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REDUCTION OF A DEGENERATED GEOMETRIC PROGRAM TO THE CANONICAL FORM

by

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1. Introduction

It is known [1] that every degenerated geometric program, which is not totally degenerated, can be reduced to an equivalent canonical geometric program, called reduced form of the corresponding geometric program. The purpose of this paper is to describe a constructive method which permits such a reduction. First some criteria for canonical geometric programs in terms of dual constraints are given. Then a simplex-like technique for the construction of the irreductible integer set [1, pag. 169] is described. To illustrate the algorithm a small numerical example is also considered.

2. Degeneracy criteria

Let

(P) $$\inf \{p_0(x) \mid p_k(x) \leqslant 1,\ k = 1, 2, \ldots, p\ ;\ x > 0\}$$

be a standard geometric program, where

$$p_k(x) = \sum_{i \in I_k} u_i(x) = \sum_{i \in I_k} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_n^{a_{in}}$$

is a posinomial, i.e. $c_i \in \mathbf{R}_+$, $a_{ij} \in \mathbf{R}$, and I_k are integer sets such that

$$I_k \cap I_h = \varnothing,\ k \neq h,\ \bigcup_{k=0}^{p} I_k = \{1, 2, \ldots, m\}.$$

As it is known [1], the dual of (P) is the following program:

(D) $$\sup \{v(y) \mid A^T y = 0, \sum_{i \in I_0} y_i = 1 \,;\, y \geqslant 0\}.$$

where A^T is transpose of the exponent matrix $A = (a_{ij})$ and v is the dual function, i.e,

$$v(y) = \prod_{i=1}^{m} \left(\frac{c_i}{y_i}\right)^{y_i} \prod_{k=1}^{p} \lambda_k(y)^{\lambda_k(y)}$$

$$\lambda_k(y) = \sum_{i \in I} y_i.$$

Let Ω and Ω^* be the sets of feasible solutions to (P) and (D) respectively. We design by

$$\Gamma = \{i \mid \exists y \in \Omega^*, y > 0\}$$

the irreductible integer set. If $\Gamma = \{1, 2, \ldots, m\}$, then program (P) (and program (D)) is called *canonical* [1, pag. 169]. Otherwise program (P) is called *degenerated*. If $\Gamma \cap I_0 = \emptyset$, then program (P) is called *totally degenerated*.

Let us put the dual constraints in the matricial form:

$$A_1 y = b, \; y \geqslant 0,$$

where

$$A_1 = \begin{pmatrix} 1 \ldots 1 & 0 \ldots 0 \\ & A^T \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In what follows we will use the notation:

$$a^{i\cdot} = (a_{i1}, a_{i2}, \ldots, a_{in}), \; a^{\cdot j} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$$

for the row vector and column vector respectively of a given matrix $A = (a_{ij})$. We will assume that rank $A = n$.

Let B and N be basic and nonbasic column vectors of the matrix A_1 respectively, and y^B, y^N corresponding basic and nonbasic variables. Then each $y \in \Omega^*$ can be written under the form: $y = (y^B, y^N)$, where

(1) $$y^B = B^{-1}b - B^{-1}Ny^N = d^{\cdot 0} + \sum d^{\cdot j} y^N \geqslant 0, \; y^N \geqslant 0,$$

where

$$d^{\cdot 0} = B^{-1}b, \; D = -B^{-1}N.$$

So $y = (d^{\cdot 0}, 0)$ is a basic feasible solution (b.f.s.) to (D) if and only if $d^{\cdot 0} \geqslant 0$. We assume that $d^{\cdot 0} \geqslant 0$.

If

(2) $$I_D = \{i \mid d_i = 0\}$$

then we have the following basic criterion for the canonical geometric programs.

THEOREM 1. *Let $y^B = d^{\cdot 0} + Dy^N$ be given in* (1) *and $d^{\cdot 0} \geqslant 0$. Geometric program* (P) *is canonical if and only if*

(3) $$\mu = \max \{\min_{i \in I_D} \{d^{i\cdot} y^N\} \mid A_1 y = b, \; y \geqslant 0\} > 0.$$

Proof. Necessity. If (P) is a canonical program then there is $\bar{y} \in \mathbf{R}^m_+$ such that

$$A_1 \bar{y} = b, \; \bar{y} > 0,$$

i.e.

$$\bar{y}^B = d^{\cdot 0} + D\bar{y}^N > 0, \; \bar{y}^N > 0.$$

So

$$\forall i \in I_D \Rightarrow d^{i\cdot} \bar{y}^N > 0$$

and, therefore,

$$\min \{d^{i\cdot} \bar{y}^N \mid i \in I_D\} > 0.$$

Because $\bar{y} \in \Omega^*$ it follows that

$$\mu = \max \{\min_{i \in I_D} \{d^{i\cdot} y^N\} \mid A_1 y = b, \; y \geqslant 0\} \geqslant \min_{i \in I_D} \{d^{i\cdot} \bar{y}^N\} > 0.$$

Sufficiency. Assume $\mu > 0$. Then there is $y^* = (y^{*B}, y^{*N})^T$ such that

$$\mu = \min_{i \in I_D} d^{i\cdot} y^{*N} > 0$$

i.e.

$$\forall i \in I_D \Rightarrow d^{i\cdot} y^{*N} > 0.$$

Without loss of generality, we can assume that $y^{*N} > 0$, since otherwise by taking $y^{\varepsilon N} = y^{*N} + \varepsilon$, $\varepsilon \in \mathbf{R}^n$, we have

$$d^{i\cdot} y^{\varepsilon N} = d^{i\cdot}(y^{*N} + \varepsilon) = d^{i\cdot} y^{*N} + d^{i\cdot}\varepsilon > 0, \; i \in I_D$$

for $\varepsilon > 0$ sufficiently small. So in this case

$$y^{\varepsilon B} = d^{\cdot 0} + Dy^{\varepsilon N} > 0.$$

Let $y^0 = (y^{0B}, y^{0N}) = (d^{\cdot 0}, 0) \in \Omega^*$. We consider

$$y(t) = (1 - t)y^0 + ty^*, \; t \in [0, 1].$$ Clearly

(4) $$y^N(t) = t\, y^{*N} > 0, \; t > 0$$

and

$$y^B(t) = d^{\cdot 0} + tDy^{*N}.$$

If $i \in I_D$, then $d_{i0} = 0$, and it follows that

$$(5) \qquad y_i^B(t) = td^{i\cdot}y^{*N} > 0, \quad \forall t \in]0, 1].$$

If $i \notin I_D$, then $d_{i0} > 0$, and we have

$$(5') \qquad y_i^B(t) = d_{i0} + t(d^{i\cdot}y^{*N}) > 0$$

for every $t > 0$ dufficiently small. Therefore, for $t > 0$ sufficiently small $y^B(t) > 0$. From (4), (5) and (5′) it follows that for $t \in]0, 1]$ sufficiently small,

$$y(t) = (y^B(t), y^N(t)) > 0.$$

According to the convexity of Ω^*, y^*, $y^0 \in \Omega^*$ implies that $y(t) \in \Omega^*$. So program (P) is canonical.

Corollary 1. *Let $y^B = d^{\cdot 0} + Dy^N$ be given in (1), $d^{\cdot 0} \geqslant 0$ and $I_D \neq \emptyset$. If $i \in I_D$ and*

$$(6) \qquad \mu_i = \max \{d^{i\cdot}y^N \mid A_1 y = b, \; y \geqslant 0\} = 0$$

then program (P) is degenerated and $y_i^B = 0$ for each $y \in \Omega^*$.

Proof. If $\mu_i = 0$, then

$$\mu = \max \{\min_{i \in I_D} \{d^{i\cdot}y^N\} \mid A_1 y = b, \; y \geqslant 0\} \leqslant$$

$$\leqslant \min_{i \in I_D} \{\max \{d^{i\cdot}y^N \mid A_1 y = b, \; y \geqslant 0\} \leqslant$$

$$\leqslant \max \{d^{i\cdot}y^N \mid A_1 y = b, \; y \geqslant 0\} = \mu_i = 0.$$

From Theorem 1 it follows that (P) is degenerated. Moreover, in this case

$$(7) \qquad \forall y \in \Omega^* \Rightarrow y_i^B = d^{i\cdot}y^N \leqslant 0,$$

i.e. $y_i^B = 0$.

Remark 1. From (7) it follows that

$$y_i^B = 0 \Leftrightarrow y_j^N = 0, \quad \forall j \in J = \{j \mid d_{ij} > 0\}$$

since $d_{ij} \geqslant 0$, $j = 1, 2, \ldots, n$.

THEOREM 2. *If dual program (D) has a nondegenerated b.f.s., then program (P) is canonical.*

Proof. Let $y^B = (d^{\cdot 0}, 0) \in \Omega^*$ be a nondegenerated b.f.s. Then $d^{\cdot 0} > 0$. So for $y^N > 0$ sufficiently small (i.e. $0 < y_i < \varepsilon$, $i = 1, 2, \ldots, n$) we have

$$y^B = d^{\cdot 0} + Dy^N > 0$$

and, therefore, program (P) is canonical.

THEOREM 3. *Let $y^B = d^{\cdot 0} + Dy^N$ be given in (1) and $I_D \neq \Phi$. If*

$$\forall i \in I_D \Rightarrow \mu_i = \max \{d^{i\cdot}y^N \mid A_1 y = b, \; y \geqslant 0\} > 0$$

then program (P) is canonical.

Proof. Let $y^i \in \Omega^*$ be the optimal solution to (6), i.e.

$$d^{i\cdot}y^i = \max \{d^{i\cdot}y^N \mid y \in \Omega^*\}, \; i \in I_D.$$

Consider $t_i \in]0, 1[$, $i \in I_D$ with $\sum t_i = 1$. From the convexity of Ω^*, we have

$$y(t) = \sum_{i \in I_D} t_i y^i \in \Omega^*.$$

So

$$d^{i\cdot}y^N(t) = \sum_{i \in I_D} t_i d^{i\cdot}y^{iN} > t_i d^{i\cdot}y^{iN} > 0, \; i \in I_d$$

and, therefore,

$$\max\{\min_{i \in I_D} \{d^{i\cdot}y^N \mid y \in \Omega^*\} \geqslant \min_{i \in I_D} \{d^{i\cdot}y^N(t)\} > 0.$$

Now from Theorem 1 it follows that (P) is canonical.

3. Statement of the algorithm

The results of the previous section are now incorporated into the following algorithm, which permits to establish whether a given geometric program is canonical or not. In the last case the procedure permits to select all components y_i which are zero for each dual feasible solutions $y \in \Omega^*$, and therefore, to reduse every degenerated geometric program (which is not totally degenerated) to a canonical one (its reduced form).

Algorithm

Step 1. Apply the simplex method to find a b.f.s. of the form (1) to the dual geometric program (D).

Step 2. Test for the degeneracy of the basic feasible solution. If b.f.s. is not degenerated then terminate, geometric program is canonical (Theorem 2). Otherwise go to Step 3.

Step 3. Construct the set I_D defined in (2).

Step 4. Solve problem (6) for the first $i \in I_D$. If $\mu_i > 0$, then go to another $i \in I_D$. Otherwise put $y_j = 0$ for all

$$j \in \{i\} \cup \{k \mid d_{ik} > 0\}$$

and go to another $i \in I_D$ or terminate if all $i \in I_D$ were tested.

4. Example

Consider the following geometric program

$$\inf \{p_0(x) = x_1x_3 + 2x_1x_2^{-1}x_3^2x_4^2 + x_1x_3^{-1}\}$$

subject to

$$p_1(x) = 2x_3 + x_2^{-1}x_3^3x_4^5 \leqslant 1,$$

$$p_2(x) = x_3^{-1}x_4 + x_1^{-1}x_3^{-1} \leqslant 1, \ x_j > 0, \ j = 1, 2, 3, 4.$$

The corresponding dual sistem is:

$$\begin{aligned} y_1 + y_2 + y_3 \quad &= 1 \\ y_1 + y_2 + y_3 \quad - y_7 &= 0 \\ - y_2 \quad - y_5 \quad &= 0 \\ y_1 + 2y_2 - y_3 + y_4 + 3y_5 - y_6 - y_7 &= 0 \\ 2y_2 \quad + 5y_5 + y_6 \quad &= 0, \ y_i \geqslant 0, \ i = 1, \ldots, 7. \end{aligned}$$

Step 1. To find a b.f.s. we start from the simplex tableau:

	$-y_1$	$-y_2$	$-y_3$	$-y_4$	$-y_5$	$-y_6$	$-y_7$	1
$0 =$	1	1	1	0	0	0	0	1
$0 =$	1	1	1	0	0	0	-1	0
$0 =$	0	-1	0	0	-1	0	0	0
$0 =$	1	2	-1	1	3	-1	-1	0
$0 =$	0	2	0	0	5	1	0	0

After five Gauss-Jordan steps, we get the following b.f.s.

(7)

	$-y_2$	$-y_3$	1
$y_7 =$	0	0	1
$y_1 =$	1	1	1
$y_5 =$	1	0	0
$y_4 =$	-5	-2	0
$y_6 =$	-3	0	0

Step 2. It is clear that b.f.s. $y = (1, 0, 0, 0, 0, 0, 1)$ is degenerated.

Step 3. $I_D = \{4, 5, 6\}$.

Step 4. To maximize y_4 we start with simplex tableau:

	$-y_2$	$-y_3$	1
$y_7 =$	0	0	1
$y_1 =$	1	1	1
$y_5 =$	1	0	0
$y_4 =$	-5	-2	0
$y_6 =$	-3	0	0
$y_4 =$	-5	-2	0

After two Gauss-Jordan steps we get the tableau:

	$-y_5$	$-y_1$	1
$y_7 =$	0	0	1
$y_3 =$	-1	1	1
$y_2 =$	1	0	0
$y_4 =$	3	2	2
$y_6 =$	3	0	0
$y_4 =$	3	2	2

i.e.

$$\max \{y_4 \mid y \in \Omega^*\} = 2 > 0.$$

From (7) it is seen that

$$y_5 = -y_2 \leqslant 0, \ \forall y_2 \geqslant 0,$$

i.e. $\max \{y_5 \mid y \in \Omega^*\} = 0$, and so $y_2 = y_5 = 0$. From (7) we also have $y_6 = 3y_2 = 0$.

Therefore, the reduced form of the initial program is:

(8) $$\inf \{\bar{p}_0(x) = x_1x_3 + x_1x_3^{-1}\}$$

subject to

(9) $$\bar{p}_1(x) = 2x_3 \leqslant 1$$

$$\bar{p}_2(x) = x_1^{-1}x_3^{-1} \leqslant 1, \ x_1, x_3 > 0.$$

The dual of (8)—(9) is:

$$\sup\left\{\bar{v}(y) = \left(\frac{1}{y_1}\right)^{y_1}\left(\frac{1}{y_2}\right)^{y_2}\left(\frac{2}{y_3}\right)^{y_3}\left(\frac{1}{y_4}\right)^{y_4}\cdot y_3^{y_3}\cdot y_4^{y_4}\right\},$$

$$\begin{aligned} y_1 + y_2 \qquad\qquad &= 1\\ y_1 + y_2 \qquad - y_4 &= 0\\ y_1 - y_2 + y_3 - y_4 &= 0,\ y_1, y_2, y_3, y_4 \geqslant 0.\end{aligned}$$

The optimal solution of this dual program is

$$y^* = (1/5,\ 4/5,\ 8/5,\ 1),\quad \bar{v}(y^*) = 5.$$

Using the duality theorem [1, chapter IV] we find that $x^* = (2, 1/2)$ is the optimal solution to the reduced program, i.e. $\bar{p}_0(x^*) = 5 = \min\{\bar{p}_0(x) \mid \bar{p}_1(x) \leqslant 1, \bar{p}_2(x) \leqslant 1, x_1, x_3 > 0\}$.

It follows that $\inf p_0(x) = 5$, but the initial program has no optimal solution, because $5 = p_0(2, 0, 1/2, 0)$, i.e. $x_2 = x_4 = 0$.

REFERENCES

[1] Duffin, R. J.; Peterson, E. L.; Zener, C., *Geometric programming-theory and application.* John Willey, New York, 1967.

[2] Maruşciac, I., *Metode de rezolvare a problemelor de programare neliniară.* Editura Dacia, 1973.

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