MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 6, N° 1, 1977, pp. 37–42

A HELLY–TYPE THEOREM FOR ARCWISE CONNECTED SETS IN THE PLANE

by

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Denote by R^m the real m-dimensional Euclidean space and by $d(x, y)$ the metric in this space. We shall use in the sequel the following two distance functions :

$$\delta(M' ; M'') = \min \{d(p', p'') : p' \in M', p'' \in M''\}$$

and

$$\rho(M', M'') = \max \{d(p', p'') : p' \in M', p'' \in M''\},$$

where M' and M'' are compact subsets in R^m.

Definition 1. *A set M in $\mathbf{R}^n$ is called nonseparable by hyperplanes, if there is no hyperplane H such that $H \cap M = \emptyset$ and M contains points in both the open half-spaces determined by H.*

The sets, which are nonseparable by hyperplanes have been called by O. HANNER and H. RADSTRÖM [1] convexly connected (see also [5] p. 174).

Definition 2. *The family $\mathfrak{M}$ of sets in R^n will be called independent, if for any k pairwise distinct members $M_1, M_2, \ldots, M_k$ of $\mathfrak{M}$; $k \leq n + 1$, any set of points $x_1, x_2, \ldots, x_k$, where $x_i \in M_i$, $i = 1, 2, \ldots, k$, determines a simplex of dimension $k - 1$, or equivalently the vectors $x_2 - x_1$, $x_3 - x_1, \ldots, x_n - x_1$ are linearly independent.*

H. KRAMER and A. B. NÉMETH have proved in [4] the following theorem, which is needed in the sequel.

THEOREM 1. *Let $M_1, M_2, \ldots, M_{n+1}$ be compact sets in R^n with the property that their convex hulls* conv (M_1), conv (M_2), ..., conv (M_{n+1}) *are independent. Then there exists at least one point $q \in R^n$ with the property that*

$$\delta(q, M_1) = \ldots = \delta(q, M_k) = \rho(q, M_{k+1}) = \ldots = \rho(q, M_{n+1}) \tag{1}$$

where k is an integer $0 \leq k \leq n+1$ (the case $k = 0$ or $k = n+1$ means that in (1) appears only ρ and respectively, only δ). In plus if each set M_i, $i = 1, 2, \ldots, n+1$ is nonseparable by hyperplanes, then the point q, with the property (1) is uniquely determined.

The following Helly-type theorem was proved in ([2], Theorem 2) by H. KRAMER and A. B. NÉMETH.

THEOREM 2. *Let $\mathcal{K} = \{K_i : i \in A\}$ be an independent family of compact convex sets in the Euclidean plane R^2 with at least five members and let r be a given positive number. If for any three distinct indices i, j, l of A, there is a point q_{ijl} such that we have*

$$\delta(q_{ijl}, K_i) = \delta(q_{ijl}, K_j) = \delta(q_{ijl}, K_l) = r, \tag{2}$$

then there exists a point $q \in R^2$ such that

$$\delta(q, K_i) = r \text{ for each } i \in A. \tag{3}$$

The aim of this note is to establish a Helly-type theorem analogous to Theorem 2 for the distance function ρ in a somewhat weaker hypothesis.

THEOREM 3. *Let $\mathcal{K} = \{K_i : i \in A\}$ be an independent family of compact arcwise connected sets in the Euclidean plane R^2 with at least five members and let r be a given positive number. If for any three distinct indices i, j, l of A there is a point q_{ijl} such that we have*

$$\rho(q_{ijl}, K_i) = \rho(q_{ijl}, K_j) = \rho(q_{ijl}, K_l) = r, \tag{4}$$

then there exists a point $q \in R^2$ such that

$$\rho(q, K_i) = r \text{ for each } i \in A. \tag{5}$$

For the proof of the theorem we need two lemmas.

Lemma 1. *Let K' and K'' be two disjoint nonempty compact arcwise connected sest of the Euclidean plane R^2 and let r be a given positive number. There are then at most two points q_1 and q_2 such that we have*

$$\rho(q_i, K') = \rho(q_i, K'') = r \text{ for } i = 1, 2. \tag{6}$$

Proof. Suppose the contrary, i.e. there exist three distinct points q_1, q_2, and q_3 such that we have

$$\rho(q_i, K') = \rho(q_i, K'') = r \text{ for } i = 1, 2, 3. \tag{7}$$

Denote by $C_i = C(q_i, r) = \{x : x \in R^2,\ d(x, q_i) = r\}$ the circle of centre q_i and radius r and by D_i the corresponding disc with boundary C_i. From the definition of the distance function ρ and from the hypothesis that K' and K'' are compact sets, follows

$$K' \cap C_i \neq \emptyset \text{ and } K'' \cap C_i \neq \emptyset,\ i = 1, 2, 3. \tag{8}$$

and

$$K' \cup K'' \subset D_i \quad i = 1, 2, 3. \tag{9}$$

From the last inclusion we get immediately

$$K' \cup K'' \subset D_1 \cap D_2 \cap D_3. \tag{10}$$

Since the sets K' and K'' are disjoint, the intersection $D_1 \cap D_2 \cap D_3$ must contain at least one straight line segment and the boundary of $D_1 \cap D_2 \cap D_3$ has to contain an arc of each circle C_i, $i = 1, 2, 3$. The only possible relative position of the three circles is that indicated in the Figure 1.

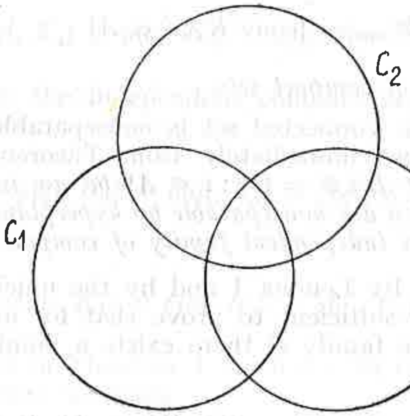

Figure 1.

It is easy to see that it is impossible to inscribe in the intersection $D_1 \cap D_2 \cap D_3$ two disjoint compact arcwise connected sets K_1 and K_2, which have to satisfy the conditions (8) and (9). This contradiction completes the proof of Lemma 1.

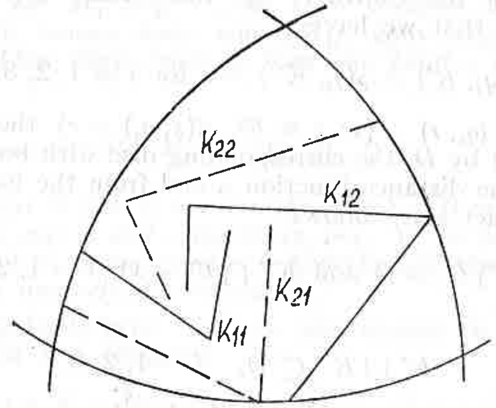

Figure 2

Remark. The hypothesis that the sets K_i are arcwise connected cannot be replaced by a weaker one requiring only that each set K_i should be nonseparable by hyperplanes. This can be seen from Figure 2 in which $K_1 = K_{11} \cup K_{12}$ and $K_2 = K_{21} \cup K_{22}$.

L e m m a 2. *Let* $\mathcal{K} = \{K_i : i \in A\}$ *be an independent family of compact arcwise connected sets in* R^n. *Then*

$$\mathcal{K}_{\text{conv}} = \{\text{conv } K_i : i \in A\}$$

is an independent family of compact sets.

Proof. As any arcwise connected set is nonseparable by hyperplanes in R^n, this lemma follows immediately from Theorem 1 in [3] which asserts us the following: *Let* $\mathcal{F} = \{F_i : i \in A\}$ *be an independent family of compact sets in* R^n, *which are nonseparable by hyperplanes. Then* $\mathcal{F}_{\text{conv}} = \{\text{conv } F_i : i \in A\}$ *is an independent family of compact sets.*

Proof of Theorem 3. By Lemma 1 and by the unicity part of Theorem 1 (with $k = 0$) it is sufficient to prove that for any four members K_i, K_j, K_l and K_m of the family $\mathcal{K}$ there exists a point p such that we have

$$\rho(p, K_i) = \rho(p, K_j) = \rho(p, K_l) = \rho(p, K_m).$$

Consider therefore five arbitrary members of the family $\mathcal{K}$, which we shall denote by $K_1, K_2, \ldots, K_5$. Corresponding to the sets K_1 and K_2 there exists by Lemma 1 at most two points q_1 and q_2 with the properties

(11) $$\rho(q_1, K_1) = \rho(q_1, K_2) = r$$

and

(12) $$\rho(q_2, K_1) = \rho(q_2, K_2) = r.$$

For a given i, $3 \leq i \leq 5$, the corresponding point q_{12i} has to coincide either with q_1 or with q_2. It results that at least two of the points q_{123}, q_{124} and q_{125} have to coincide with one and the same point of q_1 and q_2. Without loss of generality we can suppose that $q_{123} = q_{124} = q_1$. If q_{125} coincides also with q_1, for the five sets $K_1, K_2, \ldots, K_5$ results the existence of a point, namely q_1, such that

(13) $$\rho(q_1, K_i) = r \qquad i = 1, 2, \ldots, 5.$$

Consider now the case $q_{125} = q_2$. Because of the Lemma 1 there exist at most two points q_1' and q_2' such that

(14) $$\rho(q_1', K_1) = \rho(q_1', K_5) = r$$

and

(15) $$\rho(q_2', K_1) = \rho(q_2', K_5) = r.$$

It results as above that for one of this points, let him be q_1', there are two indices i and j, $2 \leq i < j \leq 4$ such that we have

(16) $$\rho(q_1', K_1) = \rho(q_1', K_i) = \rho(q_1', K_j) = \rho(q_1', K_5).$$

It results that for the independent compact arcwise connected sets K_1, K_i and K_j we have

(17) $$\rho(q_1, K_1) = \rho(q_1, K_i) = \rho(q_1, K_j) = r$$

and

(18) $$\rho(q_1', K_1) = \rho(q_1', K_i) = \rho(q_1', K_j) = r.$$

By the unicity part of Theorem 1 (with $k = 0$) the points q_1 and q_1' have to coincide. Therefore we have

(19) $$\rho(q_1, K_i) = r \qquad i = 1, 2, \ldots, 5.$$

This completes the proof of the theorem.

Remark. From the Figure 3 it can be seen that the requirement card $A \geq 5$ of the Theorem 3 is essentially. If we consider $K_i = \{p_i\}$

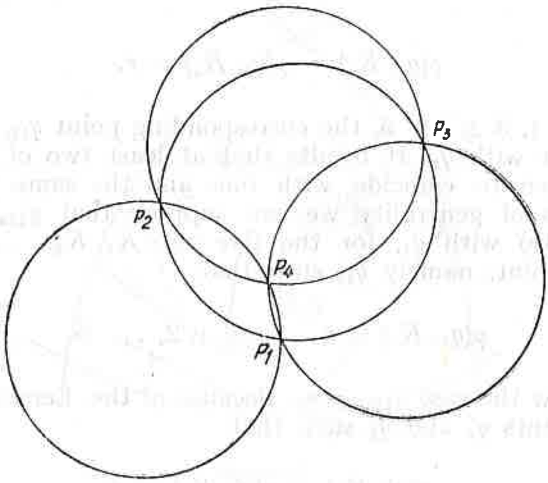

Figure 3

$i = 1, 2, 3, 4$, the family $\mathcal{K} = \{K_1, K_2, K_3, K_4\}$ is an independent family of compact arcwise connected sets, such that for every three members of $\mathcal{K}$ there is a point which satisfies the condition (4) of Theorem 3. But there is no point which verifies $\rho(q, K_i) = r$ for $i = 1, \ldots, 4$.

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Received 27.XII.1976.

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