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ON THE RADIUS OF A GRAPH

by

FLORICA KRAMER

(Cluj-Napoca)

A graph is called an 1-*graph*, if any pair of his vertices is connected by at most one edge.

In the following we shall study only finite undirected connected 1-graphs without loops.

Let $G = (V, E)$ be such a graph, where V is the vertex set and E is the edge set. The *order* of the graph G is the cardinal of the vertex set V.

The number of edges having a given endpoint $x \in V$ we shall designate by $g(x)$ and this number is called the *degree at the vertex* x. We shall designate by $D(G)$ and by $R(G)$ the diameter and respectively the radius of the graph G. If we shall denote by $d(x, y)$ the distance between the vertices x and y of V, i.e. the length of the shortest path from x to y, then the *diameter* and the *radius* can be defined by

(1) $$D(G) = \max_{y \in V} \max_{x \in V} d(x, y)$$

$$R(G) = \min_{x \in V} \max_{y \in V} d(x, y)$$

From the definitions of this two numerical characteristics follows immediately the relation

(2) $$R(G) \leqq D(G) \leqq 2R(G).$$

There are known some inferior bounds for the radius of a graph, even then when he is a directed one. If the graph is directed we shall denote by $g^+(x)$ the *out-degree* of the vertex x, i.e. the number of outgoing edges at the vertex x.

THEOREM 1. *Let $G = (V, E)$ be an 1-graph without loops, of order n with*

$$\max_{x \in V} g^+(x) = p > 1.$$

His radius R(G) verifies then the inequality

$$R(G) \geq \frac{\log (np - n + 1)}{\log p} - 1. \tag{3}$$

Another inferior bound for the radius of a graph depending on the number of vertices and edges was given in 1965 by M. K. GOLDBERG [2].

We call a directed graph *strongly connected* if for any pair of vertices x and y there is a path from x to y and inversely.

THEOREM 2. *The radius of a strongly connected graph with n vertices and m edges verifies the inequality*

$$R(G) \geq \left[\frac{n-1}{m-n+1}\right]^* \tag{4}$$

where $[r]^*$ designates the least integer $\geq r$. For each pair m and n, there is a strongly connected 1-graph G with n vertices and m edges such that we have the equality sign in (4).

In the sequel we shall try to obtain some upper bounds for the radius of an undirected graph.

We denote for $k = 1, 2, \ldots$ by $g_k(x)$ the generalized degree of order k at the vertex $x \in V$. This notion was introduced in [3] in the following way:

$$g_k(x) = \text{card } \{y,\ y \in V,\ 1 \leq d(x, y) \leq k\},$$

i.e. the number of vertices y with the distance from x satisfying the inequality $1 \leq d(x, y) \leq k$.

THEOREM 3. *Let $G = (V, E)$ be a finite connected undirected 1-graph of order n without loops such that the generalized degree of an order k verifies for every vertex x the inequality*

$$g_k(x) \geq \left[\frac{n}{h}\right]$$

where h is an integer $h \geq 2$. We have then:

$$R(G) \leq \begin{cases} 2k & \text{for } h = 2 \\ 3k + 1 & \text{for } h = 3 \\ (2k+1)(h-2) - 1 & \text{for } h \geq 4 \end{cases} \tag{5}$$

The established bounds are the best in the sense, that there are graphs verifying the hypotheses of the theorem and for which we have the equality sign in (5).

Proof. 1. First we shall settle the case $h \geq 4$. Let us suppose $R(G) > (2k+1)(h-2) - 1$. This means, that for every vertex $a \in V$ there is a vertex $b \in V$ such that $d(a, b) = (2k+1)(h-2)$. We consider a shortest path between a and b

$P = \{a = x_0, (x_0, x_1), x_1, \ldots, x_{(2k+1)(h-2)-1}, (x_{(2k+1)(h-2)-1}, x_{(2k+1)(h-2)}), x_{(2k+1)(h-2)} = b\}$ (fig. 1)

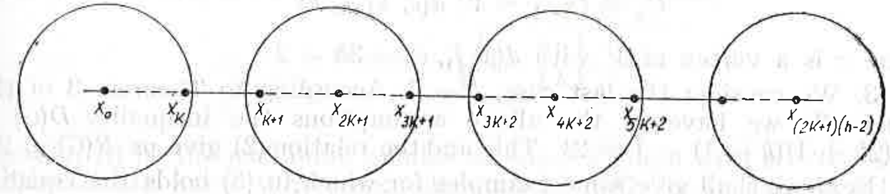

Fig. 1

We denote by C_i, $i = 1, 2, \ldots, h-1$ the following sets of vertices:

$$C_i = \{x,\ x \in V,\ d(x_{(2k+1)(i-1)}, x) \leq k\}$$

These sets are pairwise disjoint, otherwise P wouldn't be a shortest path between a and b.

Since our supposition $R(G) > (2k+1)(h-2) - 1$ implies for every vertex the existence of another vertex at the distance $(2k+1)(h-2)$ from the first, there is also a vertex c corresponding to the vertex x_{2k+1} such that $d(x_{2k+1}, c) = (2k+1)(h-2)$. We shall prove that the set

$$C = \{x,\ x \in V,\ d(x, c) \leqslant k\}$$

is disjoint from each set C_i, $i = 1, 2, \ldots, h-1$. If there would be some i, $1 \leq i \leq h-1$, such that $C \cap C_i \neq \emptyset$, we can conclude

$$d(x_{2k+1}, c) \leq d(x_{2k+1}, x_{(2k+1)(i-1)}) + d(x_{(2k+1)(i-1)}, c) \leq$$
$$\leq (2k+1)(h-3) + 2k = (2k+1)(h-2) - 1.$$

This yields a contradiction to the above results. Thus we have determined h disjoint subsets of V, each of them containing at least $\left(\left[\frac{n}{h}\right] + 1\right)$ vertices. By a comparison of the cardinals of V and of the subsets $C, C_1, C_2, \ldots, C_{h-1}$ we get

$$n = |V| \geq |C| + \sum_{i=1}^{h-1} |C_i| \geq h\left(\left[\frac{n}{h}\right] + 1\right) > n$$

which is evidently impossible.

2. Consider now the case $h = 3$. We proceed analogous to the preceding case supposing $R(G) > 3k + 1$. For each vertex $a \in V$ results then

the existence of a vertex b such that $d(a, b) = 3k + 2$. Let $P = \{a = x_0, (x_0, x_1), x_1, \ldots, x_{3k+1}, (x_{3k+1}, x_{3k+2}), x_{3k+2} = b\}$ be a shortest path between a and b. The subsets of V necessary to reason as above are:

$$C_0 = \{x, x \in V, d(x_0, x) \leq k\}$$

$$C_1 = \{x, x \in V, d(x_{2k+1}, x) \leq k\}$$

$$C_2 = \{x, x \in V, d(c, x) \leq k\}$$

where c is a vertex of V with $d(x_{k+1}, c) = 3k + 2$.

3. We consider the last case, $h = 2$. According to Theorem 3 of the paper [3] we have in the above assumptions the inequality $D(G) \leq (2k + 1)(h - 1) - 1 = 2k$. This and the relation (2) give us $R(G) \leq 2k$.

Next we shall give some examples for which in (5) holds the equality sign.

Example 1. For $h = 2$, let G be the graph formed by a circuit with $n = 4k + 1$ vertices. We have $n = 4k + 1$, $g_k(x) = 2k$ and $\left[\frac{n}{2}\right] = 2k$. Hence there are verified the hypotheses of Theorem 3. The radius of this graph is $R(G) = 2k$.

Example 2. For $h = 3$, let G be the graph formed by the circuit with $n = 6k + 2$ vertices. We have then $g_k(x) = 2k$ for each vertex x, $n = 6k + 2$, $\left[\frac{n}{3}\right] = 2k$ and $R(G) = 3k + 1$.

Example 3. For $h = 4$ let G be a circuit with $n = 8k + 2$ vertices. We have then $g_k(x) = 2k$ for each vertex x, $\left[\frac{n}{h}\right] = 2k$ and $R(G) = 4k + 1$.

We get from Theorem 3 for $k = 1$ the following:

C o r o l l a r y. *If $G = (V, E)$ is a finite connected undirected 1-graph of order n without loops and if the degree of every vertex satisfies the inequality*

$$g(x) \geq \left[\frac{n}{h}\right]$$

where h is an integer, $h \geq 2$, then we have

$$R(G) \leq \begin{cases} 2 & \text{for } h = 2 \\ 4 & \text{for } h = 3 \\ 3h - 7 & \text{for } h \geq 4. \end{cases} \tag{6}$$

THEOREM 4. *Let $G = (V, E)$ be a finite connected undirected 1-graph of order n without loops. If the degree of every vertex $x \in V$ satisfies the inequality*

$$g(x) \geq \left[\frac{n}{h}\right] \geq 2$$

then for $h \geq 4$ we have the inequality

$$R(G) \leq \min\left\{3h - 7, \frac{2n + 3 - \sqrt{4n \cdot \left[\frac{n}{h}\right] - 8n + 9}}{4}\right\} \tag{7}$$

Proof. The first part of the conclusion has been established in the Corollary. It remains to show that for $h \geq 4$, we have

$$R(G) \leq \frac{2n + 3 - \sqrt{4n \cdot \left[\frac{n}{h}\right] - 8n + 9}}{4} \tag{8}$$

Let $f(n, R)$ be the maximum number of edges a graph with n vertices and radius R can have. V. T. VIZING deduced in 1967 in the paper [4] the following values for $f(n, R)$:

$$f(n, 1) = \frac{n(n-1)}{2}, \quad f(n, 2) = \left[\frac{n(n-2)}{2}\right] \text{ and}$$
$$f(n, R) = \frac{n^2 - 4nR + 5n + 4R^2 - 6R}{2} \text{ for } R \geq 3. \tag{9}$$

As $g(x) \geq \left[\frac{n}{h}\right]$ for every vertex $x \in V$, we have for the number m of edges of the graph G the inequality

$$m \geq \frac{n}{2}\left[\frac{n}{h}\right]. \tag{10}$$

From (9) and (10) follows:

$$\frac{n}{2}\left[\frac{n}{h}\right] \leq \frac{n^2 - 4nR + 5n + 4R^2 - 6R}{2} \tag{11}$$

which will us yield the bound for $R(G)$. The last inequality can be put under the form

$$4R^2 - 2(2n + 3)R + n^2 + n - n\left[\frac{n}{h}\right] \geq 0. \tag{12}$$

Thus we get an elementary algebraic problem relative to the sign of a quadratic trinomial. The discriminant of the trinomial is

$$\Delta = 4n\left[\frac{n}{h}\right] - 8n + 9$$

and the equation in R has the roots

$$R_1 = \frac{2n + 3 - \sqrt{4n\left[\frac{n}{h}\right] - 8n + 9}}{4} \text{ and } R_2 = \frac{2n + 3 + \sqrt{4n\left[\frac{n}{h}\right] - 8n + 9}}{4}$$

The set of the values R which satisfy the inequality (11) and respectively (12) is formed by the intervals $[3, R_1] \cup [R_2, +\infty)$, but as $\left[\frac{n}{h}\right] \geq 2$, it follows that

$$R_2 = \frac{2n+3+\sqrt{4n\left[\frac{n}{h}\right]-8n+9}}{4} \geq \frac{2n+3+3}{4} = \frac{n+3}{2} > \frac{n}{2}$$

As the radius of a connected graph cannot be greater than $\frac{n}{2}$, the set of those R interesting us is the interval $[3, R_1]$ and hence $R \leq R_1$. Thus we have established the inequality (8). It is easy to verify $R_1 \geq 3$.

Indeed, as $h \geq 4$, we have

$$\frac{n}{h} \leq \frac{n}{4} \text{ and hence } \left[\frac{n}{h}\right] \leq \frac{n}{4}.$$

Then

$$\sqrt{4n\left[\frac{n}{h}\right]-8n+9} \leq \sqrt{4n\left[\frac{n}{4}\right]-8n+9} \leq \sqrt{n^2-8n+9} <$$
$$< \sqrt{n^2-8n+16} = n-4,$$

wherefrom

$$R_1 = \frac{2n+3-\sqrt{4n\left[\frac{n}{2}\right]-8n+9}}{4} > \frac{2n+3-(n-4)}{4} = \frac{n+7}{4} > \frac{15}{4} > 3.$$

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Institutul de cercetări pentru tehnica de calcul
Filiala Cluj-Napoca

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