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A METRIC OF POMPEIU-HAUSDORFF TYPE FOR THE SET OF CONTINUOUS FUNCTIONS

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1. Introduction.

Several metrics can be defined for the set of continuous functions between two metric spaces, the best-known (see [1]) being the uniform (or Chebyshev) metric and the metric which generates the compact open topology. Usually, the uniform metric is considered only for functions defined on a compact, but there is not difficult to take it more generally.

In some problems appearing in topological dynamics, these two metrics are unsatisfactory. For this, in what follows, we define a new metric for the set of continuous functions, using as example Pompeiu-Hausdorff's metric. Also we establish some relations among these metrics. Applications of this new metric in solving some problems in topological dynamics will appear elsewhere.

2. Basic notations and definitions.

Let d_1 and d_2 be two metrics for the set X. We say that d_1 is finer than d_2, and write $d_2 \leqslant d_1$, if the topology generated by d_1 is finer than that generated by d_2. It is easy to check that $d_2 \leqslant d_1$ iff the identity application

$$i : (X, d_1) \to (X, d_2)$$

is continuous. This holds, for exemple, if there is a $M > 0$ such that for every x and y in X:

$$d_2(x, y) \leqslant M d_1(x, y).$$

Let (X, d) and (Y, e) be two metric spaces, and $C(X, Y)$ the set of all continuous functions from X to Y. To define some metrics for $C(X, Y)$ we shall use the function $L : [0, \infty] \to [0, 1]$ given by:

$$L(t) = \begin{cases} \frac{t}{1+t}, & 0 \leqslant t < \infty \\ 1, & t = \infty \end{cases} \tag{1}$$

The uniform metric T for $C(X, Y)$ may be defined by:

$$T(f, g) = L(\sup \{e(f(x), g(x)) : x \in X\}) \tag{2}$$

or, as usual, with the identity instead of L, if X is a compact.

If X is a locally compact, separable metric space, then there are compact subsets K_n of X, with $K_n \subset K_{n+1}$ for every n, such that X may be represented as: $X = \bigcup_{n=1}^{\infty} K_n$. In this case, the compact open topology on $C(X, Y)$ is generated by the metric K defined as follows:

$$K(f, g) = \sum_{n=1}^{\infty} 2^{-n} L(\max\{e(f(x), g(x)) : x \in K_n\}) \tag{3}$$

S. MROWKA [2] proved that the converse is also true: if the compact open topology is metrisable, then X is locally compact.

We may also use the metric P of Pompeiu-Hausdorff (see [3]) which is defined by the relation

$$P(F, E) = L(\sup \{\sup_{x \in F} \inf_{y \in E} e(x, y), \sup_{y \in E} \inf_{x \in F} e(x, y)\}) \tag{4}$$

F and E being closed, non-void subsets of Y. Taking

$$P(f, g) = P(f(X), g(X)) \tag{5}$$

we obtain only a pseudo-metric for $C(X, Y)$, because one loses the parametrization on the set of values.

3. The metric S for C(X, Y).

With the above notations, we have the following:

THEOREM. *If for $f, g \in C(X, Y)$ we denote*

$$S_0(f, g) = \inf \{r > 0 : \forall x \in X, \ \inf \{e(f(x), g(y)) : d(x, y) < r\} < r\} \tag{6}$$

(with the convention: $\inf \emptyset = \infty$), then

$$S(f, g) = L(\sup\{S_0(f, g), S_0(g, f)\}) \tag{7}$$

defines a metric for $C(X, Y)$.

Proof. We have to verify only the triangle inequality:

$$S(f, h) \leqslant S(f, g) + S(g, h) \tag{8}$$

and this only for $S(f, g) < 1$ and $S(g, h) < 1$, i.e.

$$s_1 = \sup \{S_0(f, g), S_0(g, f)\} < \infty$$

and

$$s_2 = \sup \{S_0(g, h), S_0(h, g)\} < \infty.$$

Let $t > 0$ be arbitrary and r_1, r_2 such that:

$$s_1 < r_1 < s_1 + t/2; \quad s_2 < r_2 < s_2 + t/2.$$

If x is a fixed point in X we have:

$$e(f(x), h(y)) \leqslant e(f(x), g(z)) + e(g(z), h(y))$$

so that:

$$\inf \{e(f(x), h(y)) : d(x, y) < r_1 + r_2\} \leqslant e(f(x), g(z)) + + \inf \{e(g(z), h(y)) : d(x, y) < r_1 + r_2\}.$$

This means that for every z with $d(x, z) < r_1$ we have:

$$\inf \{e(f(x), h(y)) : d(x, y) < r_1 + r_2\} \leqslant e(f(x), g(z)) + + \inf \{e(g(z), h(y)) : d(y, z) < r_2\} \leqslant e(f(x), g(z)) + r_2.$$

Taking in the right hand the infimum on $\{z : d(x, z) < r_1\}$ we have $S_0(f, h) \leqslant r_1 + r_2$ and interchanging f and h, we get:

$$\max \{S_0(f, h), S_0(h, f)\} \leqslant r_1 + r_2 \leqslant \max \{S_0(f, g), S_0(g, f)\} + + \max \{S_0(g, h), S_0(h, g)\} + t.$$

Letting $t \to 0$ and taking in account the monotony of the function L, we obtaine (8).

Remark 1. The geometrical interpretation of the metric S is the following: if $S(f, g) = r$ then, for every x in X the function g takes on the r-vicinity of x at least one value at distance smaller than r of $f(x)$.

Remark 2. Among the metrics T, S and (if X is separable, locally compact) K, there are the following relations:

$$S(f, g) \leqslant T(f, g); \qquad K(f, g) \leqslant T(f, g) \tag{9}$$

for every f and g in $C(X, Y)$. Also we have $P(f, g) \leqslant S(f, g)$.

We have in addition the following:

Lemma. *If X is a locally compact, separable metric space, then the identity function*

$$i : (C(X, Y), S) \to (C(X, Y), K)$$

is continuous.

Proof. Let us suppose $S(f_n, f) \to 0$ for $n \to \infty$ and let an arbitrary $0 < r < 1$. Let X be represented as: $X = \bigcup_{n=1}^{\infty} K_n$, with K_n compact subsets of X, $K_n \subset \text{Int } K_{n+1}$. We fix a natural number $i_r > \log_2 3/r$, and denote by s_r the distance between K_{i_r} and Fr K_{i_r+1}. Clearly $s_r \neq 0$. Since f is uniformly continuous on K_{i_r+1}, we may find a s, $0 < s < \min \{r/3, s_r\}$, such that $u \in K_{i_r}$ and $d(u, v) < s$ implies $e(f(u), f(v)) < r/3$. Also, there is a $N(s)$ such that $n > N(s)$ implies $S(f_n, f) < s$, i.e. for every u in X there is a v_u in X with $d(v_u, u) < s$, for which $e(f_n(u), f(v_u)) < s$. So, for u in K_{i_r} we have:

$$e(f_n(u), f(u)) \leqslant e(f_n(u), f(v_u)) + e(f(v_u), f(u)) < s + r/3 < 2r/3$$

hence

$$K(f_n, f) \leqslant \sum_{i=1}^{i_r} L(\max \{e(f_n(u), f(u)) : u \in K_{i_r}\}) + \sum_{i=i_r+1}^{\infty} 1/2^i < 2r/3 + r/3 = r$$

that is $K(f_n, f) \to 0$ for $n \to \infty$.

C o n s e q u e n c e. *If X is a locally compact, separable metric space we have:*

$$K \leqslant S \leqslant T. \tag{10}$$

Remark 3. Generally the above metrics are not equivalent, that is in (10) converse relations fail to hold, as show the following exemples.

Exemple 1. Let $f, f_n : \mathbf{R} \to [-1, 1]$ be defined by:

$$f(x) = \sin \cdot \exp (x)$$

and

$$f_n(x) = f(x + t_n), \text{ with } t_n = \ln \frac{4n+3}{4n+1}.$$

For $n \to \infty$ we have $t_n \to 0$ and $S(f_n, f) \to 0$, but

$$T(f_n, f) \geqslant L\left(\left| f_n\left(\ln \frac{(4n+1)\pi}{2}\right) - f\left(\ln \frac{(4n+1)\pi}{2}\right)\right|\right) = \frac{2}{3}.$$

That is T and S are not equivalent even if Y is compact.

Exemple 2. Let

$$f(x) = \exp (x)$$

and

$$f_n(x) = f(x + t_n)$$

with $t_n \to 0$. We have:

$$K(f_n, f) \to 0 \text{ for } n \to \infty$$

but

$$T(f_n, f) = 1 \text{ for every } n.$$

Exemple 3. Let

$$f(x) = 0 \text{ for every } x \text{ in } \mathbf{R}$$

and

$$f_n(x) = t_n \exp (x)$$

with $0 < t_n \to 0$. We have:

$$K(f_n, f) \to 0 \text{ for } n \to \infty$$

but

$$S(f_n, f) = 1 \text{ for every } n.$$

Remark 4. If on $C(X, Y)$ one takes the topology generated by one of the metrics K, S and T, then the application:

$$F : C(X, Y) \times X \to Y,$$

defined by $F(f, x) = f(x)$, is continuous.

In the case of the metric K, this is proved in [1] and for the metric T it is trivial. Let us suppose $S(f_n, f) \to 0$ and $x_n \to x$ for $n \to \infty$. Because f is continuous in x, for every $r > 0$, there is a $s > 0$ such that $d(x, y) < s$ implies $e(f(x), f(y)) < \frac{r}{2}$. By assumption, there is a natural N with the property that for $n > N$, $d(x, x_n) < \frac{s}{2}$ and $S(f_n, f) < \min \left\{\frac{s}{2}, \frac{r}{2}\right\}$. For such a n, there is an y_n in X with $d(x_n, y_n) < \frac{s}{2}$ and $e(f_n(x_n), f(y_n)) < \frac{r}{2}$. That is $d(x, y_n) < s$ and so

$$e(f_n(x_n), f(x)) \leqslant e(f_n(x_n), f(y_n)) + e(f(y_n), f(x)) < r$$

hence

$$f_n(x_n) \to f(x).$$

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[3] P o m p e j u, D., Math. Ann., **63**, 326 (1907).

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