)k_1\right]g_1(t) + R_1$$

$$p_2' = \frac{a^2}{h^2}(p_1 - 2p_2 + p_3) + R_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_{n-1}' = \frac{a^2}{h^2}(p_{n-2} - 2p_{n-1} + p_n) + \bar{R}_{n-1}$$

$$\text{(8)} \qquad p_{n+1}' = \frac{b'^2}{h^2}(p_n - 2p_{n+1} + p_{n+2}) + \bar{R}_{n+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_{n+m-2}' = \frac{b'^2}{h^2}(p_{n+m-3} - 2p_{n+m-2} + p_{n+m-1}) + R_{n+m-2}$$

$$p_{n+m-1}' = \left(1 - \frac{9}{11}k_2\right)\frac{b'^2}{h^2}(-2p_{n+m-1} + p_{n+m-2} + k_2 p_{n+m-3}) + $$

$$+ \left[1 - \left(1 - \frac{6h}{11}\right)k_2\right]\frac{b}{h^2}g_2(t) + R_{n+m-1}$$

where $b' = \frac{h}{k} b$. The initial condition leads us to the conditions

$$\text{9)} \qquad p_i(0) = f(x_i),\qquad i = 1,\ 2,\ \ldots,\ n + n - 1$$

In order to eliminate the unknown function $p_n(t)$ from the system (8) we use the condition (5). For this purpose we express the derivative $\frac{\partial^2 p}{\partial x^2}$ on the lines $x = x_{n-1}$ and $x = x_{n+1}$ by a numerical differentiation formula which contains some functions $p_i(t)$ and the derivatives

$$\left.\frac{\partial p}{\partial x}\right|_{x=l-0},\quad \left.\frac{\partial p}{\partial x}\right|_{x=l+0}$$

**3.** In the following we shall determine such a formula.

Let $f(x)$ be a function belonging to the class $C^4_{[x_0, x_3]}$. Subdivide the interval $(x_0, x_3)$ into equal parts of length $h$ by means of the points $x_1, x_2, x_3$. For the function $f(x)$ we determine a formula of the following form:

$$(10)\qquad f''(x_1) = Af'(x_0) + B_0f(x_0) + B_1f(x_1) + B_2f(x_2) + B_3f(x_3) + R,$$

using the method from the paper [6]. Consider the function $\varphi$ defined on $[x_0, x_3]$ by the equations $\varphi(x) = \varphi_i(x)$ for $x \in [x_{i-1}, x_i]$, $i = 1, 2, 3$, where the functions $\varphi(x)$ are the solution of certain two point boundary value problems of differential equations

$$(11)\qquad \varphi_i^{IV}(x) = 0, \qquad i = 1, 2, 3.$$

From the formula

$$(12)\qquad \int_{x_0}^{x_3}\varphi^{IV}(x)f(x)\,dx = \sum_{i=1}^{3}\left\{[\varphi_i'''f - \varphi_i''f^{I} + \varphi_i'f'' - \varphi_i f''']_{x_{i-1}}^{x_i} + \int_{x_{i-1}}^{x_i}\varphi_i(x)f^{IV}(x)dx\right\}$$

imposing on the functions $\varphi_i(x)$ the following conditions

$$(13)\qquad \begin{aligned} &\varphi_1'(x_0) = \varphi_1'(x_0) = 0\\ &\varphi_1(x_1) = \varphi_2(x_1),\ \varphi_1''(x_1) = \varphi_2''(x_1)\\ &\varphi_2(x_2) = \varphi_3(x_2), \quad \varphi_2'(x_2) = \varphi_3'(x_2), \quad \varphi_2''(x_2) = \varphi_3''(x_2)\\ &\varphi_3(x_3) = \varphi_3'(x_3) = \varphi_3''(x_3) = 0\end{aligned}$$

and considering $\varphi^{IV}(x) = 0$, we obtain the following formula

$$(14)\qquad \begin{aligned}&\varphi_1'''(x_0)f'(x_0) + [\varphi_1'''(x_1) - \varphi_2'''(x_1)]f(x_1) +\\ &+ [\varphi_1'(x_1) - \varphi_2'(x_1)]f''(x_1) + [\varphi_2'''(x_2) - \varphi_3'''(x_2)]f(x_2) +\\ &+ \varphi_3'''(x_3)f(x_3) - \varphi_1'''(x_0)f(x_0) = -\int_{x_0}^{x_3}\varphi(x)f^{IV}(x)dx.\end{aligned}$$

For the full determination of the formula (14) it is necessary to solve the problems (11), (13). For this purpose we observe that the polynomials

$$(15)\qquad \begin{aligned}\varphi_3(x) &= \frac{(x_3 - x)^3}{3!}\\ \varphi_2(x) &= \frac{(x_3 - x)^3}{3!} + \lambda_1\frac{(x_2 - x)^3}{3!}\\ \varphi_1(x) &= \frac{(x_3 - x)^3}{3!} + \lambda_1\frac{(x_2 - x)^3}{3!} + \lambda_2\frac{(x_1 - x)^3}{3!} + \lambda_3(x_1 - x).\end{aligned}$$



satisfy to the equations (11) and to the conditions (13) attached at the points $x_1, x_2, x_3$. From the fact that the function $\varphi_1$ satisfies the conditions (13) at the point $x_0$, we obtain for the determination of the parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$ the system:

$$27h^2 + 8\lambda_1h^2 + 6\lambda_3 = 0$$

$$9h^2 + 4\lambda_1h^2 + \lambda_2h^2 + 2\lambda_3 = 0$$

which has the solutions

$$(16)\qquad \begin{aligned}\lambda_1 &= -\left(\frac{9}{2} + \lambda\right)\\ \lambda_2 &= 9 + 2\lambda\\ \lambda_3 &= \lambda h^2\end{aligned}$$

Substituting (16) into (15) we obtain

$$(17)\qquad \begin{aligned}\varphi_1(x) &= \frac{(x_3 - x)^3}{3!} - \left(\frac{9}{2} + \lambda\right)\frac{(x_2 - x)^3}{3!} + (9 + 2\lambda)\frac{(x_1 - x)^3}{3!} + \lambda h^2(x - x_1)\\ \varphi_2(x) &= \frac{(x_3 - x)^3}{3!} - \left(\frac{9}{2} + \lambda\right)\frac{(x_2 - x)^3}{3!},\end{aligned}$$

and the formula (14) becomes

$$(18)\qquad f''(x_1) = \frac{1}{\lambda h^2}\left[3hf'(x_0) + \left(\frac{11}{2} + \lambda\right)f(x_0) - (9 + 2\lambda)f(x_1) + \right.$$
$$\left. + \frac{1}{2}(9 + 2\lambda)f(x_2) - f(x_3)\right] + R,$$

where

$$(19)\qquad R = \frac{1}{\lambda h^2}\int_{x_0}^{x_3}\varphi(x)f^{IV}(x)dx.$$

The choice of the parameter $\lambda$ will be made by the reasons of error estimation; therefore, the discussion of this choice is postponed to the next section, where we find $\lambda$ as a root of the equation $1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda} = 0$.

For both roots of this equation, the function $\varphi(x) > 0$ for $x \in (x_0, x_3)$ and consequently, the formula (19) can be written as

$$R = \frac{1}{\lambda h^2} f^{IV}(\xi) \int_{x_0}^{x_3} \varphi(x)dx, \qquad \xi \in [x_0, x_3]$$

Putting in (18) $f(x) = \frac{1}{4!}(x - x_0)^2(x - x_1)(x - x_2)$

we obtain

$$\frac{1}{\lambda h^2} \int_{x_3}^{x_0} \varphi(x)dx = 2\left(\frac{9}{\lambda} - 1\right) h^2,$$

and

$$R = 2\left(\frac{9}{\lambda} - 1\right) h^2 f^{IV}(\xi),$$

and also

$$(19') \qquad |R| \leqslant 2\left(\frac{9}{\lambda} - 1\right) h^2 M_4,$$

where

$$M_4 \geqslant \max_{[x_0,\, x_3]} |f^{IV}(x)|.$$

The smaller coefficient $\frac{9}{\lambda} - 1$ in (19′) is obtained for the root $\lambda = -\frac{9 + \sqrt{103}}{2}$ namely

$$|R| < 4M_4h^2.$$

4. Now we shall use the numerical differentiation formula, just determined, expressing the derivatives $\left.\frac{\partial^2 p}{\partial x^2}\right|_{x=x_{n-1}}$ and $\left.\frac{\partial^2 p}{\partial x^2}\right|_{x=x_{n+2}}$ in (1), and obtaining in this way the following equations

$$(20) \qquad p'_{n-1} = \frac{a^2}{\lambda h^2}\left[-3h\left.\frac{\partial p}{\partial x}\right|_{x=l-0} + \left(\frac{11}{2} + \lambda\right)p_n + (9 + 2\lambda)p_{n-1} + \right.$$

$$\left. + \frac{1}{2}(9 + 2\lambda)p_{n-2} - p_{n-3}\right] + R^*_{-1},$$



$$p'_{n+1} = \frac{b'^2}{\lambda h^2}\left[3h\left.\frac{\partial p}{\partial x}\right|_{x=l-0} + \left(\frac{11}{2} + \lambda\right)p_n + (9 + 2\lambda)p_{n+1} + \frac{1}{2}(9 + 2\lambda)p - \right.$$

$$\left. - p_{n+3}\right] + R^*_{n+1}$$

Combining these equations with those of the system (8) which contain the function $p_n(t)$ and considering the condition (5), we get

$$(21) \qquad p'_{n-1} = \frac{a^2}{(1 + F)h^2}\left\{\frac{2F}{11}p_{n-3} + \left[1 - \left(\frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)F\right]p_{n-2} - \right.$$

$$- 2\left[1 - \left(\frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)F\right]p_{n-1} + 2\left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)p_{n+1} -$$

$$\left. - \left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)p_{n+2} + \frac{2}{11}p_{n+3}\right\} + \frac{11}{2\lambda}\frac{a^2}{1 + F}\overline{R}_{n+1}$$

$$- \frac{b'^2}{b^{12}(1 + F)}R^*_{n+1} - \frac{b'^2}{1 + F}R^*_{n-1} + \frac{1}{1 + F}\left[\left(\frac{11}{2\lambda} + 1\right)F + 1\right]\overline{R}_{n-I}$$

$$p'_{n+1} = \frac{b'^2}{(1 + F)h^2}\left[\frac{2F}{11}p_{n-3} - \left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)Fp_{n-2} + \right.$$

$$+ 2\left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)Fp_{n-1} - 2\left(F + \frac{1}{\lambda} - \frac{2\lambda}{11} - \frac{7}{11}\right)p_{n+1}$$

$$\left. + \left(F + \frac{1}{\lambda} - \frac{2\lambda}{11} - \frac{7}{11}\right)p_{n+2} + \frac{2}{11}p_{n+3}\right] + \frac{b'^2}{a^2}\frac{11F}{2\lambda(1 + F)}\overline{R}_{n+1} -$$

$$- \frac{b'^2}{a^2}\frac{F}{1 + F}R^*_{n-1} + \frac{1}{1 + F}\left(\frac{11}{2\lambda} + 1 + F\right)\overline{R}_{n-1} - \frac{1}{1 + F}R^*_{n+1}$$

If we add the equation (21) to the system (8), leaving out those which contain the function $p_n(t)$, we obtain the following system of ordinary differential equationas concerning the unkonwn functions $p_1(t)$, $p_2(t)$, ..., $p_{n-1}(t)$, $p_{n+1}(t)$, ..., $p_{n+m-1}(t)$:

$$p'_1 = \frac{a^2}{h^2}\left(1 - \frac{9}{11}k_1\right)(-2p_1 + p_2 + k_1p_3) + \frac{a^2}{h^2}\left[1 - \left(1 - \frac{6h}{11}\right)k_1g_1(t)\right] + R_1$$

$$p'_1 = \frac{a^2}{h^2}(p_1 - 2p_2 + p_3) + R_2$$

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$$(22)\quad p'_{n-1} = \frac{a^2}{(1+F)h^2}\left[\frac{2F}{11}\,p_{n-3} + \left(1 - \frac{2F}{11} - \frac{2\lambda}{11}F + \frac{F}{\lambda}\right)p_{n-2} - \right.$$

$$- 2\left(1 - \frac{7F}{11} - \frac{2\lambda F}{11} + \frac{F}{\lambda}\right)p_{n-1} + 2\left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)p_{n+1} -$$

$$\left. - \left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)p_{n+2} + \frac{2}{11}\,p_{n+3}\right] + R_{n-1}$$

$$p'_{n+1} = \frac{b'^2}{(1+F)h^2}\left[\frac{2F}{11}\,p_{n-3} - \left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)Fp_{n-2} + \right.$$

$$+ 2\left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)Fp_{n-1} - 2\left(F + \frac{1}{\lambda} - \frac{2\lambda}{11} - \frac{7}{11}\right)p_{n+1} +$$

$$\left. + \left(F + \frac{1}{\lambda} - \frac{2\lambda}{11} - \frac{7}{11}\right)p_{n+2} + \frac{2}{11}\,p_{n+3}\right] + R_{n+1}$$

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$$p'_{n+m-1} = \left(1 - \frac{9}{11}\,k_2\right)\frac{b'^2}{h^2}\,(-2p_{n+m-1} + p_{n+m-2} + k_2 p_{n+m-3}) +$$

$$+ \frac{b'^2}{h^2}\left[1 - \left(1 - \frac{6k}{11}\right)k_2\right]g_2(t) + R_{n+m-1}.$$

The solution of this system which satisfices the initial conditions

$$(23)\quad \begin{array}{ll} p_i(0) = f(ih), & i = 1, 2, \ldots, n-1 \\ p_j(0) = f(jk), & j = n+1, \ldots, n+m-1. \end{array}$$

represents a system of values of the solution of the problem (1), (2), (3), (5) on the lines $x = x_i$, $i = 1, 2, \ldots, n-1, n+1, \ldots, n+m-1$. Now we drop in the system (22) the remainders $R_i$, and denote by $\bar{p}_i(t)$ its solution subject to the condition (23) and by $\gamma_i(t) = p_i(t) - \bar{p}_i(t)$ $i = 1, 2, \ldots, n-1, n+1, \ldots, n-1+m$ the discretization errors.

5. For the estimation of the discretization errors we remark that $\gamma_i(t)$, $i = 1, 2, \ldots, n-1$, $n+1, \ldots, n+m-1$ represent the solution of the system:

$$\gamma'_1 = \frac{a^2}{h^2}\left(1 - \frac{9}{11}\,k_1\right)(-2\gamma_1 + \gamma_2 + k_1\gamma_3) + R_1$$

$$\gamma'_2 = \frac{a^2}{h^2}\,(\gamma_1 - 2\gamma_2 + \gamma_3) + R_2$$

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$$\gamma'_{n-1} = \frac{a^2}{(1+F)h^2}\left[\frac{2F}{11}\,\gamma_{n-3} + \left(1 - \frac{7F}{11} - \frac{2\lambda}{11}F + \frac{F}{\lambda}\right)\gamma_{n-2} - \right.$$

$$- 2\left(1 - \frac{7F}{11} - \frac{2\lambda}{11}F + \frac{F}{\lambda}\right)\gamma_{n-1} + 2\left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)\gamma_{n+1} -$$

$$\left. - \left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)\gamma_{n+2} + \frac{2}{11}\,\gamma_{n+3}\right] + R_{n-1}$$

$$(24)\quad \gamma'_{n+1} = \frac{b'^2}{(1+F)h^2}\left[\frac{2F}{11}\,\gamma_{n-3} - \left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)F\gamma_{n-2} + \right.$$

$$+ 2\left(1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right)F\gamma_{n-1} - 2\left(F + \frac{1}{\lambda} - \frac{2\lambda}{11} - \frac{7}{11}\right)\gamma_{n+1}$$

$$\left. + \left(F + \frac{1}{\lambda} - \frac{2\lambda}{11} - \frac{7}{11}\right)\gamma_{n+2} + \frac{2}{11}\,\gamma_{n+3}\right] + R_{n+1}$$

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$$\gamma'_{n+m-1} = \left(1 - \frac{9}{11}\,k_2\right)\frac{b'^2}{h^2}\,(-2\gamma_{n+m-1} + \gamma_{n+m-2} + k_2\gamma_{n+m-3}) + R_{n+m-1}$$

which satisfies the initial conditions

$$\gamma_i(0) = 0, \quad i = 1, 2, \ldots, n-1, n+1, \ldots, n+m-1$$

Considering the form of the remainders of our numerical differentiation formulas we remark that the remainders $R_i$ in (24) are of order $h^2$.

For evaluating the solution of the system (24) subject to (25), we use the following lemma [7].

L e m m a 1. *Let $x = x(t)$ be the n-dimensional vector solution of the Cauchy problem*

$$\frac{dx}{dt} = A(t)x + R(t)$$

$$x(t_0) = x_0$$

*where the elements of the matrix $A(t) = (a_{ik}(t))$, $1 \leqslant i, k \leqslant n$ and $R(t) = (R_i(t))$, $1 \leqslant i \leqslant n$ are continuous functions on $[t_0, T]$. Then*

$$(26)\quad \|x(t)\| \leqslant \|x_0\|\, e^{\int_{t_0}^{t} a(\tau)\,d\tau} + \int_{t_0}^{t} \|R(\tau)\|\, e^{\int_{\tau}^{t} a(\tau)\,d\tau}\, d\tau$$

where

$$||x(t)|| = \max_{1\leqslant i\leqslant n} |x_i(t)|, \quad ||R(t)|| = \max_{1\leqslant i\leqslant n} |R_i(t)|,$$

$$a(t) = \max_{1\leqslant i\leqslant n} \left[a_{ii}(t) + \sum_{\substack{k=1\\k\neq i}}^{n} |a_{ik}(t)|\right].$$

The coefficients of the system (24) satisfy the conditions

$$(27) \qquad a_{ii} + \sum_{\substack{s=1\\s\neq n,\, s\neq i}}^{n+m-1} |a_{is}| \leqslant 0, \qquad i = 1, 2, \ldots, n-2, n+2, \ldots, n+m-1$$

In order to satisfy the condition (27) also when $i = n-1$ and $i = n+1$ we choose $\lambda$ as a root of the equation

$$1 + \frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda} = 0.$$

In this case one obtains $1 - \left(\frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right) = 1 + F$ and $F = \left(\frac{7}{11} + \frac{2\lambda}{11} - \frac{1}{\lambda}\right) = 1 + F$, and the equations of the system (24) containing $\lambda$ become

$$\gamma'_{n-1} = \frac{a^2}{h^2}\left[\frac{2}{11}\frac{F}{1+F}\gamma_{n-3} + \gamma_{n-2} - \gamma_{n-1} + \frac{2}{11}\frac{1}{1+F}\gamma_{n+3}\right] + R_{n-1},$$

$$\gamma_{n+1} = \frac{b'^2}{h^2}\left[\frac{2}{11}\frac{F}{1+F}\gamma_{n-3} - 2\gamma_{n+1} + \gamma_{n+2} + \frac{2}{11}\frac{1}{1+F}\gamma_{n+3}\right] + R_{n+1},$$

Hence, their coefficients satisfy the conditions

$$(28) \qquad a_{n-1,\,n-1} + \sum_{\substack{s=1\\s\neq n-1,\, n}}^{n+m-1} |a_{n-1,\,s}| = -\frac{9a^2}{11\,h^2},$$

and

$$(29) \qquad a_{n+1,\,n+1} = \sum_{\substack{s=1\\s\neq n,\, n+1}}^{n+m-1} |a_{n+1,\,s}| = -\frac{9b'^2}{11\,h^2},$$



Considering (27, (28), (29) we obtain

$$(30) \qquad a(t) = \max_i \left\{a_{ii} + \sum_{\substack{s=1\\s\neq i,\, n}}^{n+m-1} a_{is}\right\} = 0,$$

and according to lemma 1 we have

$$(31) \qquad ||\gamma_i(t)|| = \int_0^t \max_i |R_i(\tau)|\, d\tau = h^2 Mt,$$

where $M$ is a constant with respect to $h$. Therefore the discretization errores $\gamma_i(t)$ tend to zero as $h$ tends to zero. From the valuation (31) it results that the solution of the problems (1), (2), (3), (5) on the lines $x = x_i$, $1 = 1, 2, \ldots, n-1, n+1, \ldots, n+m-1$ is approximated with the discretization error of order $h^2$ by the solution of the system

$$\bar{p}'_1 = \frac{a^2}{h^2}\left(1 - \frac{9}{11}k_1\right)(-2\bar{p}_1 + \bar{p}_2 + k_1\bar{p}_3)$$

$$\bar{p}'_2 = \frac{a^2}{h^2}(\bar{p}_1 - 2\bar{p} + \bar{p}_3)$$

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$$(32) \qquad \bar{p}'_{n-1} = \frac{a^2}{h^2}\left(\frac{2}{11}\frac{F}{1+F}\bar{p}_{n-3} + \bar{p}_{n-2} - 2\bar{p}_{n-1} + \frac{2}{11}\frac{1}{1+F}\bar{p}_{n+3}\right)$$

$$\bar{p}'_{n+1} = \frac{b'^2}{h^2}\left(\frac{2}{11}\frac{F}{1+F}\bar{p}_{n-3} - 2\bar{p}_{n+1} + \bar{p}_{n+2} + \frac{2}{11}\frac{1}{1+F}\bar{p}_{n+3}\right)$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$\bar{p}'_{n+m-1} = \frac{b'^2}{h^2}\left(1 - \frac{9}{11}k_2\right)(-2\bar{p}_{n+m-1} + \bar{p}_{n+m-2} + k_2\bar{p}_{n+m-3})$$

which satsifies to the initial contitions

$$(33) \qquad \begin{aligned} p_i(0) &= f(ih), & i &= 1, 2, \ldots, n-1 \\ p_j(0) &= f(x_n + jk), & j &= 1, 2, \ldots, m-1 \end{aligned}$$

*Remark* I. In order to simplify, we shall use the following matrix notation

Hence, the system (32) and the conditions (33) can be written as

$$(34)\qquad \begin{cases} \overline{P}'(t) = A\overline{P}(t) + H(t) \\ \overline{P}(0) = f_0 \end{cases}$$

In order to obtain an approximation for the function $p_n(t)$ we write the condition (5) as

$$(35)\qquad \frac{1}{h}\,[p_{n+1}(t) - p_n(t)] = \frac{F(t)}{h}\,[p_n(t) - p_{n-1}(t)] + \frac{h}{2}\,\frac{\partial^2 p}{\partial x^2}\bigg|_{nc=\eta_2} - F(t)\,\frac{h}{2}\,\frac{\partial p^2}{\partial x^2}\bigg|_{x=\eta_1},$$

where $\eta_1 \in (x_{n-1}, x_n)$, $\eta_2 \in (x_n, x_{n-1})$.

Denoting $\frac{h}{k} = r$, we have

$$(36)\qquad p_n(t)\,[1 + rF(t)] = p_{n+1}(t) + rF(t)p_{n-1}(t) - R_n(t),$$

where

$$(36')\qquad R_n(t) = \frac{h^2}{2}\left(\frac{\partial^2 p}{\partial x^2}\bigg|_{x=\eta_1} F(t) - \frac{1}{r}\,\frac{\partial^2 p}{\partial x^2}\bigg|_{x=\eta_2}\right).$$

Dropping the term $R_n(t)$ in (36) and using the known approximative values for $p_{n-1}(t)$ and $p_{n+1}(t)$ we obtain the following equation

$$(37)\qquad \overline{p}_n(t) = \frac{1}{1 + r\,F(t)}\,[\overline{p}_{n+1}(t) + rF(t)\,\overline{p}_{n-1}(t)],$$

whose solution represents an approximation for $p_n(t)$. For the purpose of evaluating the error of this approximation we subtract (37) from (36) and we denote

$$p_n(t) - \overline{p}_n(t) = \gamma_n(t).$$

In this way we obtain the equation

$$\gamma_n(t) = \frac{1}{1 + r\,F(t)}\,[\gamma_{n+1}(t) + rF(t)\,\gamma_{n-1}(t)] + \frac{R_n(t)}{1 + r\,F(t)}.$$

Considering that $|\gamma_{n+1}(t)|$ and $|\gamma_{n-1}(t)| \leqslant Mh^2t$ and that $F(t) \geqslant 0$ for $0 \leqslant t \leqslant T$ we have

$$|\gamma_n(t)| \leqslant (Mt + N)h^2,$$



where $M$ and $N$ are constants with respect to $h$, and therefore $\gamma_n(t)$ tends to zero as $h$ tends to zero.

*Remark* 2. If, from the boundary conditions one obtains the derivative $\frac{\partial p}{\partial t}$ at one of the ends of [0, 1], then, using the formulas

$$\overline{P}(t) = \left\| \begin{matrix} \overline{p}_1(t) \\ \overline{p}_2(t) \\ \vdots \\ \overline{p}_{n-1}(t) \\ \overline{p}_{n+1}(t) \\ \vdots \\ \overline{p}_{n+m-1}(t) \end{matrix} \right\|; \quad H(t) = \left\| \begin{matrix} \frac{a^2}{h^2}\left[1 - \left(1 - \frac{6h}{11}\right)k_1\right] g_1(t) \\ 0 \\ \vdots \\ 0 \\ \vdots \\ \frac{b'^2}{h^2}\left[1 - \left(1 - \frac{6h}{11}\right)k_2\right] g_2(t) \end{matrix} \right\| \quad R(t) = \left\| \begin{matrix} R_1(t) \\ R_2(t) \\ \vdots \\ R_{n-1}(t) \\ R_{n+1}(t) \\ \vdots \\ R_{n+m-1}(t) \end{matrix} \right\| \quad f_0 = \{f(h), f(2h), \ldots, f[(n-1)h], f(x_n + k), \ldots, f[x_n + (m-1)k]\}$$

$$A_F = \left\| \begin{array}{ccccccccccccc}
2a^2\left(1 - \frac{9}{11}k_1\right) & a^2\left(1 - \frac{9}{11}k_1\right) & k_1a^2\left(1 - \frac{9}{11}k_1\right) & 0\ 0\ \ldots & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
a^2 & -2a^2 & a^2 & 0\ 0\ \ldots & 0 & 0 & 0 & 0 & 0 & 0\ldots0 & 0 & & 0 \\
0 & a^2 & -2a^2 & a^2\ 0\ \ldots & 0 & 0 & 0 & 0 & 0 & 0\ldots0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0\ 0\ \ldots & 0 & a^2 & a^2 & 0 & 0 & 0\ldots0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0\ 0\ \ldots & \frac{2a^2F}{11(1+F)} & a^2 - 2a^2 & 0 & 0 & \frac{2a^2}{11(1+F)} & 0\ldots0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0\ 0\ \ldots & \frac{2b'^2F}{11(1+F)} & 0 & -2b'^2 & a^2 & \frac{2b'^2}{11(1+F)} & 0\ldots0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0\ 0\ \ldots & 0 & 0 & b'^2 & -2b'^2 & b'^2 & 0\ldots0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0\ 0\ \ldots & 0 & 0 & 0 & 0 & 0 & 0\ldots0 & b'^2 & -2b'^1 & b'^2 \\
0 & 0 & 0 & 0\ 0\ \ldots & 0 & 0 & 0 & 0 & 0 & 0\ldots0 & -2b'^2\left(1 - \frac{9}{11}k_2\right) & b'^2\left(1 - \frac{9}{11}\right)k_2 & k_2b'^2\left(1 - \frac{9}{11}k_2\right)
\end{array} \right\|$$

$$(41)\qquad (P^*)' = A_FP^* + H,$$

with the condition

$$(42)\qquad P^*(0) = f_0,$$

where $M$ and $N$ are constants with respect to $h$, and therefore $\gamma_n(t)$ tends to zero as $h$ tends to zero.

*Remark* 2. If, from the boundary conditions one obtains the derivative $\frac{\partial p}{\partial t}$ at one of the ends of $[0, 1]$, then, using the formulas

$$\bar{p}_0(t) = \bar{p}_1(t) - h\left.\frac{\partial p}{\partial x}\right|_{x=0}, \quad \bar{p}_{n+m}(t) = \bar{p}_{n+m-1}(t) + h\left.\frac{\partial p}{\partial x}\right|_{x=n+m}.$$

one obtain the approximative values with the errors of order $h^2$ for the functions $p_0(t)$ and $p_{n+m}(t)$.

6. We now give a procedure for finding an approximative solution for the problem (1)—(4). First of all, we discretizate the equation (1) and the condition (4) as above, obtaining such a system of ordinary differential equations with initial conditions which has the matrix form

$$P'(t) = A_{F[t,p_n(t)]}P(t) + H(t), \tag{38}$$

with initial condition

$$P(0) = f_0. \tag{39}$$

To determine $p_n(t)$ we shall consider the equation

$$p_{n+1}(t) - p_n(t) = rF[t, p_n(t)][p_n(t) - p_{n-1}(t)]. \tag{40}$$

If $[F[t, p_n(t)]$ sold be known, then the solution of the problem (38), (39) would represent the solution of the problem (1)—(4) on the lines $x = x_i$ with a discretization error of order $h^2$. But since $p_n(t)$ is unknown, we shall substitute the function $F(t, p(t)]$ by a step function. To this purpose we subdivide the interval $[0, T]$ into equal parts of length $\Delta t$ by means of the points $t_0 = 0, t_1, t_2, \ldots, t_N = T$ and denote

$$F_{s-1}(t) = F[t, p_n(t_{y-1})], \quad s = 1, 2, \ldots, N.$$

We replace in (38) $F[t, p_n(t)]$ by $F_{s-1}(t)$ for $t \in [t_{s-1}, t_s]$. The approximative solution of the system (38) is performed step by step beginning with the interval $[0, t_1]$, since $p(0)$ is done from the initial condition. In fact on the interval $[0, t_1]$ one solves the system

$$(P^*)' = A_F P^* + H, \tag{41}$$

with the condition

$$P^*(0) = f_0, \tag{42}$$

and the function $p_n(t)$ is obtained from the equation

$$p^*_{n+1} - p^*_n = rF(t, p_n)(p_n - p_{n-1}). \tag{43}$$

*Remark* 3. If the vectors $H(t)$ and $f_0$ have all their components non-negative, then the solution of the system (41) with the condition (42) is a vector which has non-negative components, that is $p_i(t) \geqq 0$ for $t \in [0, t_1]$, $i = 1, 2, \ldots, n-1, n+1, \ldots, n+m-1$. Indeed, from the assumption $f(ih) \geqq 0$, $i = 1, 2, \ldots, n-1, n$ and $f(x_n + jk) \geqq 0$, $k = 1\ 2, \ldots, m-1$, follows $F_0(t) \geqq 0$ for $t \in [0, t_1]$. In this case the elements of the matrx $A_{F_0}(t)$ satisfy the condition 1° and 2° from the theorem 1 of [4]. According to this theorem, the system (41) is of monotonic type on the interval $[0, t_1]$ and from this, it follows the remark 3. The condition that the vector $H(t)$ and $f_0$ are to be non-negative, is fulfilled from the technical point of view.

*Remark* 4. If the functions $p_{n+1}$ and $p_{n-1}$ are non-negative for $t \in [0, t_1]$ then under a certain assumption, the equation (43) has a unique non-negative solution on $[0, t_1]$.

Writing the equation (43) as

$$p^*_n = \frac{p^*_{n+1} + rF(t, p^*_n)}{1 + rF(t, p^*_{n-1})} \tag{44}$$

and introducing the notation

$$\Omega(u) = \frac{p^*_{n+1} + F(t, u)p^*_{n-1}}{1 + rF(t, u)}$$

the equation (43) becomes

$$p^*_n = \Omega(p^*_n)$$

Let $U$ be the space of the continuous, non-negative on $[0, t_1]$ function with norm $||u|| = \max\limits_{t \in [0, t_1]} u(t)|$. From the definition of the operator $\Omega$ one observe that if $u \in U$ then $\Omega(u) \in U$. We shall show that the operator $\Omega$ is a contraction operator on the space $U$. We have

$$\Omega(u) - \Omega(v) = \frac{rp^*_{n-1}(1 - p^*_{n+1})}{[1 + F(t,u)p^*_{n+1}][1 + F(t,v)p^*_{n-1}]}[F(t, u) - F(t, v)]$$

Since $p_{n-1}(t) \geqq 0$, for $t \in [0, t_1]$ and for $u \geqq 0$, we have $F(t, u) \geqq 0$ and

$$[1 + F(t, u)p^*_{n-1}][1 + F(t, v)p_{n-1}] \geqq 1$$



and consequently

$$|\Omega(u) - \Omega(v)| \leqq rp^*_{n-1}|1 - p^*_{n+1}||F(t, u) - F(t, v)|.$$

If

$$|F(t, u) - F(t, v)| < L|u - v|,$$

for all $u$, $v \in U$ and $t \in [0, t_1]$ then

$$||\Omega(u) - \Omega(v)|| \leqq rL||p^*_{n-1}||\ ||1 - p^*_{n+1}||\ ||u - v||, \tag{45}$$

from (45) it follows that if

$$L < \frac{1}{r||p^*_{n-1}||\ ||1 - p^*_{n+1}||}, \tag{46}$$

then, the operator $\Omega$ is a contraction operator on the space $U$. From the contraction mapping principle follows that if the condition (46) is fulfiled then the equation (41) has a unique solution in $U$.

The property of the function $p_i(t)$ to be non-negative on $[0, t_1]$ keeps its validity on each interval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$, since for each of these interval the initial functions and the vector $H(t)$ are non-negative.

In order to determine the error vector $\Gamma_1(t) = P(t) - P^*(t)$ and $\gamma^{(1)}_n(t) = p_n(t) - p^*_n(t)$, $t \in [0, t_1]$ we subtract (41) and (42) from (38) and (39) and also (43) from (40), obtaining thus

$$(P - P^*)' = A_{F(t,p_n)}P - A_{F_0}P^*, \tag{47}$$

and

$$(p_{n+1} - p^*_{n-1}) - (p_n - p^*_n) = r[F(t, p_n)(p_n - p_{n-1}) - F(t, p^*_n)(p^*_n - p_{n-1})]. \tag{48}$$

We remark that the matrix equation (47) can be written as

$$\Gamma'_1 = A_{F(t,p_n)}\Gamma_1 + [F(t, p_n) - F_0]G, \tag{49}$$

where $G$ means the column matrix $G = (g_{i1})$ with $g_{i1} = 0$ for

$i = 1, 2, \ldots, n-2, n+2, \ldots, n+m-1$, while $g_{n-1,1} = \dfrac{2a^2}{11h^2}$ and

$$g_{n+1,1} = \frac{2b^{12}}{11h^2}.$$

The solution of the system (49) subject to the condition $\Gamma(0) = 0$ can be evaluated by means of the lemma 1, as the elements of the matrix $A_F$ satisfing to the condition (30). In this way one obtains the following valuation

$$\|\Gamma_1\| \leqslant \frac{\alpha^2}{h^2} \,|F(t,\ p_n(t)] - F[t,\ p_n(0)]\,|\,\Delta t, \tag{50}$$

where

$$\alpha^2 = \max\left\{\frac{a^2}{11},\ \frac{b'^2}{11}\right\}.$$

Denoting as

$$B \geqslant \max_{[0,\ t]} \left|\frac{p_n(t_1) - p_n(t_2)}{t_1 - t_2}\right|$$

from (49) follows

$$\max_{[0,\ t_1]} \|\Gamma_1\| \leqslant \alpha^2 \left(\frac{\Delta t}{h^2}\right) LB\,\Delta t, \tag{51}$$

Denoting by

$$\gamma_i^{(1)}(t) = p_n(t) - p^*(t).$$

For the estimation of the error $\gamma_n^{(1)}(t)$ we write the equation (48) as

$$\gamma^{(1)}_{n+1} - \gamma^{(1)}_n = r[F(t,\ p_n) - F(t,\ p_n^*)(p_n - p_{n-1}) + F(t,\ p_n^*)(\gamma_n^{(1)} - \gamma_{n-1}^{(1)})]. \tag{52}$$

But

$$|F(t,\ p_n) - F(t,\ p_n^*)| \leqslant L|p_n - p_n^*|,$$

consequently

$$F(t,\ p_n) - F(t,\ p_n^*) = \mu(p_n - p_n^*),$$

where

$$|\mu| \leqslant L.$$

Then, from (52) it results

$$|\gamma_n^{(1)}| \leqslant \left|\frac{1 + rF(t,\ p_n^*)}{1 + rF(t,\ p_n^*) + \mu(p_n - p_{n-1})}\right| \|\Gamma_1\|.$$



Denoting as

$$\Phi = \max_{[0,\ T]} \left|\frac{1 + rF(t,\ p_n^*)}{1 + rF(t,\ p_n^*) + r\mu(p_n - p_{n-1})}\right|, \tag{53}$$

we have

$$\max_{[0,\ t_1]} |\gamma_n^{(1)}(t)| \leqslant \Phi \max_{[0,\ 1]} \|\Gamma_1(t)\|, \tag{54}$$

and also

$$\max_{t \in [ts-1,\ ts]} |\gamma_n^{(s)}(t)| \leqslant \Phi \max_{t \in [ts-1,\ ts]} \|\Gamma_s(t)\|. \tag{55}$$

Passing to the interval $[t_1,\ t_2]$, we solve the system

$$P_2^* = A_{F_1^*} P_2^* + H, \tag{56}$$

with the condition

$$P_2^*(t_1) = P_1^*(t_1), \tag{57}$$

where we denote by $F_1^* = F([t,\ p_n^*(t_1)])$.

The vector $\Gamma_2 = P_2 - P_2^*$ which represents the difference between the solution of the system (56), (57) and the solution of the system (38), (39) on the interval $[t_1,\ t_2]$ is obtained from the equation

$$\Gamma_2'(t) = A_F\Gamma_2(t) + G\{F[t,\ p_n(t)] - F[t,\ p_n^*(t_1)]\}, \tag{58}$$

with the condition

$$\Gamma_2(t_1) = P_1(t_1) - P_1^*(t_1). \tag{59}$$

According to lemma 1, one obtains for the error $\Gamma_2(t)$ the delimitation

$$\|\Gamma_2\| \leqslant \|\Gamma_1\| + \frac{\alpha^2}{h^2} |FL[t,\ p_n(t)] - F[t,\ p_n^*(t_1)]. \tag{60}$$

But

$$|F[t,\ p_n(t)] - F[t,\ p_n^*(t_s)] \leqslant |F[t,\ p_n(t)] - F[t,\ p_n(t_s)]| - - |F[t,\ p_n(t_s)] - F[t,\ p_n^*(t_s)]|,$$

and corresponding to the above assumption one gets

$$|F[t, p_n(t)] - F[t, p_n^*(t)]| \leqslant LB\Delta t,$$

and

$$|F[t, p_n(t_s)] - F[t, p_n^*(t_s)]| \leqslant L|p_n(t_s) - p_n^*(t_s)| = L|\gamma_n^{s-1}|,$$

and also

(61) $$|F[t, p_n(t)] - F[t, p_n^*(t_s)]| \leqslant LB\Delta t + L||\gamma^{(s)}||.$$

For $s = 1$, taking (54) into account, one obtains

$$|F[t, p_n(t)] - F[t, p_n^*(t_s)]| \leqslant L\Phi B\Delta t + L\Phi||\Gamma_1||,$$

and from (60)

$$||\Gamma_2|| \leqslant ||\Gamma_1|| + \frac{\alpha^2}{h^2}(LB\Delta t + L\Phi||\Gamma_1||)\Delta t,$$

According to the estimation (51) one obtains

(62) $$\max_{[t_1, t_2]} ||\Gamma_2|| \leqslant \left[\alpha L^2\Phi\left(\frac{\Delta t}{h}\right)^2 + 2\alpha^2 L\left(\frac{\Delta t}{h^2}\right)\right]B\Delta t.$$

Continuing this process, one solves in each interval $[t_{s-1}, t_s]$ $s = 1, 2, \dots, N$ the ordinary differential system of matrix form

(63) $$P_s^* = A_{F_{s-1}^*}P_s^* + H,$$

with the condition

(64) $$P_s^*(t_s) = P_{s-1}^*(t_s),$$

and the equation

(65) $$p_{n+1}^* - p_n^* = F(t, p_n^*)(p_n^* - p_{n-1}^*),$$

where

$$F_{s-1}^* = F(t, p_n^*(t_{s-1})].$$

In this way one obtains the vector $P^*(t)$ which is equal on the interval $[t_{s-1}, t_s]$ with the vector $P_s^*(t)$, $s = 1, 2, \dots, N$ and the function $p_n^*(t)$



defined on the interval $[t_{s-1}, t_s]$ by the equation (65). Denoting by $\Gamma(t) = P(t) - P^*(t)$ and $\gamma_n(t) = p_n(t) - p_n^*(t)$ we have, according to the notation introduced above,

$$\Gamma(t) = \Gamma_s(t),\quad \gamma_n^1(t) = \gamma_n^{(s)}(t),\quad s = 1, 2, \dots, N$$

for $t \in [t_{s-1}, t_s]$

The relationship between $\Gamma_s(t)$ and $\gamma_n^{(s)}(t)$ is given by the equation (55) and the estimation of the $\Gamma_1$ and $\Gamma_2$ by the equations (51) and (62).

Continuing the estimation of the error $\Gamma(t)$ on the successive intervals we get

(66) $$||\Gamma_s|| \leqslant (1 + \sigma)||\Gamma_{s-1}|| + \sigma\frac{B}{\Phi}\Delta t, \quad s = 2, 3, \dots, N$$

where

$$\sigma = \Phi\alpha^2 L\frac{\Delta t}{h^2}.$$

From (51) and (66) one obtains

$$\max_{[t_{s-1}, t_s]} ||\Gamma_s|| \leqq \sigma\frac{B}{\Phi}\Delta t[(1 + \sigma)^{s-1} + (1 + \sigma)^{s-2} + \dots + (1 + \sigma) + 1]$$

or

$$\max_{[t_{s-1}, t_s]} ||\Gamma_s|| \leqq \frac{B}{\Phi}\Delta t[(1 + \sigma)^s - 1].$$

But since $s \leqq N$ we also have

$$\max_{[t_{s-1}, t_s]} ||\Gamma_s|| \leqq \frac{B}{\Phi}\Delta t[(1 + \sigma)^N - 1],$$

and taking $\Delta t = \frac{T}{N}$ into account and using the notice $\frac{L\alpha^2\Phi T}{h^2} = \omega$ we obtain

$$\max_{[t_{s-1}, t_s]} ||\Gamma_s|| \leqq \frac{B}{\Phi}\Delta t\left[\left(1 + \frac{\omega}{N}\right)^N - 1\right] \leqq \frac{B}{\Phi}(e^{\omega} - 1)\Delta t.$$

This last estimation shows that $\Gamma(t)$ tends to zero as $\Delta t$ tends to zero. It also, shows, that, one obtains better results when the interval $[0, T]$ is little.

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