MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 6, N° 1, 1977, pp. 15—21

APPLICATION OF ITERATIVE METHODS FOR SOLVING OPERATOR EQUATIONS AND IMPROVING CONVERGENCE CONDITIONS

by

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1. In the present paper are studied the method of chords and its modifications. The conditions of the convergence of successive approximations given by these methods are improved, and the results are applied to solving the ordinary differential equations and algebraic systems.

We use the following notations:

$\rho_{X,Y}(L)$ for the generalized norm of operator defined on X with values in Y;

$\rho_{X^2,Y}(B)$ for the generalized norm of a bilinear operator defined on XxX with values in the space of the linear operators $L(X, Y)$;

$\rho_Y(P(x_0))$ for the generalized norm of an element $y_0 = P(x_0) \in Y$;

2. Let be the operator equation

$$P(x) = \theta \tag{1}$$

where $P(x)$ is a nonlinear continuous mapping of the complete supermetric space X in the complete supermetric space Y [1].

Supposing that the divided differences [2] of the second order of the mapping P exist, we study [4] the following iterative formula (method of chords) [3]

$$x_{n+1} = x_n - \Lambda_n P(x), \tag{2}$$

where $\Lambda_n = [P_{x_n, x_{n-1}}]^{-1}$, $n = 0, 1, \ldots$. The following theorem of existence of the solutions of equation (1) holds:

THEOREM 1. *If for initial approximations x_0, $x_{-1} \in X$ the following conditions are satisfied*

1° *There exists* $\Lambda_0 = [P_{x_0, x_{-1}}]^{-1}$ *with* $\rho_{Y,X}(\Lambda_0) \leqslant B_0$,

2° $\rho_Y P(x_i) \leqslant \eta_i$, $i = 0, -1$ *and* $\eta_0 \leqslant \frac{1}{4}\eta_{-1}$

3 $\rho_{X^3,Y}(P_{u,v,w,}) \leqslant k$ *for every* $u, v, w \in S(x, r)$, $r = \frac{5}{4} B_0\eta_{-1}$

4° $h_0 \equiv B_0^2 k\eta_{-1} \leqslant \frac{1}{4}$

then the equation (1) *has a solution* $x^* \in S$, *which is the limit of the sequence* (2), *the rapidity of the convergence being given by the inequality*

$$\rho_X(x_n - x^*) \leqslant 2^{1-s_n} q^{s_{n-1}} (4h_0)^{s_n} \eta_0.$$

where $0 < q < 1$, *and* s_n *is the general term of the sequence of the partial sums of a Fibonacci sequence* u_n, *with* $u_1 = u_2 = 1$.

For avoiding the necessity to compute the mapping $[P_{x',x''}]^{-1}$ in every step of the iteration, the algorithm (2) can be replaced by the modified method [4]

(3) $$x_{n+1} = x_n - \Lambda_0 P(x_n),$$

where we must calculate only the mapping $[P_{x_0, x_{-1}}]^{-1}$ for initial approximations x_0, x_{-1}.

For proving some theorems on the existence of solutions of a given operator equation, by using the algorithm (3), we prove the following

L e m m a. *If we have*

(4) $$\rho_X(x - x^*) \leqslant \rho_X(x_1 - x^*)$$
$$\rho_X(x - x_0) \leqslant B_0\eta_0,$$

where x, x_1, $x_0 \in X$ *with* x^* *a solution of the equation* (1), *and if the operator* P *satisfies the conditions* 1° — 4° *of the Theorem 1, then*

(5) $$\rho_X(A(x) - x^*) < q\rho_X(x_1 - x^*)$$
$$\rho_X(A(x) - x_0) \leqslant 2B_0\eta_0,$$

where

(6) $$A(x) = x - \Lambda_0 P(x).$$

Proof. We observe that the mapping of the supermetric space X defined by (6) satisfies the following properties:

a). If x^* is a solution of the equation (1) then $x^* = A(x^*)$.

b). $x_{n+1} = A(x_n)$

c). $A_{x',x''} = I - \Lambda_0 P_{x',x''}$, I being the identity $A_{x_0,x_{-1}} = \theta$

d). $A_{x',x'',x'''} = -\Lambda_0 P_{x',x'',x'''}$, $x', x'', x''' \in X$.

Hence it follows by the property a)

(7) $$A(x) - x^* = A(x) - A(x^*) = A_{x,x^*}(x - x^*).$$

Since

$$A_{x,x^*} - A_{x,x_0} = A_{x,x^*x_0}(x^* - x_0),$$

we have

$$\rho_{X,X}(A_{x,x^*} - A_{x,x_0}) \leqslant B_0^2 k\eta_{-1} = h_0$$

i.e.

(8) $$\rho_{X,X}(A_{x,x^*}) - \rho_{X,X}(A_{x,x_0}) \leqslant h_0.$$

From the property c) we can write

$$A_{x,x_0} = A_{x,x_0} - A_{x_0,x_{-1}} = A_{x,x_0,x_{-1}}(x - x_1)$$

and thus

(9) $$\rho_{X,X}(A_{x,x_0}) \leqslant B_0 k\rho_X(x - x_{-1}).$$

We have

$$\rho_X(x - x_{-1}) \leqslant \rho_X(x - x_0) + \rho_X(x_0 - x_{-1}) \leqslant$$
$$\leqslant B_0\eta_0 + B_0(\eta_{-1} + \eta_0) = (2\eta_0 + \eta_{-1})B_0;$$

consequently the relation (8) becomes

$$\rho_{X,X}(A_{x,x^*}) \leqslant h_0 + B_0^2 k(2\eta_0 + \eta_{-1}) \leqslant$$
$$\leqslant h_0 + \frac{3}{2} B_0^2 k\eta_{-1} \leqslant 3h_0$$

and from (7) we obtain

$$\rho_X(A(x) - x^*) \leqslant 3h_0\rho_X(x_1 - x^*) \leqslant q\rho_X(x_1 - x^*)$$

and the first relation of (4) is proved.

For proving the second relation we consider the inequality

$$\rho_X(A(x) - x_0) \leqslant \rho\,(A(x) - x_1) + \rho_X(x_1 - x_0)$$

what by hypothesis leads to

$$\rho_X(A(x) - x_0) \leqslant 2B_0\eta_0$$

and the lemma is proved.

Now we can prove

THEOREM 2. *If for the initial approximations* x_0, $x_{-1} \in X$, *the operator* P *satisfies the conditions* 1°—4° *of the Theorem 1, then the sequence*

$\{x_n\}$, given by (3), converges to a solution x^* of the equation (1), the rapidity of the convergence being given by.

$$\rho_X(x^* - x_n) \leqslant q^{n-1}\rho\ (x_1 - x^*) \tag{10}$$

where $q = 3h_0 < 1$.

Proof. By Theorem 1 it results that the solution x^* of the equation (1) exists. We show that x^* is the limit of the sequence $\{x_n\}$ given by (3).

For $x = x_1$ the conditions (4) of the Lemma are satisfied, and by using the property b) of the mapping (6) we have

$$\rho_X(x_2 - x^*) \leqslant q\rho_X(x_1 - x^*)$$

and

$$\rho_X(x_2 - x_0) \leqslant 2B_0\eta_0.$$

By induction we obtain

$$\rho_X(x_n - x^*) \leqslant q^{n-1}\rho_X(x_1 - x^*)$$

and since $q < 1$, it follows that $\lim\limits_{n\to\infty} x_n = x^*$.

3. Let be the nonlinear system of algebraic equations

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$$

which can be given in the form

$$P(x) = \theta \tag{12}$$

where the mapping $y = P(x)$ is defined in the supermetric space X, with values in supermetric space Y, where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X \qquad y = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \in Y. \tag{13}$$

We can write a divided difference of the mapping P in the point x_0, x_{-1} in the form

$$P_{x_0 x_{-1}} = \begin{pmatrix} \dfrac{f_1(x_1^{(0)}, x_2^{(0)}) - f_1(x_1^{(-1)}, x_2^{(0)})}{x_1^{(0)} - x_1^{(-1)}} & \dfrac{f_1(x_1^{(-1)}, x_2^{(0)}) - f_1(x_1^{(-1)}, x_2^{(-1)})}{x_2^{(0)} - x_2^{(-1)}} \\ \dfrac{f_2(x_1^{(0)}, x_2^{(0)}) - f_2(x_1^{(-1)}, x_2^{(0)})}{x_1^{(0)} - x_1^{(-1)}} & \dfrac{f_2(x_1^{(-1)}, x_2^{(0)}) - f_2(x_1^{(-1)}, x_2^{(-1)})}{x_2^{(0)} - x_2^{(1-)}} \end{pmatrix} \tag{14}$$

Considering the following generalized norms for spaces X, Y

$$\rho_X(x(t)) = \max_t \left\{ \frac{c_1|x(t)|}{1 + c_2|x(t)|} \right\} \tag{15}$$

$$\rho_Y(x(t)) = \max_t \left\{ \frac{c_3|x(t)|}{1 + c_4|x(t)|} \right\} \tag{15'}$$

with $c_2 = c_4 = 0$, $c_3 = 1$, the matrix (14) being nondegenerate, we have

$$\rho_Y(P(x_0)) \leqslant \eta_0, \quad \rho_Y(P(x_{-1})) \leqslant \eta_{-1}$$

and

$$\rho_{X,X}([P_{x_0, x-1}]^{-1}) = \rho_{X,X}(\Lambda_0) \leqslant B_0. \tag{16}$$

Supposing that the divvided differences of the second order of $f_i(x_1, x_2)$, $i = 1, 2$ are bounded with k, we can find the constant k such that

$$\rho_{X,X}(P_{x',x''} - P_{x'';x'''}) \leqslant K\rho_X(x' - x'''). \tag{17}$$

Then $K = \dfrac{2k}{c^2}$, and choosing $c > 0$ so that

$$h_0 = B_0^2 K\eta_{-1} \leqslant \frac{1}{4}$$

we can apply the Theorem 1 or 2 to solve the system.

4. Consider the differential equation of the first order

$$x'(t) - f[t, x(t)] = 0, \tag{18}$$

defined in the domain given by the inequalities $t_0 \leqslant t \leqslant T$ and $|x(t) - x_0(t)| \leqslant \delta$, with the initial condition

$$x(t_0) = \xi_0 \tag{19}$$

$f[t, x(t)]$, $x(t)$ and $x_0(t)$ being continuous functions.

The equation (18), with initial value problem (19), can be written as an operator equation

$$P(x) \equiv x(t) - \xi_0 - \int_{t_0}^{t} f[\bar{t}, x(\bar{t})]d\bar{t} = 0. \tag{20}$$

For solving this equation we use the method of chords, in a new form determined by the equation (20).

Computing a diveded difference of P, we obtain

$$P_{x_1, x_2}\eta(t) = \eta(t) - \int_{t_0}^{t} f_{x_1, x_2}\eta(\bar{t})d\bar{t},$$

where

$$f_{x_1, x_2} = \frac{f[t, x_1(t)] - f[t, x_2(t)]}{x_1(t) - x_2(t)}.$$

In order to find the inverse of P_{x_1, x_2} we determine the function $x(t)$ in the equation

$$x(t) - \int_{t_0}^{t} f_{x_1, x_2} x(\bar{t}) d\bar{t} = y(t)$$

i.e. in the equation

$$x'(t) - f_{x_1, x_2} x(t) = y'(t).$$

We obtain

$$[P_{x_1, x_2}]^{-1} y(t) = e^{\int_{t_0}^{t} f_{x_1, x_2} d\bar{t}} \left[y(t_0) + \int_{t_0}^{t} y'(\bar{t}) e^{\int_{t_0}^{t} f_{x_1, x_2} d\bar{t}} d\bar{t} \right].$$

By partial integration we get the method of chords

$$x_{n+1}(t) = x_n(t) - P[x_n(t)] -$$

$$- e^{\int_{t_0}^{t} f_{x_n, x_{n-1}} d\bar{t}} \int_{0}^{t} e^{-\int_{t_0}^{t} f_{x_n, x_{n-1}} d\bar{t}} f_{x_n, x_{n-1}} P(x_n(\bar{t})) d\bar{t}. \tag{21}$$

As generalized norms we consider (15) and (15′).

For our problem we can put $c_3 = 1$, $c_2 = c_4 = 0$, c_1 being choosen in optimal manner.

By the method of chords, the formula (21), and the generalized Newton formula, we can obtain K satisfying the hypothesis of Theorem 1.

We have

$$P(x_n) = P(x_n) - P(x_{n-1}) - P_{x_{n-1}, x_{n-2}}(x_n - x_{n-1}) =$$

$$= - \int_{t_0}^{t} \left\{ f[\bar{t}, x_n(\bar{t})] - f[\bar{t}, x_{n-1}(\bar{t})] - f_{x_{n-1}, x_{n-2}}(x_n(\bar{t}) - x_{n-1}(\bar{t}) \right\} d\bar{t}$$

and with $|f_{x', x'', x'''}| \leqslant k$ it results

$$\rho_Y P(x_n) \leqslant \frac{T - t_0}{c^2} k \rho_X(x_n(t)) - x_{n-1}(t)) \cdot \rho_X(x_n(t) - x_{n-2}(t))$$

i.e. $K = \frac{T - t_0}{c^2}$.

By (21) we obtain

$$\rho_{Y, X}(\Lambda_0) \leqslant [1 + M(T - t_0) e^{2M(T - t_0)}] \equiv B_0,$$

where $M = |f_{x', x''}|$.

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Received 15.XII.1976.