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A GENERALIZED INTERPOLATION PROBLEM WITH VARIABLE NODES

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1. Introduction

In this paper we study a simple, yet intuitively appealing, interpolation problem. An interesting feature of the problem is that some of the interior nodes at which the interpolation takes place are variabile. The interpolation problem has an important application to the electrical engineering problem of designing digital filters.

We will assume that the basis functions $h_0, \ldots, h_n$ for the generalized polynomials of interpolation satisfy the following hypotheses:

(H1) $h_0, \ldots, h_n$ are continuously differentiable on $[x_0, x_n]$.

(H2) Any linear combination $\sum_{i=0}^{n} a_i h_i'(x)$ of the derivatives which is not identically zero has at most $n-1$ zeros in (x_0, x_n).

Using Rolle's Theorem it is easy to see that $h_0, \ldots, h_n$ form a Chebyshev system on $[x_0, x_n]$, i.e. every nontrivial linear combination $\sum_{i=0}^{n} a_i h_i(x)$ has at most n zeros in $[x_0, x_n]$. Conditions (H1), (H2) are similar to but not the same as requiring $h_0, \ldots, h_n$ to be an extended Chebyshev system of order two [6]. We can now state the interpolation problem.

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Generalized Interpolation Problem. Let $y_0, \ldots, y_n$ be given real numbers satisfying $(-1)^i(y_i - y_{i-1}) < 0$ for $i = 1, \ldots, n$. Let $I = \{i_0, \ldots, i_n\}$ be given integers satisfying $0 = i_0 < i_1 < \ldots < i_k = n$. Let

$$x_0 = \xi_{i_0} < \xi_{i_1} \ldots < \xi_{i_k} = x_n$$

be given nodes (fixed nodes). Assume $h_0, \ldots, h_n$ satisfy (H1), (H2). Does there exist a generalized polynomial $p(x) = \sum_{i=0}^{n} a_i h_i(x)$ and nodes $x_0 < x_1 < \ldots < x_n$ such that

1) $x_{i_j} = \xi_{i_j}$, $j = 0, \ldots, k$

2) $p(x_i) = y_i$, $i = 0, \ldots, n$

3) $p'(x_i) = 0$, $i \notin I$.

Condition (3) is equivalent to requiring that $p(x)$ attain a relative extreme value at the variable nodes. The interpolation problem was first introduced and solved by C. DAVIS [2] for the case $I = \{0, n\}$, $h_i(x) = x^{i-1}$; i.e. all interior nodes variable and using ordinary polynomials. In [5] W. KAMMERER gave an iterative method which converges to a solution of the interpolation problem for the special case introduced by Davis. In Section 2 we give an algorithm similar to Kammerer's for the construction of a solution to the Generalized Interpolation Problem. Our analysis is different than Kammerer's, which utilized divided differences. In [3] FITZGERALD and SCHUMAKER used a differential equations approach to prove existence and uniqueness of the solution to the Generalized Interpolation Problem under slightly different assumptions on the basis functions $h_0, \ldots, h_n$.

2. Description and analysis of the algorithm

Algorithm. Make an initial guess $x_0^{(0)} < x_1^{(0)} < \ldots < x_n^{(0)}$ for the nodes with $x_{i_j}^{(0)} = \xi_{i_j}$, $j = 0, \ldots, k$. Let $p^{(0)}$ be the unique generalized polynomial satisfying $p^{(0)}(x_i^{(0)}) = y_i$, $i = 0, \ldots, n$. Let $z_1 < \ldots < z_{n-1}$ be the $n-1$ zeros of $[p^{(0)}]'$ in (x_0, x_n). If $i \in I$ set $x_i^{(1)} = x_i^{(0)}$ (fixed nodes). If $i \notin I$ set $x_i^{(1)} = z_i$. Then calculate $p^{(1)}$ satisfying $p^{(1)}(x_i^{(1)}) = y_i$, $i = 0, \ldots n$ and continue.

We will now prove a series of lemmas to show that the Algorithm converges to a solution of the Generalized Interpolation Problem. It is convenient to have a name for a generalized polynomial which interpolates at the fixed nodes and successively attains the prescribed y-levels; such a pol ynomial will be called a feasible polynomial.

Definition. *A feasible polynomial p is any linear combination $p = \sum_{i=0}^{n} a_i h_i$ such that there exists $x_1, \ldots, x_{n-1}$ with $x_0 < x_1 < \ldots < x_{n-1} < x_n$ and*

1) $x_{i_j} = \xi_{i_j}$, $j = 0, \ldots, k$

2) $p(x_i) = y_i$, $i = 0, \ldots, n$.

The first lemma states some basic facts to be used (often implicitly); the proof is straightforward and will be omitted.

Lemma 1. *The derivative p' of any feasible polynomial p has exactly $n-1$ zeros $z_1 < \ldots < z_{n-1}$ in (x_0, x_n). The polynomial p is monotonic in each subinterval $[x_0, z_1]$, $[z_1, z_2]$, $[z_{n-1}, x_n]$. Also p achieves its relative extreme values in (x_0, x_n) precisely at $z_1, \ldots, z_{n-1}$ and these are alternately relative maxima and relative minima.*

A crucial property of the iterative method which will be exploited heavily in the analysis is that each iterate starts out below its predecessor; the next lemma expresses this precisely.

Definition. *For any feasible polynomial p we define $z(p) = \inf\{x \in [x_0, x_n] : p(x) \geq y_1\}$. That is $z(p)$ is the first point where p reaches the y_1-level.*

Lemma 2. *Let p be a feasible polynomial. Let p_I be the polynomial obtained by iterating once on p using the Algorithm. Then*

1) *$p_I(x) < p(x)$ for all x in $(x_0, z(p))$ unless $p_I(x) \equiv p(x)$*

2) *$z(p_I) \geq z(p)$.*

Proof. To prove (1) assume there exists x^* in $(x_0, z(p))$ such that $p_I(x^*) \geq p(x^*)$. Let $x_0 < s_1 < \ldots < s_{n-1} < x_n$ be the abscissas obtained from p at which the interpolation to get p_I takes place (these are either fixed nodes or zeros of p'). Then $p - p_I$ is alternately nonnegative and nonpositive at the $n+2$ points $x_0, x^*, s_1, \ldots, s_{n-1}, x_n$. Hence $p - p_I$ has at least $n+1$ zeros in $[x_0, x_n]$, counting interior zeros where $p - p_I$ does not change sign as two [7, p. 63]. Hence $p_I(x) \equiv p(x)$. The fact that $z(p_I) \geq z(p)$ follows immediately from (1).

Lemma 3. *Let $\{p_k\}$ be a sequence of polynomials generated by the Algorithm. Then $\{p_k\}$ is uniformly bounded on $[x_0, x_n]$.*

Proof. By Lemma 2 for all k, $z(p_k) \geq z(p_0)$ and $p_k(x) \leq p_0(x)$ for all x in $[x_0, z(p_0)]$. Hence each p_k is monotonically increasing in $[x_0, z(p_0)]$ and $y_0 \leq p_k(x) \leq y_1$ for all x in $[x_0, z(p_0)]$. Let $t_0, \ldots, t_n$ be $n+1$ fixed points in $[x_0, z(p_0)]$. Then $p_k(x) = \sum_{i=0}^{n} p_k(t_i) \frac{D(t_0, \ldots, t_{i-1}, x, t_{i+1} \ldots, t_n)}{D(t_0, \ldots, t_n)}$ where

$$D(t_0, \ldots, t_n) = \det \begin{vmatrix} h_0(t_0) & \ldots & h_n(t_0) \\ \vdots & & \\ h_0(t_n) & \ldots & h_n(t_n) \end{vmatrix}.$$

Since $t_0, \ldots, t_n$ are in $[x_0, z(p_0)]$, $y_0 \leq p_k(t_i) \leq y_1$. Using the continuity of $h_0, \ldots, h_n$ it is easily seen that $\{p_k\}$ is uniformly bounded on $[x_0, x_n]$.

In the subsequent analysis let $\|p\|$ denote the uniform norm on $[x_0, x_n]$, $\|p\| = \max_{x_0 \leq x \leq x_n} |p(x)|$.

L e m m a 4. *Let p be a feasible polynomial. Given $\varepsilon > 0$ there exists $\delta > 0$ such that if q is any feasible polynomial with $\|p - q\| < \delta$ then $\|p_I - q_I\| < \varepsilon$ where p_I is the polynomial obtained by iterating once on p and q_I is obtained by iterating once on q using the Algorithm.*

Proof. Let $\varepsilon > 0$ be given and let $x_0 < s_1 < \ldots < s_{n-1} < x_n$ be the abscissas obtained from p at which the interpolation to get p_I takes place. The proof of Lemma 4 follows immediately from the following two statements:

(i) For $\varepsilon > 0$ given, there exists $\delta_1 > 0$ such that if $u_1 < \ldots < u_{n-1}$ satisfy $\max_{1 \leq i \leq n-1} |s_i - u_i| < \delta_1$ then $\|p_I - q_u\| < \varepsilon$ where q_u is the polynomial interpolating to $(x_0, y_0), (u_1, y_1), \ldots, (u_{n-1}, y_{n-1}), (x_n, y_n)$.

(ii) For $\delta_1 > 0$ given, there exists $\delta > 0$ such that if q is a feasible polynomial satisfying $\|p - q\| < \delta$ then $\max_{1 \leq i \leq n-1} |s_i - u_i| < \delta_1$ where $x_0 < u_1 < \ldots < u_{n-1} < x_n$ are the abscissas of q at which the interpolation to get q_I takes place.

To prove (i) let

$$q_u(x) = \sum_{i=0}^{n} y_i \frac{D(x_0, u_1, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_{n-1}, x_n)}{D(x_0, u_1, \ldots, u_{n-1}, x_n)}$$

where D is as defined in the proof of Lemma 3. Then q_u is a continuous function of $x, u_1, \ldots, u_{n-1}$ on the compact set $[x_0, x_n] \times [s_1 - \eta, s_1 + \eta] \times \ldots \times [s_{n-1} - \eta, s_{n-1} + \eta]$ where $\eta = \frac{1}{3} \min_{1 \leq i \leq n-2} (s_{i+1} - s_i)$ and (i) follows. The proof of statement (ii) is straightforward and will be omitted. The lemma then follows from (i) and (ii).

The next Theorem is the main result of the paper; it guarantees the existence of a solution to the Generalized Interpolation Problem.

THEOREM 1. *Assume $h_0, \ldots, h_n$ satisfy hypotheses* (H1), (H2). *The sequence $\{p_k\}$ generated by the Algorithm converges uniformly to a solution of the Generalized Interpolation Problem.*

Proof. Since $\{p_k\}$ is uniformly bounded, it has a uniformly convergent subsequence $\{p_{k_j}\}$ with limit, say p_*. Clearly, p_* is a feasible polynomial.

We first show p_* is a solution of the Generalized Interpolation Problem. Assume it is not and let $(p_*)_I$ be the polynomial obtained by doing one iteration of the Algorithm on p_*. Let $z(p_*)$ be the point where p_* first reaches the y_1-level and let $z(p_{k_j})$ be similarly defined for p_{k_j}. Since p_* was assumed not to be a solution we have $(p_*)_I \not\equiv p_*$. Let

$x_c \in (x_0, z(p_*))$. Then by Lemma 2, $\varepsilon \equiv p_*(x_c) - (p_*)_I(x_c) > 0$. By Lemma 4 there exists a $\delta > 0$ such that if q is a feasible polynomial with $\|p_* - q\| < \delta$, then $\|(p_*)_I - q_I\| < \varepsilon/3$. Let J be such that $\|p_{k_j} - p_*\| < \min\{\varepsilon/3, \delta\}$ for all $j \geq J$ and also such that $z(p_{k_j}) > x_c$. Hence by Lemma 2 the iterates of p_{k_j} lie below p_{k_j} for x in $(x_0, z(p_{k_j}))$ and in particular for $x = x_c$. Thus

$$p_{k_{J+1}}(x_c) \leq p_{k_{J+1}}(x_c) \leq (p_*)_I(x_c) + \varepsilon/3$$

$$= p_*(x_c) - \frac{2\varepsilon}{3}, \text{ a contradiction.}$$

Hence, p_* is a solution of the Generalized Interpolation Problem.

Now assume there exists another uniformly convergent subsequence $\{p_{m_j}\}$ with limit, say, p_{**} different from p_*. Since $p_* \not\equiv p_{**}$, there is some point x_* in $(x_0, \min\{(z(p_*), z(p_{**})\})$ where one of these is above the other, say $p_*(x_*) - p_{**}(x_*) \equiv \varepsilon > 0$. There exists a J such that if $j \geq J$, then

$$\|p_{k_j} - p_*\| < \varepsilon/3 \quad (1)$$

$$\|p_{m_j} - p_{**}\| < \varepsilon/3 \quad (2)$$

and also $z(p_{m_J}) > x_*$. Then

$$p_{m_J}(x_*) \leq p_{**}(x_*) + \varepsilon/3$$

$$= p_*(x_*) - 2\varepsilon/3.$$

Since all iterates of p_{m_J} lie below p_{m_J} for x in $(x_0, x_*]$ we have a contradiction to (1). Hence $p_* = p_{**}$. Thus all uniformly convergent subsequences of $\{p_k\}$ have the same limit and so $\{p_k\}$ is uniformly convergent. This completes the proof of the theorem.

3. Concluding Remarks

We have tested the algorithm on the computer. In the examples we ran, convergence to the solution was very rapid.

The algorithm has been used by electrical engineers to design so-called extra-ripple digital filters [4]. This application is a special case of the General Interpolation Problem with $I = \{0, n\}$, $h_k(x) = \cos 2\pi kx$, $(x_0, x_n) = (0, \cdot 5)$. Their numerical experience with the algortihm has been very satisfactory.

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